



## PROBABILISTIC ANALYSIS OF A SPACE TRUSS BY MEANS OF A MULTIDIMENSIONAL VARIABLE DESCRIPTION

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Probabilistic analysis of a space truss is presented in the paper. Reliability of such a structure is sensitive to geometrical and material imperfections. The objective of this paper is to present a variant of the point estimate method (PEM) to determine mean values and standard deviations of limit loads of engineering structures. The main advantage presented by this method is the small number of sample calculations required to obtain estimators of investigated parameters. Thus the method is straightforward, requiring only preliminaries of probability theory. This approach is illustrated by limit state analysis of a space truss, considering geometric and material imperfections. The calculations were performed for different random models, so the influence of random parameters on the limit load of the truss can be determined. A realistic snow load was imposed.

*Keywords:* reliability, point estimated method, truss, imperfections, limit load analysis

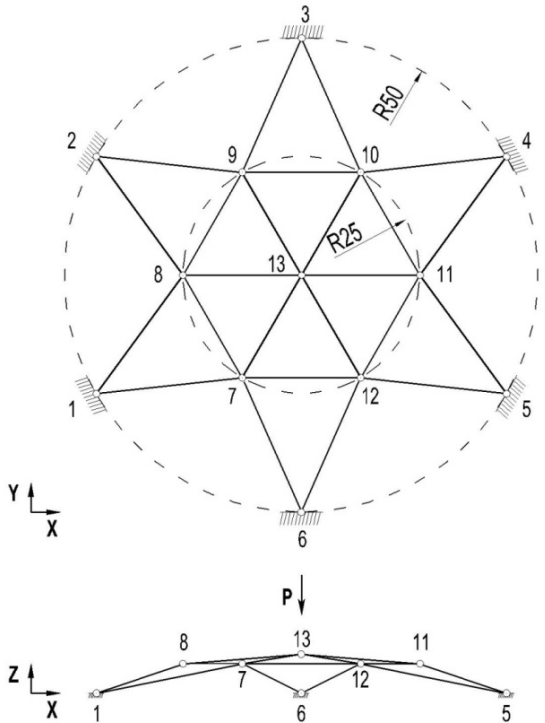
A typical example of a reticulated structure is analyzed.

Tubular sections RO 647.8x20 were designed for the structural elements.

All elements were made of S355 steel.

The elements were connected by means of ball joints.

Figure. Three dimensional truss structure (reticulated shell)



## ONE-DIMENSIONAL RANDOM VARIABLES

A simplest type of loading was assumed, i.e. a single force placed at the highest point of the structure – node no. 13 (Figure).

Calculations were made using the MSC Nastran code (2001).

The geometric and material nonlinearities were taken into account.

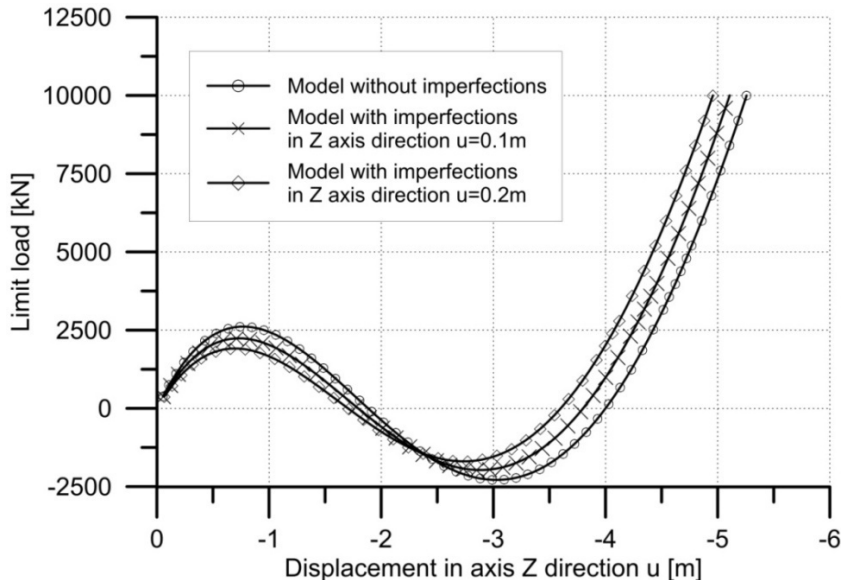


Figure. The equilibrium path – the ideal structure

The limit load for the ideal structure was calculated  $R = 2612.3$  kN

The limit state function was defined as the value of the admissible limit load.

Only geometric imperfections were considered.

Several preliminary examples were made, proving that even small changes in the geometric description, i.e. the nodal vertical coordinates – resulted in considerable changes of the limit load.

For example, displacement of the highest structure point (no. 13) only 0.10 m down led to a 14.3% drop of the limit load.

It should be pointed out that the horizontal displacements do not influence visibly the results.

**Next the probabilistic analysis was performed.**

The geometric discrepancy – the vertical displacement – was assumed having following parameters mean value  $m_u = 0.0$  m and standard deviation  $\sigma_u = 0.08$  m.

The majority of the generated values of imperfections belong to the following interval  $(-0.13, 0.13)$  m.

The assumed range of imperfections fulfills the maximum limits of discrepancies which are allowed during the production of the tubular elements.

The initial vertical displacements of the nodes 7 – 12 were calculated proportionally to the discrepancies of the node no. 13.

**The direct Monte Carlo method was applied.**

The calculations were performed for the generated 55 samples – vertical displacements of the node 13.

The accuracy of the results depends on the number of the samples.

To determine the minimal but sufficient number of field samples, a convergence analysis of the outcomes was performed.

After each sample calculations, the improved expected values and standard deviations of the outcomes were estimated.

