Mechanical vibrations due to random excitations

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Typical actions on mechanical systems and structures have a considerable temporal variation.

This leads to time-dependent dynamic responses.

- The mathematical framework to describe actions and reactions is based on random process theory: the basic description of random processes in both time and frequency domains, including the mechanical transfer functions.
- Methods to compute the response statistics in stationary and non-stationary situations are described and applied to several problem classes.
- Analytical methods based on Markov process theory are discussed as well a numerical methods based on Monte Carlo simulation.
- Stochastic stability.

Basic definitions

A random process $X(t)$ is the ensemble of all possible realizations (sample functions) $X(t,\sigma)$ as shown in Fig. 4.1.

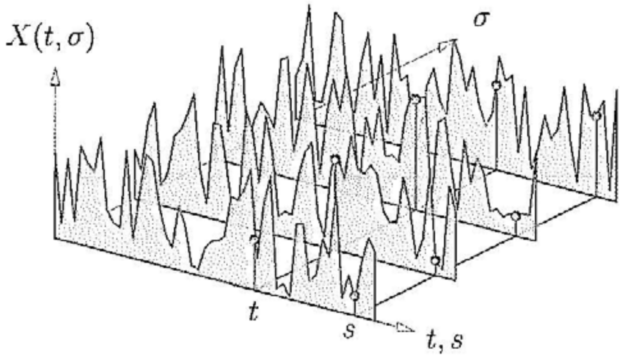

Figure 4.1 Ensemble of realizations of a random process.

Here t denotes the independent variable (usually identified with time) and σ denotes chance (randomness).

For any given value of t, $X(t)$ is a random variable.

By taking ensemble averages at a fixed value of *t*, we can define expected values of the random process $X(t)$.

These expectations are first of all the mean value function

$$\bar{X}(t) = \mathbf{E}\left[X(t) \right] \tag{1}$$

and the auto-covariance function

$$R_{XX}(t,s) = \mathbf{E}\left[\left(X(t) - \bar{X}(t) \right)\left(X(s) - \bar{X}(s) \right) \right] \tag{2}$$

From (2) we obtain as a special case for $t=s$

$$R_{XX}(t,t) = \mathbf{E}\left[\left(X(t) - \bar{X}(t) \right)^2 \right] = \sigma_X^2(t) \tag{3}$$

An enhanced description of a random process involves the probability distribution functions.

This includes the one-time distribution function

$$F_X(x,t) = \Pr\left[X(t) < x \right] \tag{4}$$

and, moreover, all multi-time distribution functions

$$\begin{aligned} & F_X\left(x_1, t_1; x_2, t_2; \ldots; x_n, t_n \right) \\ & \quad = \Pr\left[\left(X(t_1) < x_1 \right) \wedge \left(X(t_2) < x_2 \right) \wedge \ldots \wedge \left(X(t_2) < x_2 \right) \right] \end{aligned} \tag{5}$$

for arbitrary $n \in \mathbb{N}$.

If all these distribution functions are (multidimensional) Gaussian distributions, then the process $X(t)$ is called *Gaussian process.*

This class of random processes has received particular attention in stochastic dynamics since its properties are easily described in terms of the mean value function and the auto-covariance function only.

A random process is called *weakly stationary* if its mean value function $X(t)$ and auto-covariance function $R_{XX}(t,s)$ satisfy the relations

$$\begin{aligned} &\bar{X}(t) = \bar{X} = const. \\ &R_{XX}(t,s) = R_{XX}(s-t) = R_{XX}(\tau) \end{aligned} \tag{6}$$

For weakly stationary processes we have

$$\begin{aligned} &R_{XX}(\tau) = R_{XX}(-\tau) \\ &\max_{t \in R} \left| R_{XX}(t) \right| = R_{XX}(0) = \sigma_X^2 \end{aligned} \tag{7}$$

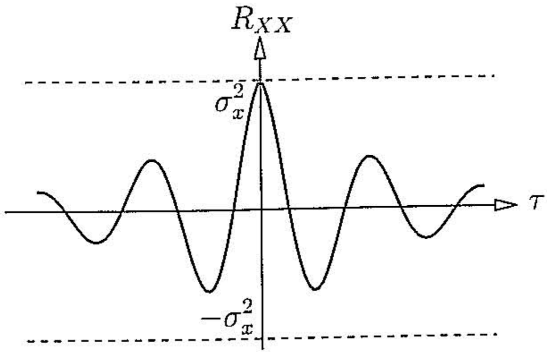

Figure 4.2 Auto-covariance function of a weakly stationary random process.

Intuitively, one may expect that for large time separation (i.e. for $\tau \to \pm\infty$) the autocovariance function should approach zero.

If this is actually the case, then the Fourier transform of the auto-covariance functions exists, and we define the *auto-power spectral density* $S_{XX}(\omega)$ of the weakly stationary random process $X(t)$ in terms of

$$S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\tau) e^{i\omega t} d\tau \qquad (8)$$

By inverting this transformation we can recover the auto-covariance function in terms of

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) e^{-i\omega\tau} d\omega \tag{9}$$

These equations are frequently called Wiener-Khintchine-relations.

Specifically, for $\omega = 0$ we obtain from the previous equation

$$\sigma_X^2 = R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega \tag{10}$$

This leads to the interpretation of the power spectral density (PSD) as the distribution of the variance of a process over the frequency axis. It forms the basis of the so-called power spectral method of random vibration analysis.

According to the range of frequencies covered by the PSD, the extreme cases of wide-band and narrow band random processes may be distinguished.

The qualitative relation between the PSD and the respective auto-covariance functions is shown in Fig. 4.3.

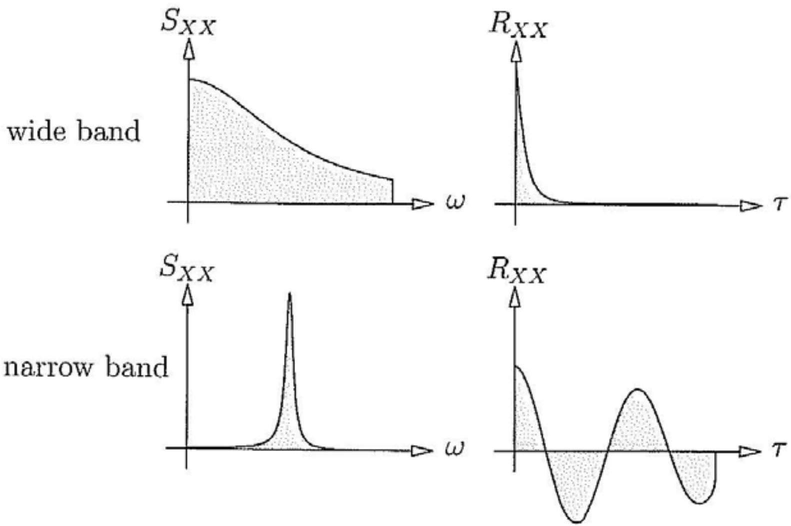

Figure 4.3 Qualitative relation beween PSD and auto-covariance functions for wide band and narrow band random processes.

