Introduction to Laboratory 3

Laboratory 3 concerns the implementation of a **Hybrid method** to find the root of a function. On one hand, in Laboratory 1 we have seen that the **Bisection method** is a very robust method (i.e. it always works) to find the root but it is rather slow (i.e. it requires many iterations to reach convergence). On the other hand, in Laboratory 2 we have seen that the **Newton-Raphson (NR) method** is much faster (i.e. it requires less iterations) but it may fail when the derivative $f'(x)$ of the function f is close to zero. Thus, the idea of the **Hybrid method** is to combine the advantages of both methods in order to obtain a method that is both fast and robust. In simple words, the Hybrid method will do a NR step when NR is accurate and a Bisection step when NR is not accurate enough.

Similarly to the Bisection method we start with an initial interval $[x_L, x_R]$ comprising the root (see the Figure 1 for an example). Starting from the initial value $x_0 = x_L$ (we could also start from $x_0 = x_R$ instead) the results of a NR step is evaluated using the expression of Laboratory 2, i.e. $x_0 - f(x_0)/f'(x_0)$. In the case of Figure 1, this leads to a value of x that is outside the initial interval $[x_L, x_R]$ (see red line). In that case, NR is not accurate enough and we will do instead a Bisection step giving the next approximation to the root $x_1 = (x_L + x_R)/2$ (see Figure 1). Then, the size of the interval is decreased to $[x_L, x_1]$ like in the Bisection method (i.e. new x_R is equal to x_1). Next, we evaluate the results of a NR step starting from x_1. This gives a value of x that falls inside the current interval $[x_L, x_1]$ (see black line). In that case NR is accurate enough and we keep this results (x_2) as the next approximation to the root. Similarly to a Bisection step, the interval should be decreased to$[x_2, x_1]$ after the NR step, i.e. new x_L is equal to x_2. Then, the process should be iterated until convergence is obtained.

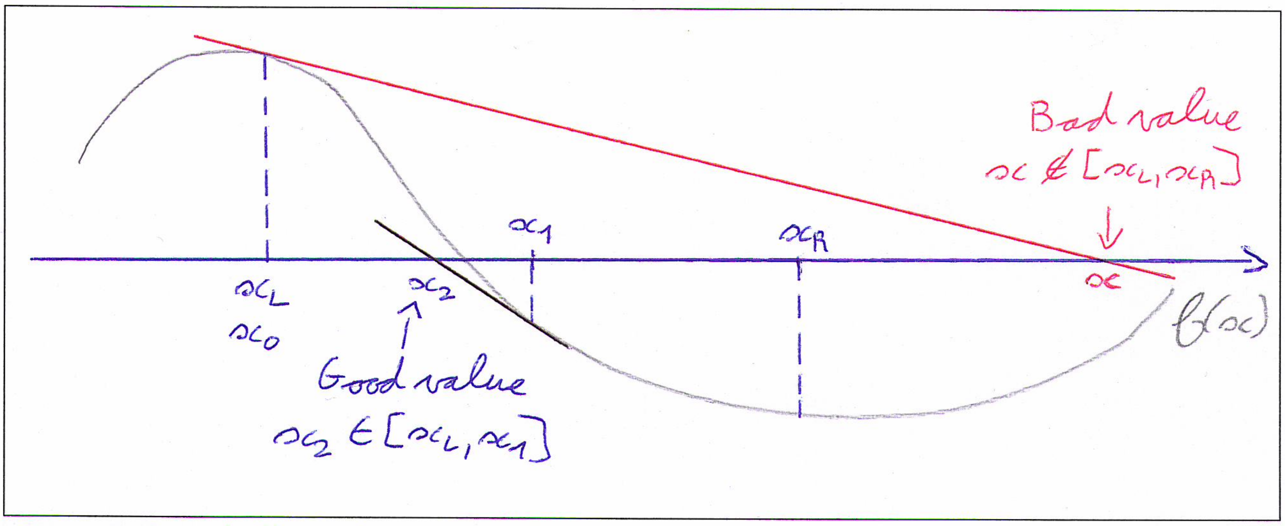

Figure 1: Example illustrating the Hybrid method.

In summary, we do a NR step if the inequality (1): $x_L \leq x_0 - f(x_0)/f'(x_0) \leq x_R$ is true, else we do a Bisection step. After each step, the interval $[x_L, x_R]$ should be updated.

To check if the inequality (1) is true or false we can write it as:

$$LT \geq 0 \geq RT \text{ if } f'(x_0) > 0$$

$$LT \leq 0 \leq RT \text{ if } f'(x_0) < 0$$

With

$$LT = (x_0 - x_L)f'(x_0) - f(x_0)$$

$$RT = (x_0 - x_R)f'(x_0) - f(x_0)$$

This means that the inequality (1) is true when LT and RT have different signs and (1) is false when LT and RT have the same sign.

Solutions:

1) For the polynomial $P(x)$, starting with $x_L = x_0 = 0.4$, $x_R = 0.7$ and *Tolerance*=10^{-8}, the program does 1 Bisection step and then 3 NR steps. The root is equal to 0.525 532 41.

2) For the function f(x), starting with $x_L = x_0 = 0$, $x_R = 3$ and *Tolerance*=10^{-8}, the program does 2 Bisection steps and then 5 NR steps. The root is equal to 2.732 050 81.