

THE POINT ESTIMATE METHOD

The expected value, standard deviation or probabilistic moments of higher order of any random function can be efficiently determined using **Point Estimate Method (PEM)**; Rosenblueth (1975).

The method is straightforward, easy to use, and requires little knowledge of probability theory.

PEM involves the use of a Gaussian quadrature for determining the probabilistic moments of random function.

It can be readily applied to response functions that are not closed-form or explicit, and to the results of existing deterministic programs.

A limitation of the Rosenblueth's point estimate method for multiple variables is that it requires calculations at 2^n points. This significantly increases computational time and effort.

Rosenblueth (1975) also proposed a technique for reducing the number of calculation points to $2n+1$ in a case of uncorrelated variables and when skewness can be ignored.

DESCRIPTION

In the point estimate method, a continuous random variable is generally replaced by a discrete random variable consisting of N pulses (Fig. 1), with a probability distribution

$$p_x(x) = \sum_{i=1}^N \delta(x - x_i) p(x_i) \quad (1)$$

where: $\delta(x - x_i)$ is Dirac delta, and $p(x_i)$ is the probability assigned to points x_i , for fixed values of random variables.

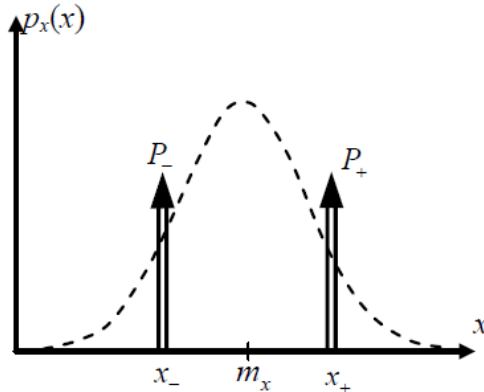


Fig.1. Distribution and probability density function

There can be infinite number of distributions described by (1).

It is most important to find a small number of points N , that have the same probabilistic moments of order k , as the probability density function $f_x(x)$.

Most frequently two points are chosen.

Comparing the first three probabilistic moments, determined in accordance with the definition of density or distribution probability, one can write:

$$\begin{aligned}m_x &= \int_{-\infty}^{\infty} x f_x(x) dx = \sum_{i=1}^N x_i p(x_i) \\ \sigma_x^2 &= \int_{-\infty}^{\infty} (x - m_x)^2 f_x(x) dx = \sum_{i=1}^N (x_i - m_x)^2 p(x_i) \\ \gamma_1 \sigma_x^3 &= \int_{-\infty}^{\infty} (x - m_x)^3 f_x(x) dx = \sum_{i=1}^N (x_i - m_x)^3 p(x_i)\end{aligned}\tag{2}$$

where: m_x – expected value, σ_x – standard deviation, γ_1 – asymmetry coefficient of random variable X .

While transforming the random variable X into Y using function $y = g(x)$, a probability distribution is sought $p_y(y)$, where probabilistic moments can be easily computed.

These values approximate moments of the continuous function $f_y(y)$.

The expected value of the k^{th} -power of the discrete random variable Y , which replaces the continuous function $y = g(x)$, can be approximated as follows:

$$E[Y^k] \approx \sum_{i=1}^N g^k(x_i) p(x_i) \quad (3)$$

In the simplest case, a continuous random variable with given values m_x , σ_x and γ_1 can be represented by two probabilities $p(x_1) = P_-$ and $p(x_2) = P_+$ (called weights) are assigned to the points $x_1 \equiv x_-$ and $x_2 \equiv x_+$ (Fig. 1).

Taking into account that the sum of the probabilities must be equal to unity, and including notations introduced in (2), one can write:

$$\begin{aligned}
 P_- + P_+ &= 1 \\
 m_x &= x_- P_- + x_+ P_+ \\
 \sigma_x^2 &= (x_- - m_x)^2 P_- + (x_+ - m_x)^2 P_+ \\
 \gamma_1 \sigma_x^3 &= (x_- - m_x)^3 P_- + (x_+ - m_x)^3 P_+
 \end{aligned} \tag{4}$$

Its solution is the discretization of the points and the assigned weights:

$$\begin{aligned}
 x_- &= m_x + \left[\frac{\gamma_1}{2} - \sqrt{1 + \left(\frac{\gamma_1}{2} \right)^2} \right] \sigma_x, & x_+ &= m_x + \left[\frac{\gamma_1}{2} + \sqrt{1 + \left(\frac{\gamma_1}{2} \right)^2} \right] \sigma_x \\
 P_- &= \frac{1}{2} \left[1 + \frac{\gamma_1}{2} \frac{1}{\sqrt{1 + \left(\frac{\gamma_1}{2} \right)^2}} \right], & P_+ &= 1 - P_-
 \end{aligned} \tag{5}$$