Markov processes

A continuous Markov process is defined in terms of conditional probability density functions, i.e.

$$f_X\left(x_n,t_n \middle| x_{n-1},t_{n-1};\ldots;x_1,t_1\right)= f_X\left(x_n,t_n \middle| x_{n-1},t_{n-1}\right);\; t_n > t_{n-1} > \ldots > t_1 \tag{11}$$

This states that the probability density function of a Markov process at time $t_n, t_{n-1}, \ldots t_1$ depends only on the value x_{n-1} at tIme t_{n-1}, i.e. only on the immediate past.

A Markov process called a “one-step-memory random process".
Using properties of the correlation coefficient function

$$\rho_{XX}\left(t,s\right)=\frac{R_{XX}\left(t,s\right)}{\sigma_X\left(t\right)\sigma_X\left(s\right)} \tag{12}$$

there is a weaker form of defining a Markov process (Markov process in the wide sense) by

$$\rho_{XX}\left(t,s\right)=\rho_{XX}\left(t,u\right)\rho_{XX}\left(u,s\right),\;\; t\le u\le s \tag{13}$$

If a random process is both wide-sense Markovian and weakly stationary, then

$$\rho_{XX}\left(t-s\right)=\rho_{XX}\left(t-u\right)\rho_{XX}\left(u-s\right),\ \ t\le u\le s \tag{14}$$

or equivalently

$$\rho_{XX}\left(\tau\right)=\rho_{XX}\left(\varphi\right)\rho_{XX}\left(\tau-\varphi\right) \tag{15}$$

Taking derivatives with respect to τ we get

$$\dot{\rho}_{XX}\left(\tau\right)=\rho_{XX}\left(t,u\right)\dot{\rho}_{XX}\left(\tau-\varphi\right) \tag{16}$$

and upon setting $\tau=\varphi$

$$\dot{\rho}_{XX}\left(\tau\right)=\rho_{XX}\left(\tau\right)\dot{\rho}_{XX}\left(0\right)=-\beta\rho_{XX}\left(\tau\right) \tag{17}$$

which due to $\rho_{XX}\left(0\right)=1$ has the unique solution

$$\rho_{XX}\left(\tau\right)=\exp\left(-\beta\tau\right);\ \beta>0 \tag{18}$$

Hence the auto-covariance function of a weakly stationary wide-sense Markov process is of the exponential type:

$$R_{XX}\left(\tau\right)=\sigma_X^2\exp\left(-\beta\tau\right) \tag{19}$$

This can easily be Fourier-transformed to give the power spectral density as a rational function

$$S_{XX}(\omega) = \frac{\beta\sigma_X^2}{\pi\left(\beta^2 + \omega^2\right)} \qquad (20)$$

The relation between these functions is shown in Fig. 4.4 for $\sigma_X^2 = 1$ and for different numerical values of β.

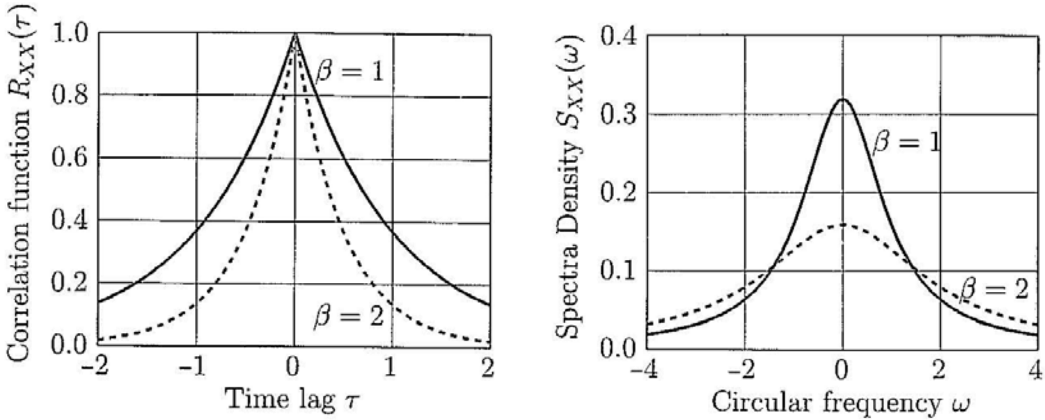

Figure 4.4 Correlation function and power spectral density function of Markov process.

The concept of Markov processes can be extended to vector-valued random processes $X(t)$.

In this case, the defining equation becomes

$$f_X\left(x_n, t_n \middle| x_{n-1}, t_{n-1}; \ldots; x_1, t_1\right) = f_X\left(x_n, t_n \middle| x_{n-1}, t_{n-1}\right);\; t_n > t_{n-1} > \ldots > t_1 \tag{21}$$

The matrix of correlation coefficients $\rho(\tau)$ consequently has the property

$$\rho(t-s) = \rho(t-u)\,\rho(u-s) \tag{22}$$

which implies

$$\rho(\tau) = \begin{cases} \exp(-Q\tau);\; \tau > 0 \\ \exp\left(Q^T\tau\right);\; \tau < 0 \end{cases} \tag{23}$$

in which Q is a constant matrix whose eigenvalues have positive real parts.

Note that the matrix exponential typically contains both exponential and trigonometric functions, thus the correlation coefficients of Markov vector processes will usually be oscillating functions of the time lag τ.

Upcrossing rates

For the design of a structure or structural element it is essential to deal with extreme responses to dynamic loading.

This means that the probability of exceeding large, possibly dangerous levels ξ of the response $X(t)$ should be kept small.

Different types of failure can be associated with the exceedance of the threshold value.

One possible failure mechanism is sudden failure once the threshold is crossed (ultimate load failure), another possibility is the accumulation of damage due to repeated exceedance of the threshold (fatigue failure).

For both types of failure it is essential to investigate the average number of crossings over the threshold per unit time (the upcrossing rate v_{ξ}).