The calculations were finished when the scatter of the estimated values reached the assumed low level.

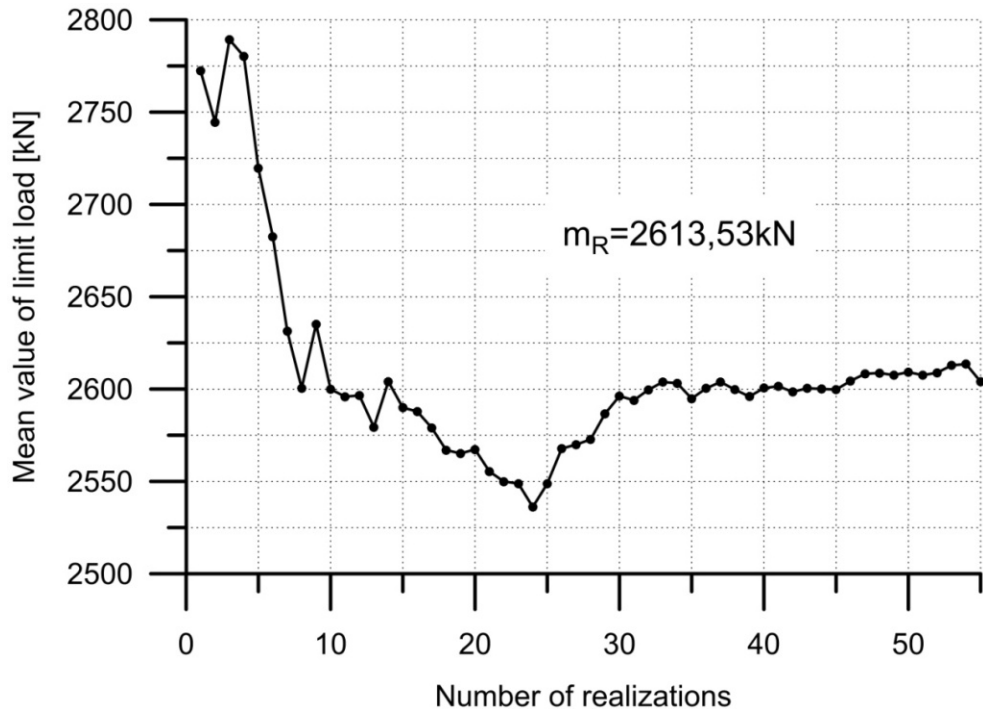


Figure. The direct Monte Carlo method – expected value

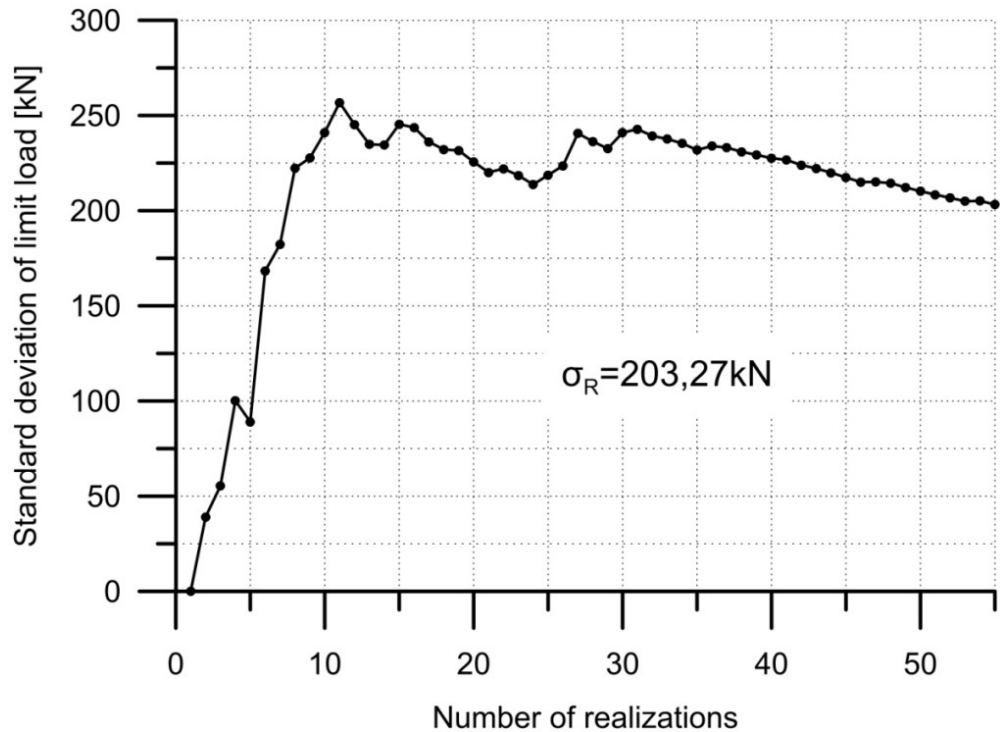


Figure. The direct Monte Carlo method – standard deviations

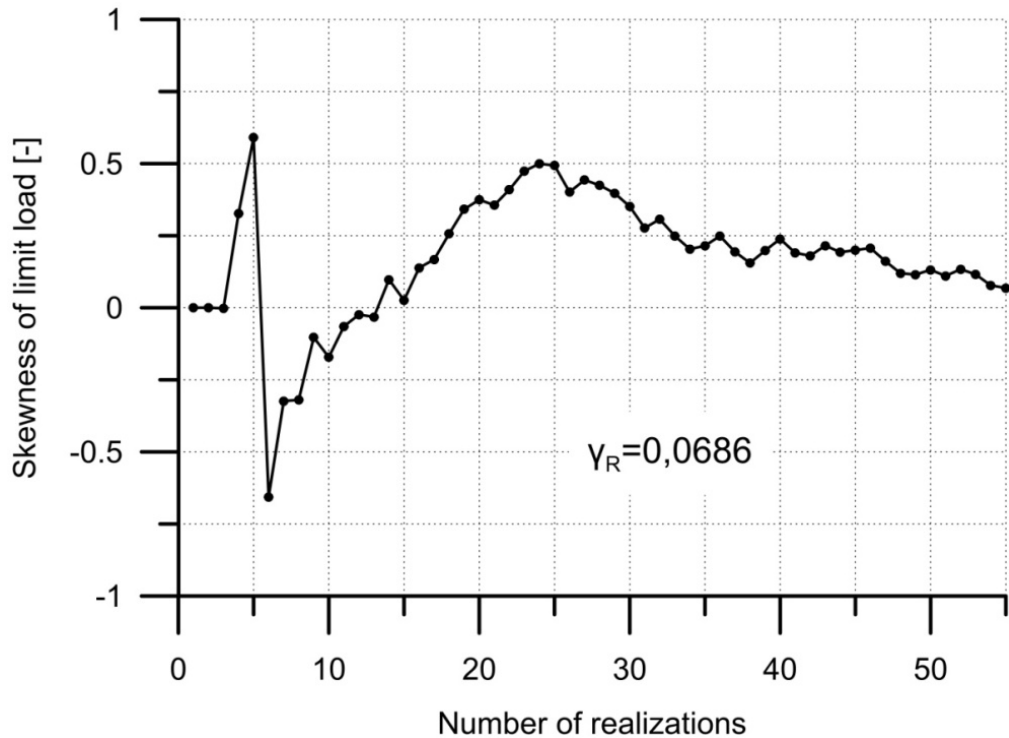


Figure. The direct Monte Carlo method – skewness



According to these calculations the following values of the critical load are assumed as a reference load: mean value  $\mu_R = 2613.53$  kN, standard deviation  $\sigma_R = 203.27$  kN and skewness  $\gamma_R = 0.0686$ .

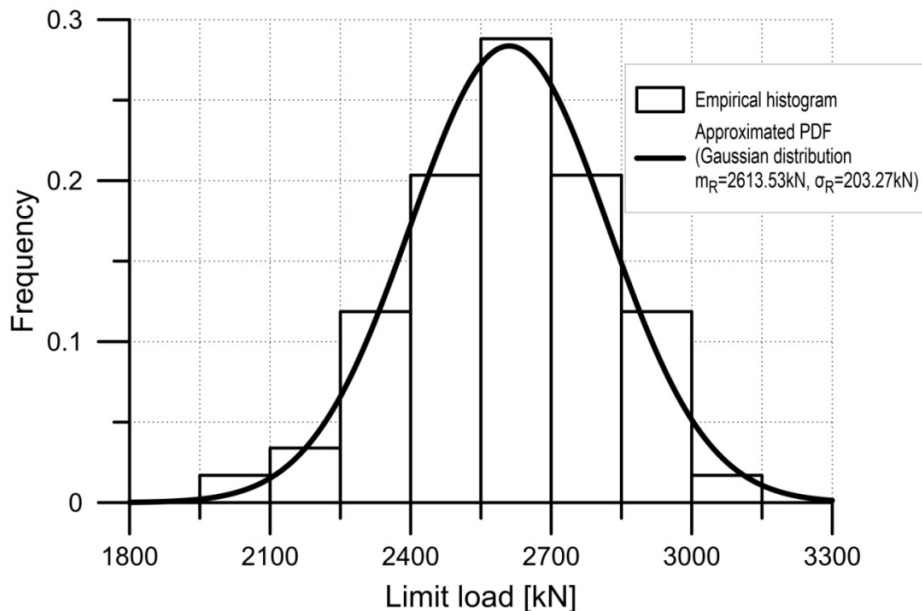


Figure. The direct Monte Carlo method – PDF and histogram of limit load

A significance **reductions of the number of the random samples** can be achieved using the Monte Carlo reduction methods.

**The method of stratified sampling was applied.**

The set of 55 random initial displacements identical to the case of the direct Monte Carlo method was used.

The generated samples were classified and arranged according to the magnitude of the displacements.

**Two methods were applied:** the whole space of the samples was divided into equal subsets or subsets of equal probability.

From each subset only one sample was chosen for the analysis.

The results of the calculations are presented in the next Figures and in Table.

