Gaussian Elimination:

The goal of the exercise is to solve a tridiagonal system of n equations using Gaussian elimination:

$$\begin{bmatrix} b_1 & c_1 & 0 & \cdots & \cdots & \cdots & 0 \\ a_2 & b_2 & c_2 & & & & \vdots \\ 0 & a_3 & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ \vdots \\ \vdots \\ r_{n-1} \\ r_n \end{bmatrix}$$

The solutions x_j (for j=1,…,n) of this system can be found using the recursive relations:

$$\begin{cases} \beta_1 = b_1 \\ \beta_j = b_j - \dfrac{a_j}{\beta_{j-1}} c_{j-1} \quad j = 2,\ldots,n \end{cases} \quad ; \quad \begin{cases} \rho_1 = r_1 \\ \rho_j = r_j - \dfrac{a_j}{\beta_{j-1}} \rho_{j-1} \quad j = 2,\ldots,n \end{cases}$$

and

$$\begin{cases} x_n = \dfrac{\rho_n}{\beta_n} \\ x_{n-j} = \dfrac{\rho_{n-j} - c_{n-j} x_{n-j+1}}{\beta_{n-j}} \quad j = 1,\ldots,n-1 \end{cases}$$

Exercise:

- Use Gaussian elimination to solve the following system of equations:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

- Verify that the obtained solutions satisfy the original equations.