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Cubic Spline Interpolation:

The goal of the exercise is to implement the so-called *natural spline* interpolation method to approximate a function $f(x)$ by a polynomial $p(x)$.
It is assumed that we know the function $f(x)$ at n points $[x_j, f(x_j)]$ with j=1,…,n. The step size $h_j = x_{j+1} - x_j$ between the points is assumed to be constant $h_j = h$.
The interpolating polynomial for $x \in [x_j, x_{j+1}]$ can be calculated from

$$p(x) = p_j + \left[\frac{p_{j+1} - p_j}{h} - \frac{1}{6} h p''_{j+1} - \frac{1}{3} h p''_j \right](x - x_j) + \frac{p''_j}{2}(x - x_j)^2 + \frac{p''_{j+1} - p''_j}{6h}(x - x_j)^3$$

where

$$p_j \equiv f(x_j)$$

and the p''_j (for j=1,…,n) can be obtained by solving the tridiagonal system of equations:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 4h & h & & & & \vdots \\ 0 & h & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & h & 0 \\ \vdots & & & & h & 4h & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p''_1 \\ p''_2 \\ \vdots \\ \vdots \\ \vdots \\ p''_{n-1} \\ p''_n \end{bmatrix} = \begin{bmatrix} 0 \\ 6(p_3 - 2p_2 + p_1)/h \\ \vdots \\ \vdots \\ \vdots \\ 6(p_n - 2p_{n-1} + p_{n-2})/h \\ 0 \end{bmatrix}$$

Exercise:

- Interpolate the function $f(x) = \frac{1}{1+x^2}$ with the points x_j = {-5.0 ; -4.0 ; -3.0 ;…; 4.0 ; 5.0}.

Calculate the *natural spline* polynomial $p(x)$ in the interval $-5 \le x \le 5$ with a step of 0.1 and plot the results.
Use Gaussian elimination to solve the tridiagonal system of equations.