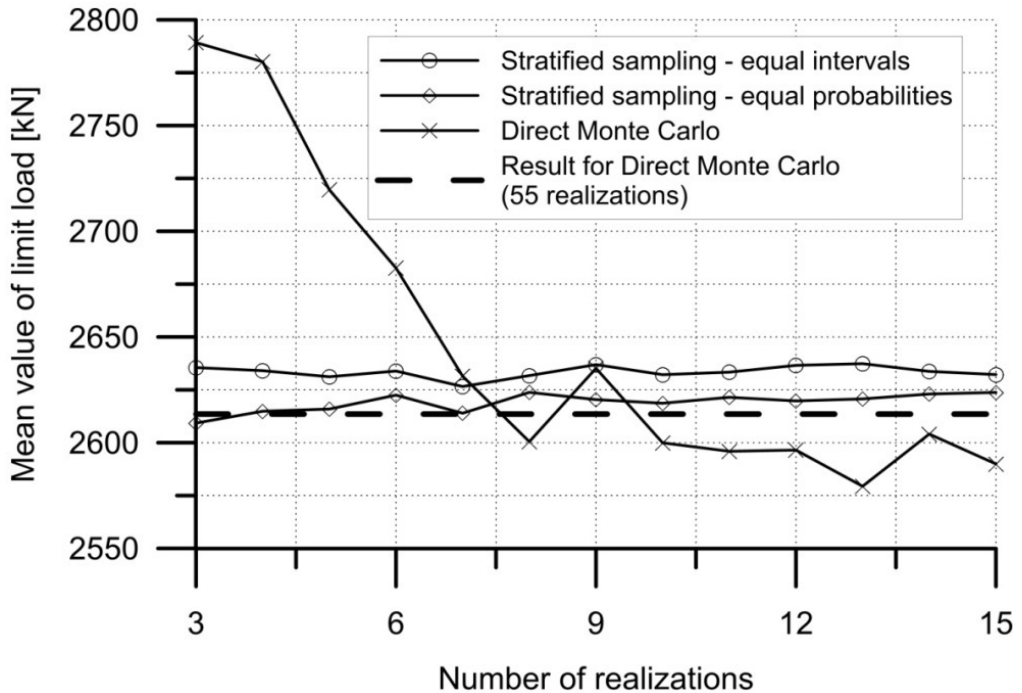


Figure. Comparison of the Monte Carlo and stratified sampling – mean values

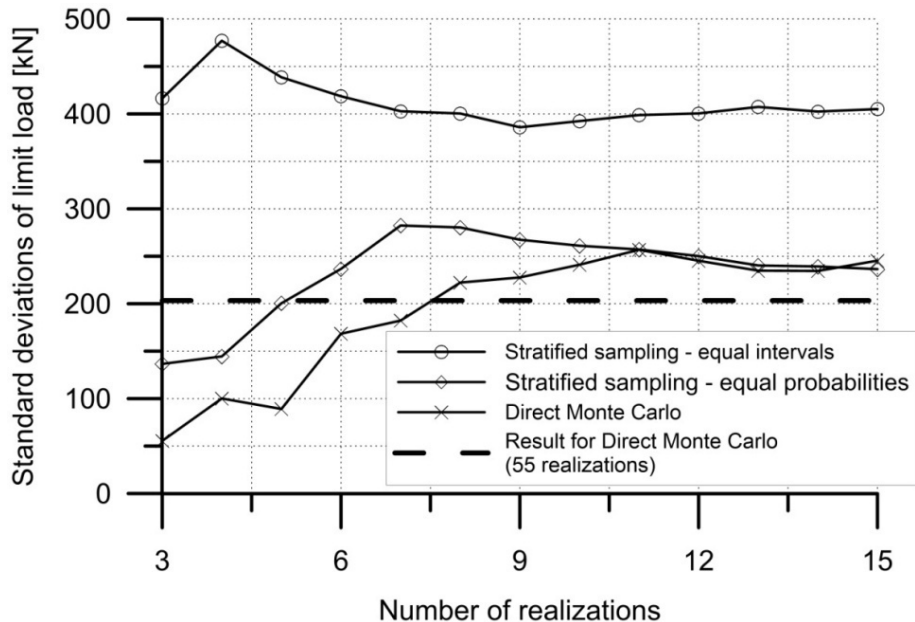


Figure. Comparison of the Monte Carlo and stratified sampling – standard deviations

Table. Results of limit load analysis of the reticulated shell  
– one dimensional case

Method used	Number of realizations $n$	Mean value of limit load $R$ [kN]	Standard deviation of limit load $R$ [kN]
Direct Monte Carlo	59	2610.40	212.74
Direct Monte Carlo	15	2604.08	245.38
Stratified sampling – equal intervals	15	2632.22	405.12
Stratified sampling – equal prob.	15	2623.79	236.43
Direct Monte Carlo	10	2635.18	241.06
Stratified sampling – equal intervals	10	2632.18	392.44
Stratified sampling – equal prob.	10	2618.66	261.10
Point Estimate Method	2	2605.12	239.26

A significant reduction of the samples was expected **using the point estimate method**.

According to (6) two displacement and their weights are calculated:

$$x_- = m_u - \sigma_u = 0.0 - 0.08 = -0.08 \quad P_- = 0.5$$

$$x_+ = m_u + \sigma_u = 0.0 + 0.08 = 0.08 \quad P_+ = 0.5$$

Then the reticulated shell capacities related to  $x_-$  and  $x_+$  are calculated

$$y_- = R(x_-) = 2844.38 \text{ kN}$$

$$y_+ = R(x_+) = 2362.86 \text{ kN}$$

Substituting the above results to (8) the expected value and standard deviation were determined for the limit load:

$$m_y = E[Y^1] \approx 2844.38 \cdot 0.5 + 2365.86 \cdot 0.5 = 2605.12 \text{ kN}$$

$$\sigma_R^2 = E[Y^2] - \mu_y^2 = 2844.38^2 \cdot 0.5 + 2365.86^2 \cdot 0.5 - 2605.12^2 = 57245.35 \text{ kN}^2$$

$$\sigma_R = \sqrt{57245.35} = 239.26 \text{ kN}$$

The results were compared with the Monte Carlo solution (Table).

It should be stressed that using the point estimated method only two calculations performed by the finite element program allows estimating the mean value and standard deviation of the limit load of the reticulated shell.

The errors of these values with the respect to the direct Monte Carlo methods are 0.2% and 12.5% respectively.

## Two-dimensional random variables

The next step of the analysis concerns two random variables – geometric and material parameters.

The second parameter – Young's modulus – was assumed Gaussian, its mean value  $m_E = 210.0$  GPa and standard deviation  $\sigma_E = 4.0$  GPa.

Two-dimensional image – the so called “ant hill” is presented in Figure.