Upcrossing is an event at time t in which the value of the random process X is below the threshold immediately before t and above the threshold immediately after t.

The event occurs while the time-derivative $\dot{X}$ is positive.

In order to derive an expression for ν_ξ it is necessary to know the joint probability density function of the random process and its time derivative at any time *t*, i.e. $f_{X,\dot{X}}(x,\dot{x},t)$.

If $\dot{X}$ exists in the mean square sense, then its mean value is zero

$$E\left[\dot{X}\right] = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[X\left(t+\Delta t\right)\right] = \dot{\bar{X}} = 0 \tag{24}$$

and its covariance function is given by

$$R_{\dot{X}\dot{X}} = E\left[\dot{X}\left(t\right)\dot{X}\left(t\right)\right] =$$

$$= \lim_{u\to 0}\lim_{v\to 0} E\left[\frac{X\left(t+\Delta t\right)-X\left(t\right)}{u}\frac{X\left(s+v\right)-X\left(s\right)}{v}\right] = -R''_{XX}\left(t-s\right) \tag{25}$$

This means that the differentiability of a random process in the mean square sense requires that its auto-covariance function is twice differentiable. This shows that a scalar Markov process is not mean-square differentiable. However, individual components of a vector Markov process may be mean-square differentiable.

In a stationary process, the process itself and its time derivative are uncorrelated if taken at the same time *t*.

This is readily shown by taking

$$E\left[\dot{X}(t)\dot{X}(t)\right]=\lim_{\Delta t\to 0}\frac{1}{\Delta t}E\left[X(t)X(t+\Delta t)-X(t)\right]= \\ =\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(R_{XX}(\Delta t)-R_{XX}(0)\right)=R'_{XX}(0) \tag{26}$$

Due to the required symmetry of the auto-covariance function $R_{XX}(\tau)$ with respect to the time separation argument τ, its derivative R_{XX} is either zero or it does not exist.

However, since the existence of the derivative requires the differentiability of $R_{XX}(\tau)$ we conclude that $X(t)$ and $\dot{X}(t)$ are uncorrelated.

In the case of a Gaussian random process this implies that $X(t)$ and $\dot{X}(t)$ are independent when evaluated at the same time.

The joint probability density function of the process and its time derivative is

$$f_{XX}(x,\dot{x})=\frac{1}{2\pi}\frac{1}{\sigma_X\sigma_{\dot{X}}}\exp\left[-\frac{\left(x-\bar{X}\right)^2}{2\sigma_X^2}\right]\exp\left[-\frac{\dot{x}^2}{2\sigma_X^2}\right] \tag{27}$$

The upcrossing rate v_ξ of a random process $X(t)$ over a threshold ξ from below can be computed as:

$$v_\xi = \int_0^\infty \dot{x} f_{X\dot{X}}\left(\xi, \dot{x}\right) d\dot{x} \tag{28}$$

For a Gaussian process as defined in Eq. (27), this evaluates to

$$v_\xi = \frac{1}{2\pi} \frac{\sigma_{\dot{X}}}{\sigma_X} \exp\left[-\frac{\left(x - \bar{X}\right)^2}{2\sigma_X^2} \right] \tag{29}$$

By studying the joint probability density function of the process $X(t)$ and its first and second derivatives $\dot{X}(t)$ and $\ddot{X}(t)$, an expression for the probability density function $f_A(a)$ of the peaks A of a random process can be obtained.

For the limiting case of a narrow-band process, $f_A(a)$ becomes

$$f_A\left(a\right) = \frac{a}{\sigma_X^2} \exp\left[-\frac{\left(x - \bar{X}\right)^2}{2\sigma_X^2} \right], \quad a \geq \bar{X} \tag{30}$$

Single-degree-of-freedom system response
Mean and variance of response

Consider a mechanical system consisting of a mass *m*, a viscous damper *c* and an elastic spring *k* as shown in Fig. 4.17.

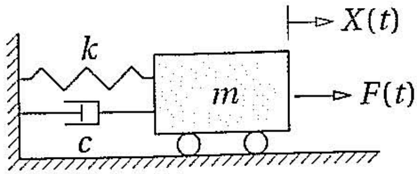

Figure 4.17 SDOF system subjected to random load $F(t)$.

The equation of motion (dynamic equilibrium condition) for this system is

$$m\ddot{X} + c\dot{X} + kX = F(t) \tag{31}$$

For this system, we can derive the natural circular frequency ω_0 and the damping ratio ζ as

$$\omega_0 = \sqrt{\frac{k}{m}}; \quad \zeta = \frac{c}{2\sqrt{mk}} \tag{32}$$

We assume that the load $F(t)$ acting on the system is a random process.

At present we specifically assume that $F(t)$ is a weakly stationary process with a given mean value *j;* and a given autocovariance function $R_{FF}(\tau)$.

We want to compute the statistical properties of the displacement response $X(t)$, which will be a random process, too.

From structural dynamics, we can apply the so-called Duhamel's integral:

$$X(t) = \int_0^t h(t-w) F(w) dw \tag{33}$$

$h(u)$ denotes the impulse response function given by

$$h(u) = \begin{cases} \frac{1}{m\omega'} \exp(-\zeta\omega_0 u) \sin \omega' ü & u \geq 0 \\ 0 & u < 0 \end{cases} \tag{34}$$

with $\omega' = \omega_0 \sqrt{1-\zeta^2}$.

Applying the expectation operator on Eq. (33), we obtain

$$E\left[X(t)\right]=\tilde{X}(t)=E\left[\int_0^t h(t+w)\,dw\right]= \\ =\int_0^t h(t+w)\,E\left[F(w)\right]dw=\bar{F}\int_0^t h(t+w)\,dw \tag{35}$$

By substituting the variable $u=t-w$ we immediately get

$$\bar{X}(t)=\bar{F}\int_0^t h(u)\,du= \\ =\frac{\bar{F}}{m\omega_0^2}\left[1-\exp\left(-\zeta\omega_0 t\right)\left(\zeta\sin\omega'+\sqrt{1-\zeta^2}\right)\cos\omega' t\right] \tag{36}$$

From this it is easily seen that in the limit as $t\to\infty$, we obtain the static solution as the stationary solution

$$\lim_{t\to\infty}\bar{X}(t)=\frac{\bar{F}}{k}=\bar{X}_\infty \tag{37}$$

If the damping ratio is not too small, the limit is approached quite rapidly.

