CHAPTER 12

Improper Integrals

DEFINITION OF AN IMPROPER INTEGRAL

The functions that generate the Riemann integrals of Chapter 6 are continuous on closed intervals. Thus, the functions are bounded and the intervals are finite. Integrals of functions with these characteristics are called *proper integrals*. When one or more of these restrictions is relaxed, the integrals are said to be *improper*. Categories of improper integrals are established below.

The integral $\int_a^b f(x)\,dx$ is called an *improper integral* if

1. $a = -\infty$ or $b = \infty$ or both, i.e., one or both integration limits is infinite,
2. $f(x)$ is unbounded at one or more points of $a \leqq x \leqq b$. Such points are called *singularities* of $f(x)$.

Integrals corresponding to (1) and (2) are called *improper integrals of the first and second kinds*, respectively. Integrals with both conditions (1) and (2) are called *improper integrals of the third kind.*

EXAMPLE 1. $\int_0^\infty \sin x^2\,dx$ is an improper integral of the first kind.

EXAMPLE 2. $\int_0^4 \frac{dx}{x-3}$ is an improper integral of the second kind.

EXAMPLE 3. $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ is an improper integral of the third kind.

EXAMPLE 4. $\int_0^1 \frac{\sin x}{x}\,dx$ is a *proper integral* since $\lim\limits_{x\to 0+} \frac{\sin x}{x} = 1$.

IMPROPER INTEGRALS OF THE FIRST KIND (Unbounded Intervals)

If f is an integrable on the appropriate domains, then the indefinite integrals $\int_a^x f(t)\,dt$ and $\int_x^a f(t)\,dt$ (with variable upper and lower limits, respectively) are functions. Through them we define three forms of the improper integral of the first kind.

Definition

(*a*) If f is integrable on $a \leqq x < \infty$, then $\int_a^\infty f(x)\,dx = \lim\limits_{x\to\infty} \int_a^x f(t)\,dt$.

(*b*) If f is integrable on $-\infty < x \leqq a$, then $\int_{-\infty}^a f(x)\,dx = \lim\limits_{x\to-\infty} \int_x^a f(t)\,dt$.

(*c*) If f is integrable on $-\infty < x < \infty$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$
$$= \lim_{x\to-\infty}\int_{x}^{a} f(t)\,dt + \lim_{x\to\infty}\int_{a}^{x} f(t)\,dt.$$

In part (*c*) it is important to observe that

$$\lim_{x\to-\infty}\int_{x}^{a} f(t)\,dt + \lim_{x\to\infty}\int_{a}^{x} f(t)\,dt.$$

and

$$\lim_{x\to\infty}\left[\int_{-x}^{a} f(t)\,dt + \int_{a}^{x} f(t)\,dt\right]$$

are not necessarily equal.

This can be illustrated with $f(x) = xe^{x^2}$. The first expression is not defined since neither of the improper integrals (i.e., limits) is defined while the second form yields the value 0.

EXAMPLE. The function $F(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x^2/2)}$ is called the *normal density function* and has numerous applications in probability and statistics. In particular (see the bell-shaped curve in Fig. 12-1)

$$\int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e - \frac{x^2}{2} : dx = 1$$

(See Problem 12.31 for the trick of making this evaluation.)

Perhaps at some point in your academic career you were "graded on the curve." The infinite region under the curve with the limiting area of 1 corresponds to the assurance of getting a grade. C's are assigned to those whose grades fall in a designated central section, and so on. (Of course, this grading procedure is not valid for a small number of students, but as the number increases it takes on statistical meaning.)

In this chapter we formulate tests for convergence or divergence of improper integrals. It will be found that such tests and proofs of theorems bear close analogy to convergence and divergence tests and corresponding theorems for infinite series (See Chapter 11).

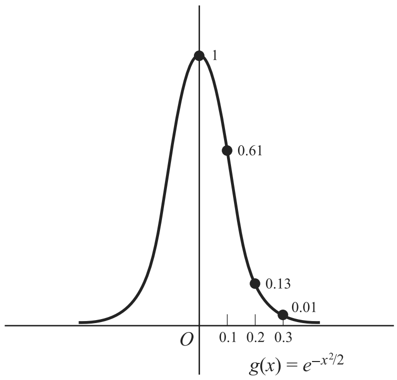

Fig. 12-1

CONVERGENCE OR DIVERGENCE OF IMPROPER INTEGRALS OF THE FIRST KIND

Let $f(x)$ be bounded and integrable in every finite interval $a \leqq x \leqq b$. Then we define

$$\int_{a}^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_{a}^{b} f(x)\,dx \tag{1}$$

where b is a variable on the positive real numbers.