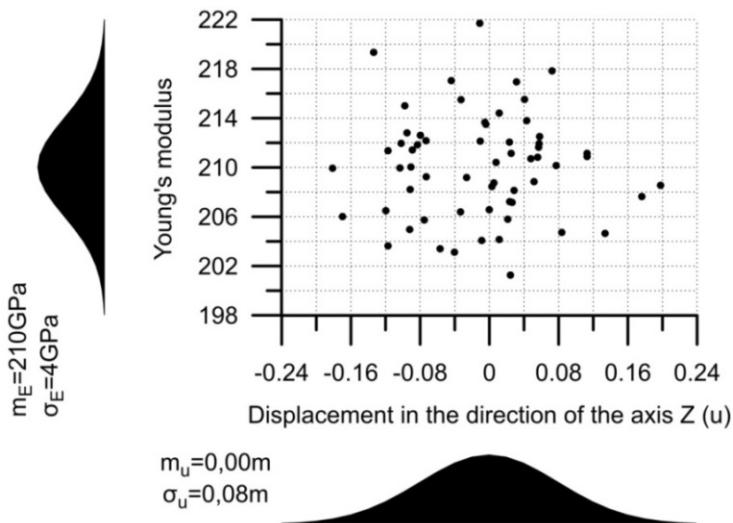


Figure. Monte Carlo method – distribution of two random variables



In the case of the Monte Carlo method 60 samples were generated.

The following results were obtained: mean value  $m_R = 2653.21$  kN, standard deviation  $\sigma_R = 232.54$  kN and skewness  $\gamma_R = 0.1004$ .

Similar calculation were performed for 36, 24, 16 and 9 samples (see Table).

Table. Results of limit load analysis of a reticulated shell – two dimensional case

Method used	Number of realizations $n$	Mean value of resistance $R$ [kN]	Standard deviation of resistance $R$ [kN]
Direct Monte Carlo	60	2653,21	232,54
Direct Monte Carlo	36	2658,71	229,21
Stratified sampling – equal intervals 6x6	36	2637,43	388,28
Stratified sampling – equal probabilities 6x6	36	2604,47	255,01
Direct Monte Carlo	25	2651,48	227,16
Stratified sampling – equal intervals 5x5	25	2645,16	410,19
Stratified sampling – equal probabilities 5x5	25	2630,21	237,22
Direct Monte Carlo	16	2602,15	253,78
Stratified sampling – equal intervals 4x4	16	2647,99	389,89
Stratified sampling – equal probabilities 4x4	16	2614,67	299,57
Direct Monte Carlo	9	2728,93	309,69
Stratified sampling – equal intervals 3x3	9	2633,06	348,67
Stratified sampling – equal probabilities 3x3	9	2656,06	328,89
Point Estimate Method	4	2620,48	229,38

The results were compared with the stratified sampling method outcomes.

The number of subsets was adjusted in order obtain the number of samples equal to the direct Monte Carlo method variant i.e. 6x6, 5x5, 4x4 and 3x3.

Similarly to a single random variable case two versions of stratified sampling were applied: using equal intervals and equal probabilities.

The results may be considered chaotic. It is difficult to decide which of the method is the most efficient and what the number of samples should produce satisfactory results.

In the case of the point estimate method computation was made for four truss models only (Table).

Relative errors, with respect to the direct Monte Carlo method are 1.2% and 1.4%, respectively.

Tab. 3. The point estimate method – two random variables

Point	$x_1$ [m]	$x_2$ [GPa]	$y$ [kN]	$p(x_{i,j})$	$yp(x_{i,j})$	$y^2 p(x_{i,j})$
1	$x_{1-} = -0.08$	$x_{2-} = 206$	2790.18	0.25	697.545	1946276
2	$x_{1+} = 0.08$	$x_{2+} = 214$	2443.23	0.25	610.8075	1492343
3	$x_{1-} = -0.08$	$x_{2+} = 214$	2898.30	0.25	724.575	2100036
4	$x_{1+} = 0.08$	$x_{2-} = 206$	2350.19	0.25	587.548	1380848
					$\sum = 2620.475$	$\sum = 6919503$
$E[R] = 2620.475kN, Var[R] \approx 6919503 - 2620.475^2 = 52614.07kN^2, \sigma_R \approx 229.38kN$						

## Multidimensional random variables

Next the truss model with three and four random variables was considered (Table).

Table. The considered cases of multivariate random variables

Set	Variable dimension	Variable type	Mean value of random variable	Standard deviation of random variable
Set "a"	2	Initial displacement of node 7-13 calculated proportionally to the displacement of node 13	0.0 m	0.08 m
		Young's modulus	210.0 GPa	4.0 GPa
Set "b"	2	Initial displacement of node 7-13 calculated proportionally to the displacement of node 13	0.0 m	0.08 m
		Cross-sectional area	394.458 cm <sup>2</sup>	2.000 cm <sup>2</sup>
Set "c"	2	Young's modulus	210.0 GPa	4.0 GPa
		Cross-sectional area	394.458 cm <sup>2</sup>	2.000 cm <sup>2</sup>
Set "d"	3	Initial displacement of node 7-13 calculated proportionally to the displacement of node 13	0.0 m	0.08 m
		Young's modulus	210.0 GPa	4.0 GPa
		Cross-sectional area	394.458 cm <sup>2</sup>	2.000 cm <sup>2</sup>
Set "e"	4	Initial displacement of node 13	0.0 m	0.08 m
		Initial displacement of node 7-12 calculated proportionally to the displacement of node 12	0.0 m	0.03 m
		Young's modulus	210.0 GPa	4.0 GPa
		Cross-sectional area	394.458 cm <sup>2</sup>	2.000 cm <sup>2</sup>

The standard and Rosenblueth version of the point estimate method and the Hong method were applied.

The results are presented in Table.