For numerical values of $\omega_0 = 1$ and $\zeta = 0.05$, this is shown in Fig. 4.7.

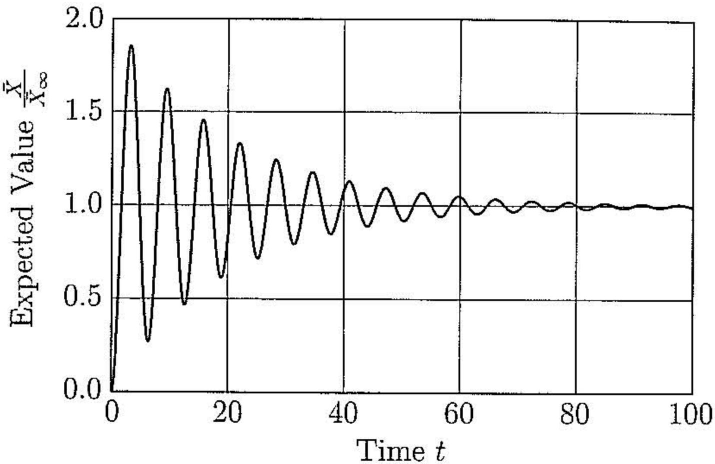

Figure 4.7 Expected displacement response for SDOF systems.

There is an initial overshoot which in structural analysis is commonly called dynamic load factor.

It is less or equal to 2.

For processes with a sufficiently long duration, the stationary limit $\bar{X}_{\infty}$ is of main interest.
This so-called stationary mean response can also be obtained for finite values of t by assuming that the excitation started in the infinite past.

Based on this Duhamel's integral can be written as

$$X(t) = \int_{-\infty}^{t} h(t-w) F(w) dw \tag{38}$$

Actually, due to the fact that $h(u) = 0$ for $u < 0$, we can write just as well

$$X(t) = \int_{-\infty}^{\infty} h(t-w) F(w) dw \tag{39}$$

From this, we can easily get

$$E\left[X(t)\right] = \bar{X}(t) = E\left[\int_{-\infty}^{\infty} h(t-w) F(w) dw\right] = \bar{F}\int_{0}^{\infty} h(u) du = \frac{\bar{F}}{k} \tag{40}$$

In the following, we assume that the excitation has been acting since the infinite past.

The autocovariance function $R_{XX}(t,s)$ of the response $X(t)$ can then also be computed from Eq. (39):

$$E\left[X(t)X(t)\right]=E\left[\int_{-\infty}^{\infty}h(t-w)F(w)dw\cdot\int_{-\infty}^{\infty}h(s-z)F(z)dz\right]=$$

$$=E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(t-w)h(s-z)F(w)F(z)dwdz\right]= \qquad (41)$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(t-w)h(s-z)E\left[F(w)F(z)\right]dwdz$$

Subtracting the expected values $\bar{F}$ and $\bar{X}$, respectively, this becomes

$$R_{XX}(t,s)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(t-w)h(s-z)R_{FF}(w,z)dwdz \qquad (42)$$

With the application of the Wiener-Khintchine-relations Eq. (8) we obtain the power spectral density function $S_{XX}(\omega)$ of the response

$$S_{XX}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(t-w)h(t+\tau-z)R_{FF}(z-w)e^{i\omega\tau}dwdzd\omega \qquad (43)$$

Using the substitutions

$$u_1 = z - w; \qquad u_2 = t - w; \qquad u_3 = t + \tau - z \tag{44}$$

with the absolute Jacobian determinant III of the coordinate transformation

$$|J| = \left| \frac{\partial(u_1, u_2, u_3)}{\partial(z, w, \tau)} \right| = \begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |-1| = 1 \tag{45}$$

this can be rewritten as

$$\begin{aligned} S_{XX}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{FF}(u_1) e^{i\omega u_1} du_1 \\ &\cdot \int_{-\infty}^{\infty} h(u_2) e^{-i\omega u_2} du_2 \cdot \int_{-\infty}^{\infty} h(u_3) e^{i\omega u_3} du_3 \end{aligned} \tag{46}$$

The first line on the right hand side of this equation apparently represents the power spectral density $S_{FF}(\omega)$ of the excitation.

The remaining two integrals are the complex transfer function $H(\omega)$ and its complex conjugate $H^*(\omega)$:

$$H(\omega) = \int_{-\infty}^{\infty} h(u) e^{-i\omega u} du \tag{47}$$

So we may conclude that the power spectral density of the response is given by the simple relation

$$S_{XX}(\omega) = S_{XX}(\omega) H(\omega) H^*(\omega) = S_{XX}(\omega) \left| H(\omega) \right|^2 \tag{48}$$

Evaluation of Eq. (47) yields

$$H(\omega) = \frac{1}{k - m\omega^2 + ic\omega} \tag{49}$$

so that $$S_{XX}(\omega) = S_{FF}(\omega) \frac{1}{k^2 + \left(c^2 - 2km\right)\omega^2 + m^2\omega^4} \tag{50}$$

Using Eq. (10), the variance σ_X^2 can be computed from

$$\sigma_X^2 = \int_{-\infty}^{\infty} S_{FF}(\omega) \left| H(\omega) \right|^2 d\omega \tag{51}$$

Example 4.1

Cantilever subjected to lateral loading.

As an example, consider a simple cantilever structure subjected to random lateral loading (cf. Fig. 4.8).

Structural data is H=4m, EI=3600 kN/m^2, m= 1 t.

From this, the lateral stiffness is $k = \dfrac{3EI}{H^3} = 400\ \text{kN/m}$.

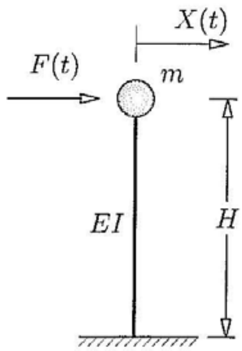

Figure 4.8 Cantilever structure subjected to lateral loading.

The load model uses a constant mean value $\bar{F}$ and power spectral density

$$S_{FF}(\omega) = \frac{\sigma_F^2 a}{\pi\left(a^2 + \omega^2\right)} \tag{52}$$

We assume that $\sigma_F = 0.2\bar{F}$ and a = 12 rad/s.
The mean response $\bar{X}$ is readily computed to be

$$\bar{X} = \frac{\bar{F}}{k} = 0.0025\bar{F} \tag{53}$$

The power spectral densities (on the positive frequency axis) of load and response are shown in Fig. 4.9.