Significant simplification is achieved for zero skewness, i.e. symmetric distribution. Substituting $\gamma_1 = 0$ into (5), one obtains:

$$x_- = m_x - \sigma_x, \quad x_+ = m_x + \sigma_x, \quad P_- = \frac{1}{2}, \quad P_+ = \frac{1}{2} \quad (6)$$

Considering (3), and substituting $N = 2$, this expression takes the form:

$$E[Y^k] \approx g^k(x_-)P_- + g^k(x_+)P_+ \quad (7)$$

So, two first probabilistic moments are given by the formulas:

$$\begin{aligned} m_y &= E[Y^1] \approx y_-P_- + y_+P_+ \\ \sigma_y^2 &= E[Y^2] - m_y^2 \approx y_-^2P_- + y_+^2P_+ - m_y^2 \end{aligned} \quad (8)$$

where: $y_+ = g(x_+)$, $y_- = g(x_-)$.

In the case where Y is a function of n , uncorrelated random variables X_1, X_2, \dots, X_n , each with zero asymmetry, i.e. $y = g(x_1, x_2, \dots, x_n)$, replace each continuous random variable with a discrete variable.

Probabilities are defined for the two values x_- and x_+ (smaller and higher than the expected value of the standard deviation).

Together one should select 2^n points for estimation.

For each point, a certain probability, that can be determined from the general formula below, is assigned:

$$P_{(s_1 s_2 \dots s_n)} = \frac{1}{2^n} \left[1 + \sum_{i=1}^{n-1} \sum_{j=i+1}^n (s_i)(s_j) r_{x_i x_j} \right] \quad (9)$$

$$\text{where: } s_i = \begin{cases} -1 & \text{dla } x_{i-} = m_{x_i} - \sigma_{x_i} \\ +1 & \text{dla } x_{i+} = m_{x_i} + \sigma_{x_i} \end{cases},$$

$r_{x_i x_j}$ – cross-correlation coefficient between random variables X_i and X_j .

In the given approach, probabilistic moments of random functions are written as the sum of all possible combinations multiplied by the products of probabilities.

Generalizing (3) to the case of n random variables, one can write:

$$E[Y^k] \approx \sum (y_i)^k P_i \quad (10)$$

where: y_i – value of the function designated for x_i , wherein i is a suitable combination of characters + i – describing the position of the point x_i , P_i – the probability assigned to the point x_i , according to (9).

For example, for two ($n = 2$) random variables X_1 i X_2 , formula (9) takes the form:

$$P_{(s_1, s_2)} = \frac{1}{4} \left[1 + (s_1)(s_2)r_{x_1, x_2} \right] \quad (11)$$

The probabilities can be denoted with a series of + or – indicators, in a way that the first sign refers to the variable X_1 , the second to X_2 , the third to X_3 , etc.

Probabilities $P_{--}, P_{+-}, P_{-+}, P_{++}$ should be determined at four points (Fig. 2), with the coordinates: $(m_{x_1} - \sigma_{x_1}, m_{x_2} - \sigma_{x_2}), (m_{x_1} + \sigma_{x_1}, m_{x_2} - \sigma_{x_2}), (m_{x_1} - \sigma_{x_1}, m_{x_2} + \sigma_{x_2}), (m_{x_1} + \sigma_{x_1}, m_{x_2} + \sigma_{x_2})$, respectively.

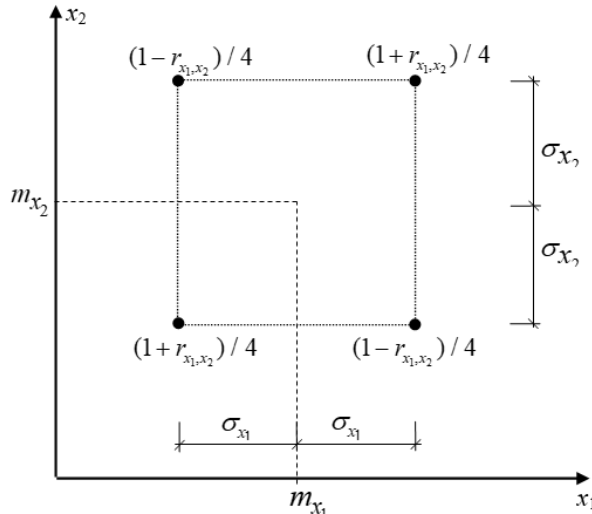


Fig. 2. The probabilities for the two correlated random variables

According to the formula (11) and the introduced indicators, the probabilities are equal to:

$$P_{++} = P_{--} = \frac{1}{4} [1 + r_{x_1 x_2}] , \quad P_{+-} = P_{-+} = \frac{1}{4} [1 - r_{x_1 x_2}] \quad (12)$$

In the case that X_1 and X_2 are uncorrelated ($r_{x_1 x_2} = 0$), the probabilities at all the points are the same and equal to 0.25.