Table. Results of limit load estimation of the space truss using PEM

Set	PEM – original method proposed by Rosenblueth			PEM with reduction of evaluation point – Rosenblueth proposition			PEM with reduction of evaluation point –Hong proposition		
	Number of realization $n$	Mean value of limit load $R$ [kN]	Standard deviation of limit load $R$ [kN]	Number of realization $n$	Mean value of limit load $R$ [kN]	Standard deviation of limit load $R$ [kN]	Number of realization $n$	Mean value of limit load $R$ [kN]	Standard deviation of limit load $R$ [kN]
“a”	4	2620.49	229.38	5	2605,94	246,09	4	2620.70	228.94
“b”	4	2620.24	223.64	5	2603,39	241,03	4	2620.69	223.85
“c”	4	2614.52	51.63	5	2603,39	51,00	4	2613.94	51.78
“d”	9	2620.58	265.34	7	2605,76	246,35	6	2606.38	245.94
“e”	16	2628.49	363.36	9	2658,55	341,40	8	2628.09	363.64

The mean values – results of various methods(see Table) are convergent.

The dispersion described by standard deviations change when various set of random variables are analyzed.

Generally, a greater number random variables produces an increment of standard deviations of the results.

Values of standard deviation in set “c” suggest that the specific random variables defined in this set do not act considerably upon the results.

It is easy to notice that the Hong method in four sets gives almost the same estimators of the limit load and standard deviations as in the case of the direct Monte Carlo method.

The former method seems to be the most efficient in real structure analysis.

Note that a great number of random variables used makes the results diverge from the true solutions.

That is because the values of random variables defined by Hong method are not the realistic.

On the other hand the evaluation points calculated by the Rosenblueth method are always realistic.

There is another advantage of the Rosenblueth method compare to the Hong method: an additional random variable  $n + 1$  introduced leaves the remaining  $n$  results accessible.

This is because the sample calculations are always performed at points defined by their mean values and standard deviations  $m_{x_n} \pm \sigma_{x_n}$ .

## Snow load – a reliability calculation

The presented so far calculations concern space truss with simply load – a single force.

The obtained results proved that the point estimate method can be implemented in realistic engineering analysis. Such an example – a reliability of the space truss with snow loading was investigated.

To this end the snow load  $s$  [kN/m<sup>2</sup>] was described by PN-EN 1990-1-3 formula

$$s = \mu_i C_e C_t s_k$$

where

$s_k$  – characteristic values of the snow load on the ground [kN/m<sup>2</sup>]

$\mu_i$  – roof shape factor

$C_e$  – exposure coefficient (here  $C_e = 1.0$  has been assumed)

$C_t$  – thermal coefficient (here  $C_t = 1.0$  has been assumed)

Symmetric and asymmetric snow loads were analysed (see Figure).

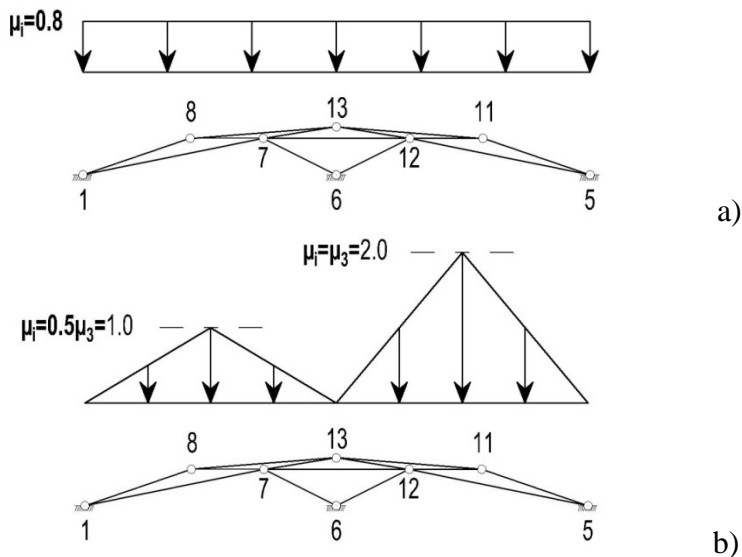


Figure. Symmetric (a) and asymmetric (b) snow loads

The space truss was described using four and five random values.

In the first case the following parameters were assumed as random: initial horizontal displacement of node 13, initial horizontal displacement of node 7-14, Young modulus and cross-section area.

The seven random values included: initial horizontal displacement for node 13 initial horizontal displacement for 7, node 7-12.

In each case the random parameters were independent. They were described by the same values as in the case of single force load.

The version of Hong point estimate method was applied.

The details of the calculations performed for the symmetric load and four random variables are presented in Table.

Table. Symmetric snow load – 4 random parameters – PEM (Hong method)

Point	$x_1$ [m]	$x_2$ [m]	$x_3$ [GPa]	$x_4$ [cm <sup>2</sup> ]	$y$ [kN/m <sup>2</sup> ]	$p(x_{i,j,k})$	$yp(x_{i,j,k})$	$y^2p(x_{i,j,k})$
1	$x_{1-} = -0.16$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	26.09	0.125	3.261	85.09
2	$x_{1+} = 0.16$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	26.16	0.125	3.270	85.54
3	$x_{1,mean} = 0.0$	$x_{2-} = -0.06$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	13.41	0.125	1.676	22.48
4	$x_{1,mean} = 0.0$	$x_{2+} = 0.06$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	14.69	0.125	1.836	26.97
5	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3-} = 202$	$x_{4,mean} = 394.458$	25.12	0.125	3.140	78.88
6	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3+} = 218$	$x_{4,mean} = 394.458$	27.14	0.125	3.393	92.07
7	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4-} = 354.458$	23.45	0.125	2.931	68.74
8	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4+} = 434.458$	28.79	0.125	3.599	103.61
$\Sigma$							23.106	563.38
$E[R] = 2620.475kN, Var[R] \approx 563.38 - 23.106^2 = 29.49kN^2, \sigma_R \approx 5.43kN$								



Similar results in the case of asymmetric snow load are given in Tab. 7.