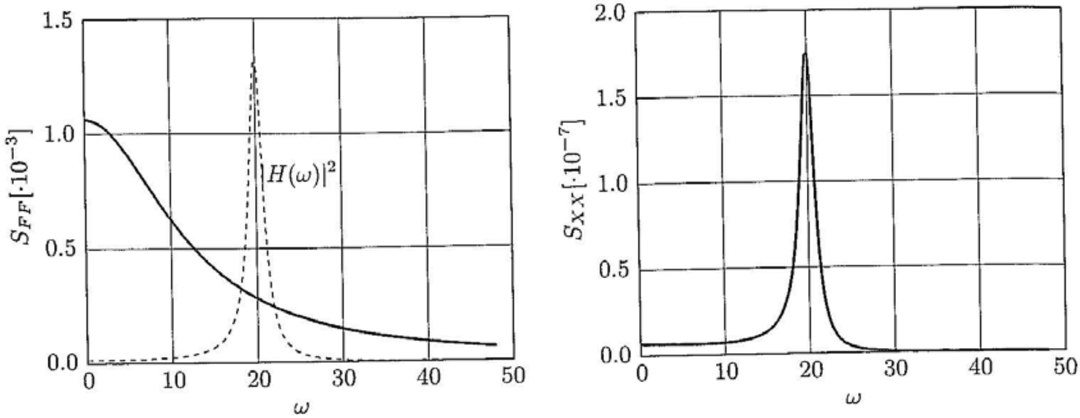

Figure 4.9 Power spectral densities of load and response for cantilever structure.

Integration over ω from $-\infty$ to ∞ yields the variance of the displacement response

$$\sigma_X^2 = \frac{13}{5560000}\bar{F}^2 = 2.338 \cdot 10^{-6}\bar{F}^2 \quad \rightarrow \quad \sigma_X = 1.529 \cdot 10^{-3}\bar{F} \tag{54}$$

The coefficient of variation of the response is 0.61.

This clearly indicates the magnification of the randomness due to dynamic effects.

White noise approximation

In view of the integral, as given in Eq. (51), it is quite obvious that the major contribution to the value of σ_X^2 will most likely come from the frequency range near the natural circular frequency ω_0.

Based on this observation, the integral can be approximated by

$$\sigma_X^2 \approx \int_{-\infty}^{\infty} S_{FF}\left(\omega_0\right)\left|H\left(\omega\right)\right|^2 d\omega = S_{FF}\left(\omega_0\right)\int_{-\infty}^{\infty}\left|H\left(\omega\right)\right|^2 d\omega \qquad (55)$$

The integral over the squared magnitude of the complex transfer function can be evaluated in closed form:

$$\int_{-\infty}^{\infty}\left|H\left(\omega\right)\right|^2 d\omega = \int_{-\infty}^{\infty}\frac{1}{k^2+\left(c^2-2km\right)\omega^2+m^2\omega^4}d\omega = \frac{\pi}{kc} \qquad (56)$$

This approximation procedure can be interpreted as replacing the actual loading process $F(t)$ by another process $W(t)$ which has a constant power spectral density function $S_{WW}(\omega) = const = S_{FF}(\omega_0)$.

Applying this to the previous example with the cantilever under lateral loading, we obtain the approximate result

$$\sigma_X^2 = \frac{3}{1360000}\bar{F}^2 = 2.206\cdot 10^{-6}\bar{F}^2 \qquad \sigma_X = 1.485\cdot 10^{-3}\bar{F} \tag{57}$$

It should be noted, however, that such a process with constant power spectral density cannot exist in reality since according to Eq. (10) its variance σ_W^2 should be infinite.

Due to the equally distributed frequency components, such a fictitious process is called "white noise" (in analogy to white light containing all visible frequencies in equal intensity).

Formally, the autocorrelation function $R_{WW}(\tau)$ of a process with constant power spectral density S_0 can be constructed from Eq. (9):

$$R_{WW}(\tau) = \int_{-\infty}^{\infty} S_0(\omega)e^{-i\omega\tau}d\omega = 2\pi S_0\delta(\tau) \tag{58}$$

Here, $\delta(\cdot)$ denotes the so-called Dirac's Delta function with the properties

$$\delta(u) = 0 \ \forall u \neq 0; \quad \int_{-\infty}^{\infty} \delta(u)g(u)du = g(0) \tag{59}$$

The latter property is true for all functions $g(u)$ which are continuous in a vicinity of $u = 0$.

The Delta function can be interpreted for example as the limiting case of a rectangular function $\delta_\varepsilon(u)$ (d. Fig. 4.10).

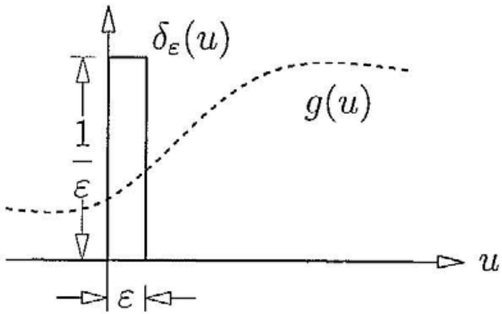

Figure 4.10 Dirac Delta function as limit of rectangular function.

We define δ_ε as

$$\delta(u) = \begin{cases} \frac{1}{\varepsilon} & 0 \le u \le \varepsilon \\ 0 & \text{else} \end{cases} \tag{60}$$

The function g can be expanded in a Taylor series about $u = 0$

$$g(u) = g(0) + g'(0)u + \frac{1}{2}g''(0)u^2 \tag{61}$$

so that

$$\begin{aligned} \int_{-\infty}^{\infty} \delta_\varepsilon(u)\, g(u)\, du &= \int_{-\infty}^{\infty} \delta_\varepsilon(u)\left[g(0) + g'(0)u + \ldots\right] du \\ &= \int_{-\infty}^{\infty} \frac{1}{\varepsilon}\left[g(0) + g'(0)u + \ldots\right] du = \frac{1}{\varepsilon}\left[g(0) + g'(0)\frac{\varepsilon^2}{2} + \ldots\right] \end{aligned} \tag{62}$$

In the limit as $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \delta_\varepsilon(u)\, g(u)\, du = g(0) \tag{63}$$

Returning to the expression for the autocovariance function of the response as given in Eq. (44), the above property of the Delta function allows the computation of an expression for the time-dependent variance $\sigma_X^2(t)$ of the response to white noise excitation

$$\sigma_X^2(t) = R_{XX}(t,t)\int_0^t\int_0^t h(t-w)h(t-z)2\pi S_0\delta(z-w)dwdz = 2\pi S_0\int_0^t h(t-z)^2 dz \tag{64}$$

By substituting the variable $u = t - z$ we obtain

$$\sigma_X^2(t) = 2\pi S_0\int_0^t h(u)^2 du =$$
$$= \frac{\pi S_0}{kc}\left[1-\exp(-2\zeta\omega_0 t)\left(\frac{\omega_0^2}{\omega'^2} - \frac{\zeta^2\omega_0^2}{\omega'^2}\cos 2\omega' t + \zeta\sin 2\omega' t\right)\right] \tag{65}$$

For numerical values of k = 1 N/m, m = 1 kg, ζ = 0.05, S_0 = 1 N^2s, the result of Eq. 4.65 is shown with the label "exact" in Fig. 4.11.

