Introduction to Laboratory 10

Laboratory 10 concerns the implementation of the **Richardson method** to calculate the first and second derivatives of a function $f(x)$, as well as of the **Romberg method** to calculate the integral $\int_{x_0}^{x_N} f(x)dx$. Starting with the **central difference approximation (CDA)** for the derivatives (see Laboratory 8) and from the **Trapezoid method** for the integral (see Laboratory 9), the idea of these methods is to remove successively the different terms in the error $E(h)$. The **CDA** and the **Trapezoid method** have an error that contains only the even power of h, i.e. $E(h) = Ah^2 + Bh^4 + Ch^6 + \cdots$. For example, the **CDA** for the first derivative can written as:

$$f'(x) = \frac{1}{2h}[f(x+h) - f(x-h)] + Ah^2 + Bh^4 + \cdots$$

$$f'(x) = D_1(h) + Ah^2 + Bh^4 + \cdots \quad (1)$$

where $D_1(h)$ is the **CDA** calculated with a step size h.

Using a step size of $2h$ we can write:

$$f'(x) = \frac{1}{4h}[f(x+2h) - f(x-2h)] + 4Ah^2 + 2^4Bh^4 + \cdots$$

$$f'(x) = D_1(2h) + 4Ah^2 + 2^4Bh^4 + \cdots \quad (2)$$

where $D_1(2h)$ is the **CDA** calculated with a step size $2h$.

Multiplying equation (1) by 4 and subtracting equation (2) leads to

$$(4-1)f'(x) = 4D_1(h) - D_1(2h) + 4Ah^2 - 4Ah^2 + 4Bh^4 - 2^4Bh^4 + \cdots$$

$$f'x) = \frac{4D_1(h) - D_1(2h)}{4-1} + \frac{4-2^4}{3}Bh^4 + \cdots$$

$$f'x) = D_2(h) + \frac{4-2^4}{3}Bh^4 + \cdots$$

This operation allows to remove the leading term in the error (i.e. $4Ah^2$) and provides an approximation to the derivative $D_2(h)$ that has a first term in the error proportional to h^4. By similar operations one can remove the next terms in the error and obtain a recursive relation:

$$D_{i+1}(h) = \frac{4^iD_i(h) - D_i(2h)}{4^i - 1}$$

$D_i(h)$ has a leading term in the error that depends on h^{2i}.

The same expression can be employed to calculate the second derivatives starting from the **CDA**.

The results can be represented in a **Richardson table**, where the first column contains the **CDA** and the next columns contain the higher-order approximation obtained from the recursive relation (see Laboratory 10).

Similarly, starting with the **Trapezoid method**

$$I_{m,0} = h\left(\frac{f_0}{2} + f_1 + f_2 + \cdots + f_{N-1} + \frac{f_N}{2}\right) \text{ with } N = 2^m$$

Better approximation to the integral can be calculated with the recursive relation:

$$I_{m,k} = \frac{4^k I_{m,k-1} - I_{m-1,k-1}}{4^k - 1}$$

$I_{m,k}$ has a leading term in the error that depends on $h^{2(k+1)}$.

The results can be represented in a **Romberg table** (see Laboratory 10).

In Laboratory 10 you have to generate the **Richardson** and **Romberg tables** for the first and second derivatives of $f(x) = xe^x$ at $x = 2$ and the integral $\int_0^\pi \sin x \, dx$.