The results – limit loads of the space truss are presented in Tab. 8.

Table. Asymmetric snow load – 4 random parameters – PEM (Hong method)

Point	$x_1 [m]$	$x_2 [m]$	$x_3 [GPa]$	$x_4 [cm^2]$	$y [kN/m^2]$	$p(x_{i,j,k})$	$yp(x_{i,j,k})$	$y^2p(x_{i,j,k})$
1	$x_{1-} = -0.16$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	4.90	0.125	0.6125	3.001
2	$x_{1+} = 0.16$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	5.01	0.125	0.6263	3.138
3	$x_{1,mean} = 0.0$	$x_{2-} = -0.06$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	3.62	0.125	0.4525	1.638
4	$x_{1,mean} = 0.0$	$x_{2+} = 0.06$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	3.48	0.125	0.4350	1.514
5	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3-} = 202$	$x_{4,mean} = 394.458$	4.76	0.125	0.5950	2.832
6	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3+} = 218$	$x_{4,mean} = 394.458$	5.15	0.125	0.6438	3.315
7	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4-} = 354.458$	4.44	0.125	0.5550	2.464
8	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4+} = 434.458$	5.47	0.125	0.6838	3.740
$\Sigma$							4.6039	21.642

$$E[R] = 2620.475kN, Var[R] \approx 21.642 - 4.6039^2 = 0.446kN^2, \sigma_R \approx 0.668kN$$

Table. Results of limit load estimation

Load type	Variable dimension	Variable type	Number of realization $n$	Mean value of resistance $R$ [kN]	Standard deviation of resistance [kN]
Snow symmetric load	4	Initial horizontal displacement of node 13	8	23,106	5,43
		Initial horizontal displacement of node 7-14			
		Young modulus			
		Cross-section area			
Snow asymmetric load	4	Initial horizontal displacement of node 13	8	4,6039	0,668
		Initial horizontal displacement of node 7-14			
		Young modulus			
		Cross-section area			
Snow symmetric load	7	Initial horizontal displacement for node 13	4	6,024	3,599
		Initial horizontal displacement for 7node 7-12 (each independently)			

It should be pointed out that the Hong method can give unrealistic results when many random variables, e.g. over 20, are taken into consideration.

Although in such cases the efficiency of the direct Monte Carlo method could be similar to any version of the reduction methods.

It is easy to noticed the difference between the results obtained for the single force and the realistic snow load. In the last case the initial geometric displacement of node 13 do not influences so visible the limit load value.

But the truss is more sensitive when the nodes 7-12 change their positions.

To estimate the reliability of the truss the distribution of the snow load should be assumed. The Gumbel distribution was applied

$$F_X(x) = e^{-e^{-\alpha(x-u)}} \quad \text{for} \quad -\infty \leq x \leq \infty$$