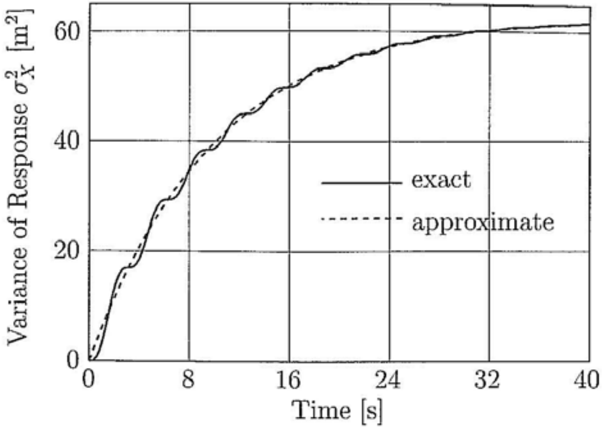

Figure 4.11 Variance of response to white noise excitation.

From Eq. (65) it can be seen that the contributions from the trigonometric functions are of the order of the damping ratio ζ or smaller.

Therefore, omitting these terms yields a simple approximation in the form of

$$\sigma_X^2(t) = \frac{2\pi S_0}{kc} = \left[1 - \exp\left(-2\zeta\omega_0 t\right)\right] \qquad (66)$$

The result from this equation is shown with the label "approximate" in Fig. 4.11.

Multi-degree-of-freedom response
Equations of motion

For a linear multi-degree-of-freedom system the equations of motion can be written in matrix-vector form as

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F}(t) \qquad (67)$$

together with appropriate initial conditions for $\mathbf{X}$ and $\dot{\mathbf{X}}$.

Here, the vectors $\mathbf{X}$ and $\dot{\mathbf{X}}$ have the dimension n, the symmetric and non-negative matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ have the size $n \times n$, and $\mathbf{F}(t)$ is an n-dimensional vector valued random process.

We assume that at least the second order statistics of $\mathbf{F}$ are known. For an important class of nonlinear structural systems, the equations of motion can be written as

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{g}\left(\mathbf{X}, \dot{\mathbf{X}}\right) = \mathbf{F}(t) \qquad (68)$$

The nonlinearity in Eq. (68) is present in the function g involving both restoring forces and damping.

Covariance analysis

In the case of delta-correlated excitation processes, there is a direct way of deriving equations for the covariance matrix of the response vector components.

This is especially advantageous for multi-degree-of-freedom systems. We assume that the matrix of auto- and cross-covariance functions of the components of the excitation vector $\mathbf{F}(t)$ are given in the form of

$$\mathbf{R}_{FF}(t,t+\tau)=\mathbf{D}(t)\delta(\tau) \tag{69}$$

Here, **D** is an arbitrary cross intensity matrix of the size $n\times n$, possibly depending on time *t*.

This means that the excitation process $\mathbf{F}(t)$ is a multi-dimensional white noise process.
Now the equation of motion is represented in phase space, i.e. the response is described in terms of a state vector **Y** containing the displacements **X** and the velocities $\dot{\mathbf{X}}$.
In phase space, the equation of motion becomes (cf. Eq. (67)):

$$\dot{\mathbf{Y}}-\mathbf{G}\mathbf{Y}=\mathbf{g}(t) \tag{70}$$

The $2n \times 2n$-matrix **G** is assembled from the mass, stiffness and damping matrices as

$$\mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \tag{71}$$

The covariance matrix $\mathbf{R}_{YY}$ also has the size of $2n \times 2n$.
This matrix satisfies the differential equation

$$\dot{\mathbf{R}}_{YY} = \mathbf{G}\mathbf{R}_{YY} + \mathbf{R}_{YY}\mathbf{G}^T + \mathbf{B}(t) \tag{72}$$

Here, the matrix **B** is defined by

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{M}^{-1}\mathbf{D}(t)\mathbf{M}^{-1} \end{bmatrix} \tag{73}$$

This equation can be solved, for instance, by arranging the elements of $\mathbf{R}_{YY}$ into a vector **r**.

Hereby, the symmetry of $\mathbf{R}_{YY}$ can be utilized to reduce the problem size.
The vector **r** then contains $n(n+1)/2$ elements.

Correspondingly, the coefficients of the matrix **G** are then arranged into-another matrix **H**, and the matrix **B** is put into a vector **b**.

Thus we obtain the system of linear equations

$$\dot{\mathbf{R}}_{YY} = \mathbf{G}\mathbf{R}_{YY} + \mathbf{R}_{YY}\mathbf{G}^{T} + \mathbf{B}(t) \tag{74}$$

which can be solved using standard methods.

The covariance matrix $\mathbf{R}_{XX}$ of the displacements is a sub-matrix of the size $n \times n$.

An important special case is the *stationary state*, i.e. the case in which **B** = const. and $\dot{\mathbf{R}}_{YY} = 0$.

The excitation process in this case possesses a constant power spectral density matrix $\mathbf{S}_0 = \mathbf{B} / 2\pi$.