The expected value and variance of random value Y according to (10), are given by the formulas:

$$\begin{aligned} m_y &\approx y_{++} P_{++} + y_{+-} P_{+-} + y_{-+} P_{-+} + y_{--} P_{--} \\ \sigma_y^2 &\approx y_{++}^2 P_{++} + y_{+-}^2 P_{+-} + y_{-+}^2 P_{-+} + y_{--}^2 P_{--} - m_y^2 \end{aligned} \quad (13)$$

where: $y_{++} = g(x_{1+}, x_{2+})$, $y_{+-} = g(x_{1+}, x_{2-})$, $y_{-+} = g(x_{1-}, x_{2+})$, $y_{--} = g(x_{1-}, x_{2-})$

In the above notation $g_{++}(x_{1+}, x_{2+})$ means that a value of the function y is calculated for $x_{1+} = m_{x_1} + \sigma_{x_1}$, $x_{2+} = m_{x_2} + \sigma_{x_2}$, etc.

ROSENBLUETH'S $2k+1$ POINT ESTIMATE METHOD

Complex problems may also be solved by using a point estimate method.

Although there are many such methods, the $2K + 1$ method proposed by Rosenblueth (1975) is one of the easiest to implement.

Loosely speaking, this method may be thought of as a simulation technique in which the number of simulations is $N = 2K + 1$ where K is the number of input random variables.

The basic idea is to evaluate a function of random variables at $2K + 1$ key points and then to use this information to estimate the mean and variance (or coefficient of variation) of the function.

However, the CDF of the function cannot be obtained by this method.

Consider a function Y described by

$$Y = f(X_1, X_2, \dots, X_k) \quad (14)$$

where $f(\bullet)$ is some deterministic function (but possibly not known in closed form) and the X_i ($i = 1, 2, \dots, k$) are the random input variables.

The steps in Rosenblueth's $2k+1$ method are as follows:

1. Determine the mean value (μ_{X_i}) and standard deviation (σ_{X_i}) for each of the k input random variables.

2. Define y_0 as the value of (14) when all input variables are equal to their mean values, that is,

$$y_0 = f(\mu_{X_1}, \mu_{X_2}, \dots, \mu_{X_k}) \quad (15)$$

3. Evaluate the function Y at $2K$ additional points as follows. For each random variable X_i evaluate the function at two values of X_i which are shifted from the mean value μ_{X_i} by $\pm\sigma_{X_i}$ while all other variables are assumed to be equal to their mean values. These values of the function will be referred to as y_i^+ and y_i^- . The subscript denotes the variable which is shifted, and the superscript indicates the direction of the shift. In mathematical notation,

$$y_i^+ = f(\mu_{X_1}, \mu_{X_2}, \dots, \mu_{X_i} + \sigma_{X_i}, \dots, \mu_{X_k}) \quad (16a)$$

$$y_i^- = f(\mu_{X_1}, \mu_{X_2}, \dots, \mu_{X_i} - \sigma_{X_i}, \dots, \mu_{X_k}) \quad (16b)$$

4. For each random variable, calculate the following two quantities based on y_i^+ and y_i^- :

$$\bar{y}_i = \frac{y_i^+ + y_i^-}{2} \quad (17a)$$

$$v_{y_i} = \frac{y_i^+ - y_i^-}{y_i^+ + y_i^-} \quad (17b)$$

5. Calculate the estimated mean and coefficient of variation of Y as follows:

$$\bar{Y} = y_0 \prod_{i=1}^k \left(\frac{y_i}{y_0} \right) \quad (18a)$$

$$v_Y = \sqrt{\left\{ \prod_{i=1}^k (1 + v_{y_i}^2) \right\} - 1} \quad (18a)$$

There are two distinct advantages to this method.

First, it is not necessary to know the distributions of the input random variables; only the first two moments are needed.

Second, the number of function evaluations (i.e., "simulations") is relatively small compared to Latin hypercube sampling or general Monte Carlo simulation.