$$f_X(x) = \alpha e^{-e^{-\alpha(x-u)}} e^{-\alpha(x-u)}$$

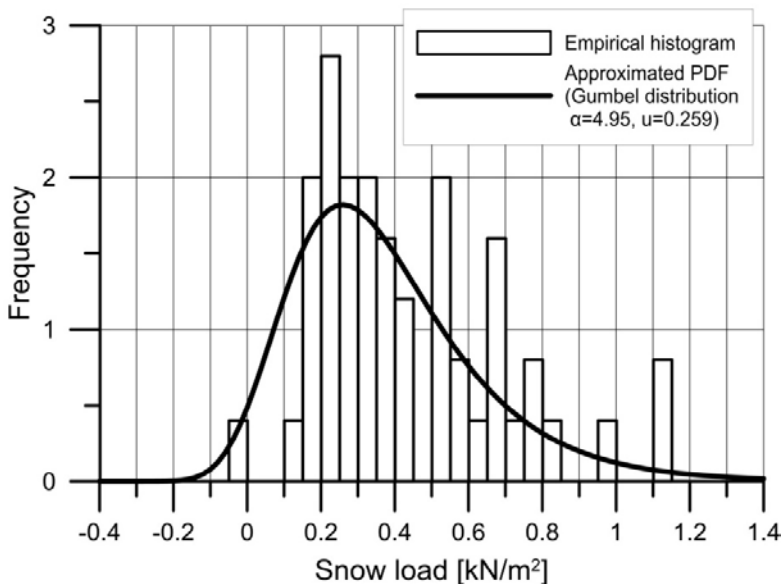


Figure. Approximation of the snow load using Gumbel distributions

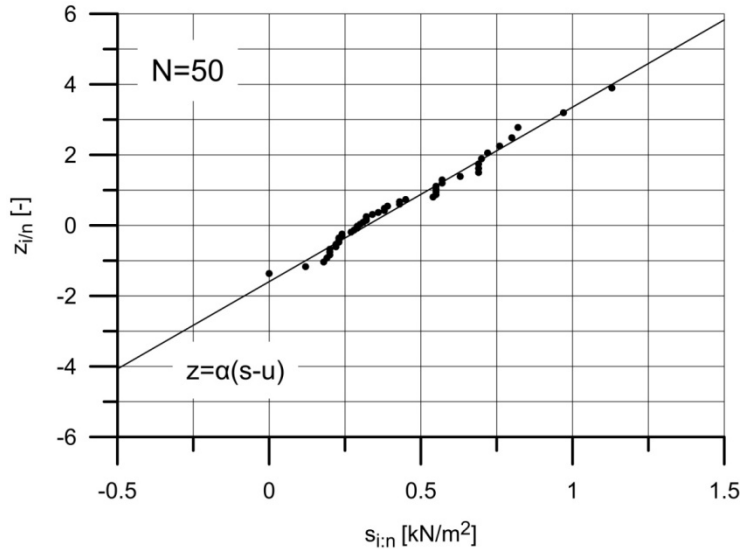


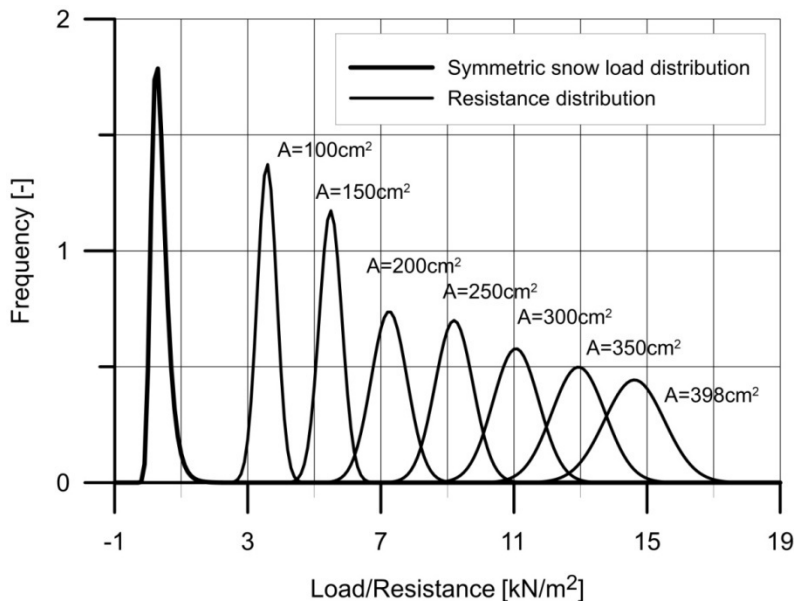
Figure. Approximation of the snow load using Gumbel distributions

The truss reliability was estimated accruing the following formula

$$P_f = \int_{-\infty}^{\infty} F_R(q_i) f_Q(q_i) dq_i$$

To define the sensitivity of the reliability to the change of the cross section parameters.

The distribution of real load and the dispersion of the truss resistance is presented in Figure.



It is possible to approximate the reliability by the following function

$$y = \alpha x^\beta$$

The parameters  $\alpha$  and  $\beta$  were calculated using standard regression analysis (Table) and the results are presented in Fig.

Load type	$\alpha$	B	Coef of determination
Symmetric snow load	4,2965x1015	-7,5	0,9975
Asymmetric snow load	4,5319x107	-3,4851	0,99

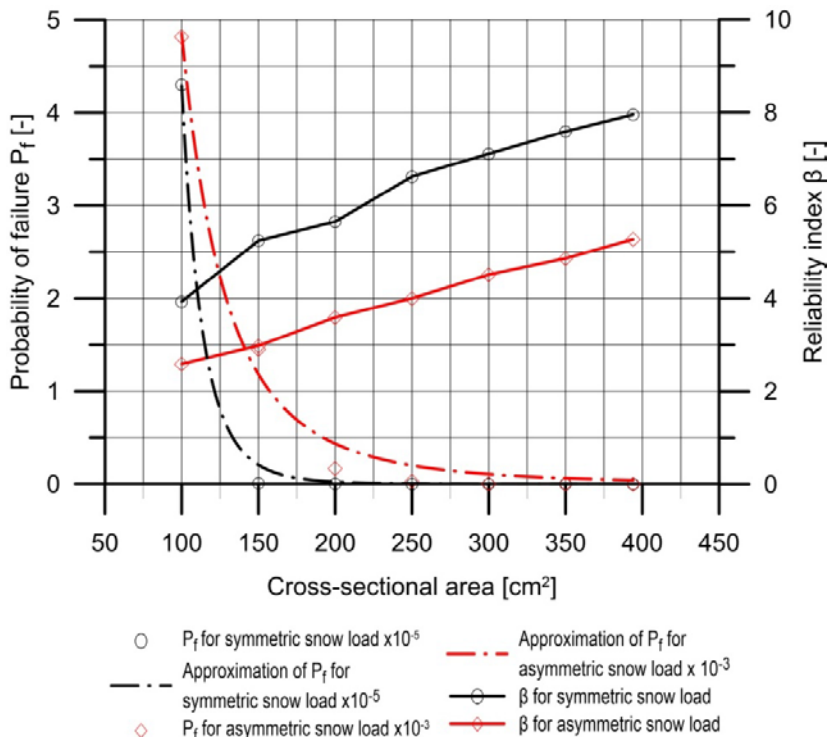


Figure. Approximation of the reliability calculation results

## CONCLUSIONS

The important advantage of the point estimate method is that probability density functions of particular random variables are not required in reliability analysis.

Similarly to the FOSM method the first two probabilistic moments are required only, i.e. expected values and standard deviations of random variables.

Unlike methods FOSM or SORM, it is not necessary to carry on laborious differentiations or iterations in PEM.

This helps to avoid errors in the results, and greatly simplifies the analysis. In contrast to simulation methods, deterministic calculations are performed only several times.

The investigation proved that probabilistic methods can be easily applied in standard engineering applications and the obtained results can substantially influence the design process.

The point estimate method was shown to be an efficient tool applied for probabilistic calculations. In the case of greater numbers of structural random variables the PEM allows for a significant reduction of a sample space, not losing the accuracy of results.

An important advantage of the point estimate method is neglecting probability density functions of particular random variables in reliability analysis. Thus the method can be used to analyze a variety of engineering problems.

The investigation proved that probabilistic methods can be easily applied for the standard engineering problems. The results of random analysis may substantially influence the design process.