If we write the matrix $\mathbf{R}_{YY}$ for this case in block notation

$$\mathbf{R}_{YY} = \begin{bmatrix} \mathbf{R}_{XX} & \mathbf{R}_{X\dot{X}} \\ \mathbf{R}_{\dot{X}X} & \mathbf{R}_{\dot{X}\dot{X}} \end{bmatrix}, \; \mathbf{R}_{\dot{X}X} = \mathbf{R}_{X\dot{X}}^{T} \tag{75}$$

we obtain for the stationary state

$$
\begin{aligned}
&\mathbf{R}_{X\dot{X}} + \mathbf{R}_{\dot{X}X} = \mathbf{0} \\
&\mathbf{KR}_{XX} + \mathbf{CR}_{\dot{X}X} - \mathbf{MR}_{\dot{X}\dot{X}} = \mathbf{0} \\
&\mathbf{R}_{XX}\mathbf{K} + \mathbf{R}_{X\dot{X}}\mathbf{C} - \mathbf{R}_{\dot{X}\dot{X}}\mathbf{M} = \mathbf{0} \\
&\mathbf{K}(\mathbf{R}_{X\dot{X}} + \mathbf{R}_{\dot{X}X}) + 2\mathbf{CR}_{\dot{X}\dot{X}} = \mathbf{MB}
\end{aligned}
\qquad (76)
$$

From this we can immediately get the covariance matrix of the displacements

$$
\mathbf{R}_{XX} = \frac{1}{2}\mathbf{K}^{-1}\mathbf{MC}^{-1}\mathbf{MB} \qquad (77)
$$

Note: For a SDOF-system this reduces to

$$
\sigma_X^2 = \frac{\pi S_0}{kc}; \quad \sigma_{\dot{X}}^2 = \frac{\pi S_0}{mc} \qquad (78)
$$

The effort required for the numerical solution can be reduced substantially if modal damping is assumed and approximate validity of some of the relations in Eq. (76) is assumed as well.

For this case, we can obtain decoupled approximate equations for the elements R_{ij} of the modal covariance matrix

$$a_{ij} = \dot{R}_{ij} + b_{ij} R_{ij} = B_{ij}\left(t\right) \tag{79}$$

in which the constants a_{ij} and b_{ij} are given by

$$a_{ij} = \omega_i \omega_j \frac{\zeta_i \omega_j + \zeta_j \omega_i}{\zeta_i \omega_i + \zeta_j \omega_j} + \frac{\omega_i^2 + \omega_j^2}{2}$$

$$\begin{aligned} b_{ij} &= \omega_i \omega_j \zeta_i + \zeta_j \omega_i \zeta_i \omega_i + \\ &+ \zeta_i \omega_j \left(\frac{\omega_i}{4\zeta_i} + \frac{\omega_j}{4\zeta_j} - \frac{\omega_i^2}{4\zeta_j \omega_j} - \frac{\omega_j^2}{4\zeta_i \omega_i} - 2\zeta_i \omega_i - 2\zeta_j \omega_j \right) \\ &+ \frac{\omega_i^2 \omega_j}{4\zeta_j} + \frac{\omega_i \omega_j^2}{4\zeta_i} - \frac{\omega_i^3}{4\zeta_i} - \frac{\omega_j^3}{4\zeta_j} \end{aligned} \tag{80}$$

This leads directly to the modal covariance matrix.

For the case $i = j$, Eq. (79) reduces to

$$\dot{R}_{ij} + 2\zeta_i \omega_i R_{ii} = \frac{B_{ii}}{2\omega_i^2} \tag{81}$$

Filtered white noise excitation

The limitation to delta-correlated processes (i.e. generalized white noise) can be lifted by introducing filters.

In this approach, the output responses of linear filters to white noise are applied as loads to the structure.

A well-known example from earthquake engineering is the Kanai-Tajimi filter.

Here, the ground acceleration $a(t)$ is defined as a linear combination of the displacement and velocity of a system with a single degree of freedom.

This SDOF system is characterized by its natural circular frequency ω_g and the damping ratio ζ_g.

The input $w(t)$ to this filter system is white noise with a power spectral density S_0.

The equation of motion for the filter system is

$$\ddot{z} + 2\zeta_g \omega_g \dot{z} + \omega_g^2 z = w(t) \tag{82}$$

The ground acceleration process $a(t)$ is then defined as

$$a(t) = 2\zeta_g \omega_g \dot{z} + \omega_g^2 z \tag{83}$$

It can be seen that the power spectral density $S_{aa}(\omega)$ has a significant frequency dependence

$$S_{aa}(\omega) = S_0 \frac{4\zeta_g^2 \omega_g^2 \omega^2 + \omega_g^4}{(\omega_g^2 - \omega^2)^2 + 4\zeta_g^2 \omega_g^2 \omega^2} \tag{84}$$

In Fig. 4.12 this function is plotted for numerical values of
S_0 = 1 m^2/s, ω_g = 17 rad/s and ζ_g = 0.3.

Fig. 4.12 clearly shows that significant contributions are present from the frequency range near $\omega = 0$.

In a nonlinear analysis these low-frequency components may lead to excessively large plastic drift.

In order to avoid this, the Kanai-Tajimi model can be modified such that low-frequency components are reduced.

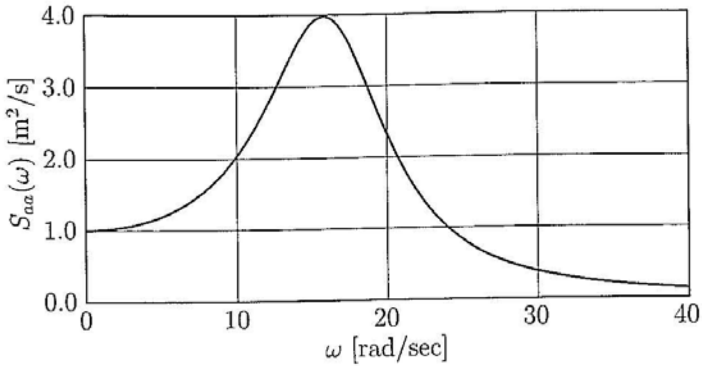

Figure 4.12 Power spectral density of ground acceleration using the Kanai/Tajimi model.

Example 4.2

(Covariance analysis for SDOF-system)

For such a system the covariance matrix of the state vector contains only four (three mutually different) entries

$$\mathbf{R}_{YY} = \begin{bmatrix} r_{XX} & r_{X\dot{X}} \\ r_{X\dot{X}} & r_{\dot{X}\dot{X}} \end{bmatrix}; \quad \mathbf{G}_{YY} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \tag{85}$$

We assume that the excitation is a non-stationary (amplitude modulated) white noise excitation with a power spectral density S_0 and a time envelope $e(t)$.

Then, the matrix **B** is defined by

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 2\pi S_0 / m \end{bmatrix} e^2(t) \tag{86}$$

The differential equation (72) according to (74) can be written as

$$\frac{d}{dt}\begin{bmatrix} r_{XX} \\ r_{X\dot{X}} \\ r_{\dot{X}\dot{X}} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ -k/m & -c/m & 1 \\ 0 & -2k/m & -2c/m \end{bmatrix} \begin{bmatrix} r_{XX} \\ r_{X\dot{X}} \\ r_{\dot{X}\dot{X}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2\pi S_0 e^2(t)/m \end{bmatrix} \tag{87}$$

which can be written in the symbolic form

$$\dot{\mathbf{r}} = \mathbf{H}\mathbf{r} + h(t) \tag{88}$$

with the initial condition $\mathbf{r}(0) = 0$.

The general solution to this equation can be obtained by quadrature

$$\mathbf{r}(t) = \exp(\mathbf{H}t)\mathbf{r}(0) + \int_0^t \exp\left[\mathbf{H}(t-\tau)\right]\mathbf{h}(\tau)\,d\tau \tag{89}$$

which, for **h** constant in a time interval Δt, evaluates to

$$\mathbf{r}(\Delta t) = \exp(\mathbf{H}\Delta t)\mathbf{r}(0) + \left[\exp(\mathbf{H}\Delta t) - \mathbf{I}\right]\mathbf{H}^{-1}\mathbf{h} \tag{90}$$

Note that the matrix exponential exp(**A**) is readily computed by means of diagonalizing the matrix **A** such that

$$\mathbf{A} = \mathbf{T}\Lambda\mathbf{T}^{-1} \tag{91}$$

in which Λ is a diagonal matrix containing the (possibly complex) eigenvalues λ_i of **A** and **T** is the matrix of corresponding right eigenvectors.

This step is always possible if the eigenvalues are distinct.

Using Eq. (91), the matrix exponential can be computed using the standard series expansion for exp (.) i.e.

$$\begin{aligned}\exp(\mathbf{A}) &= \sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{A}^k = \mathbf{I} + \mathbf{T}\Lambda\mathbf{T}^{-1} + \frac{1}{2}\mathbf{T}\Lambda\mathbf{T}^{-1}\mathbf{T}\Lambda\mathbf{T}^{-1} + \ldots \\ &= \mathbf{T}\mathbf{T}^{-1} + \mathbf{T}\Lambda\mathbf{T}^{-1} + \frac{1}{2}\mathbf{T}\Lambda^2\mathbf{T}^{-1} + \ldots. \\ &= \mathbf{T}\left(\mathbf{I} + \Lambda + \frac{1}{2}\Lambda^2 + \ldots\right)\mathbf{T}^{-1} = \mathbf{T}\exp(\Lambda)\mathbf{T}^{-1}\end{aligned} \tag{92}$$

The matrix exponential of the diagonal matrix Λ is simply a diagonal matrix containing the exponentials of the eigenvalues λ_i.

The time-dependent intensity (envelope) is assumed to be (Fig. 4.13)

$$e(t) = 4 \cdot \left[\exp(-0.25t) - \exp(-0.50t) \right] \tag{93}$$

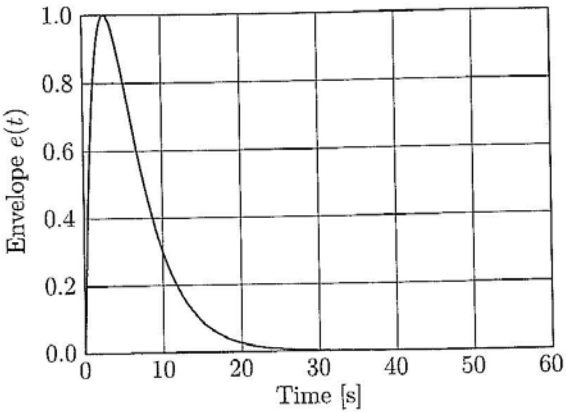

Figure 4.13 Time-dependent amplitude-modulating function (envelope).

The exact solution from the differential equation (72) is obtained and the results for $\sigma_X(t)$ are shown in Fig. 4.14.

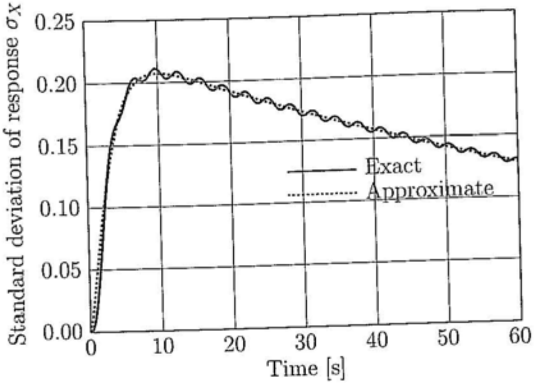

Figure 4.14 Transient standard deviation of the displacement response.

Upon application of the approximations as mentioned above we obtain the first-order differential equation

$$\frac{d}{dt}\sigma_X^2 = -\frac{c}{m}\sigma_X^2 + \frac{S_0 \pi m}{k} e^2(t) \tag{94}$$

The solution to this equation is also shown in Fig. 4.14.

It can be seen that the approximate solution does not contain oscillations.

However, apart from that it matches the exact solution quite well.

Exercise 4.1

(Transient stochastic response)

Compute the variance of the displacement response for a system as defined in the previous example to a nonstationary white noise with an amplitude-modulating function

$$e(t) = \begin{cases} 1; & 0 \le t \le T \\ 0; & else \end{cases} \tag{95}$$

for the time interval [0, *3T*] with *T* = 20.

Solution: The resulting standard deviation $\sigma_X(t)$ is shown in Fig. 4.15.

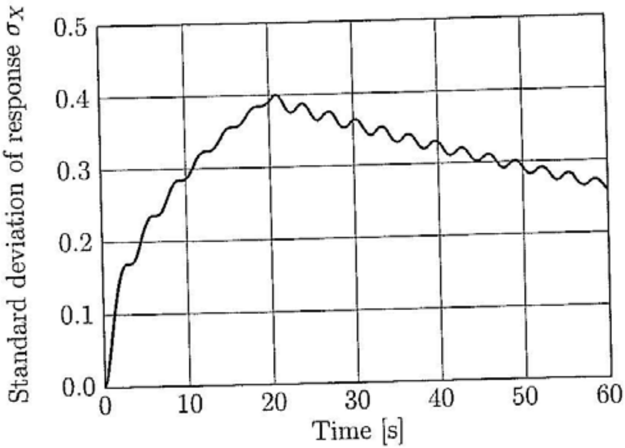

Figure 4.15 Transient standard deviation of the displacement response to pulse modulated white noise.