The integral on the left is called *convergent* or *divergent* according as the limit on the right does or does not exist. Note that $\int_a^{\infty} f(x)\,dx$ bears close analogy to the infinite series $\sum_{n=1}^{\infty} u_n$, where $u_n = f(n)$, while $\int_a^b f(x)\,dx$ corresponds to the partial sums of such infinite series. We often write M in place of b in (*1*).

Similarly, we define

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)\,dx \tag{2}$$

where a is a variable on the negative real numbers. And we call the integral on the left convergent or divergent according as the limit on the right does or does not exist.

EXAMPLE 1. $\displaystyle\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1$ so that $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges to 1.

EXAMPLE 2. $\displaystyle\int_{-\infty}^{u} \cos x\,dx = \lim_{a\to-\infty}\int_a^u \cos x\,dx = \lim_{a\to-\infty}(\sin u - \sin a)$. Since this limit does not exist, $\displaystyle\int_{-\infty}^{u}\cos x\,dx$ is divergent.

In like manner, we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{x_0} f(x)\,dx + \int_{x_0}^{\infty} f(x)\,dx \tag{3}$$

where x_0 is a real number, and call the integral convergent or divergent according as the integrals on the right converge or not as in definitions (*1*) and (*2*). (See the previous remarks in part (*c*) of the definition of improper integrals of the first kind.)

SPECIAL IMPROPER INTEGRALS OF THE FIRST KIND

1. **Geometric or exponential integral** $\displaystyle\int_a^\infty e^{-tx}\,dx$, where t is a constant, converges if $t > 0$ and diverges if $t \leqq 0$. Note the analogy with the geometric series if $r = e^{-t}$ so that $e^{-tx} = r^x$.
2. **The p integral of the first kind** $\displaystyle\int_a^\infty \frac{dx}{x^p}$, where p is a constant and $a > 0$, converges if $p > 1$ and diverges if $p \leqq 1$. Compare with the p series.

CONVERGENCE TESTS FOR IMPROPER INTEGRALS OF THE FIRST KIND

The following tests are given for cases where an integration limit is ∞. Similar tests exist where an integration limit is $-\infty$ (a change of variable $x = -y$ then makes the integration limit ∞). Unless otherwise specified we shall assume that $f(x)$ is continuous and thus integrable in every finite interval $a \leqq x \leqq b$.

1. **Comparison test** for integrals with non-negative integrands.

   (*a*) *Convergence*. Let $g(x) \geqq 0$ for all $x \geqq a$, and suppose that $\displaystyle\int_a^\infty g(x)\,dx$ converges. Then if $0 \leqq f(x) \leqq g(x)$ for all $x \geqq a$, $\displaystyle\int_a^\infty f(x)\,dx$ also converges.

   EXAMPLE. Since $\dfrac{1}{e^x+1} \leqq \dfrac{1}{e^x} = e^{-x}$ and $\displaystyle\int_0^\infty e^{-x}\,dx$ converges, $\displaystyle\int_0^\infty \frac{dx}{e^x+1}$ also converges.

   (*b*) *Divergence*. Let $g(x) \geqq 0$ for all $x \geqq a$, and suppose that $\displaystyle\int_a^\infty g(x)\,dx$ diverges. Then if $f(x) \geqq g(x)$ for all $x \geqq a$, $\displaystyle\int_a^\infty f(x)\,dx$ also diverges.

   EXAMPLE. Since $\dfrac{1}{\ln x} > \dfrac{1}{x}$ for $x \geqq 2$ and $\displaystyle\int_2^\infty \frac{dx}{x}$ diverges (p integral with $p = 1$), $\displaystyle\int_2^\infty \frac{dx}{\ln x}$ also diverges.

2. Quotient test for integrals with non-negative integrands.

   (*a*) If $f(x) \geqq 0$ and $g(x) \geqq 0$, and if $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = A \neq 0$ or ∞, then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

   (*b*) If $A = 0$ in (*a*) and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

   (*c*) If $A = \infty$ in (*a*) and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

This test is related to the comparison test and is often a very useful alternative to it. In particular, taking $g(x) = 1/x^p$, we have from known facts about the p integral, the following theorem.

Theorem 1. Let $\lim_{x\to\infty} x^p f(x) = A$. Then

(i) $\int_a^\infty f(x)\,dx$ converges if $p > 1$ and A is finite

(ii) $\int_a^\infty f(x)\,dx$ diverges if $p \leqq 1$ and $A \neq 0$ (A may be infinite).

EXAMPLE 1. $\displaystyle\int_0^\infty \frac{x^2\,dx}{4x^4+25}$ converges since $\displaystyle\lim_{x\to\infty} x^2 \cdot \frac{x^2}{4x^4+25} = \frac{1}{4}$.

EXAMPLE 2. $\displaystyle\int_0^\infty \frac{x\,dx}{\sqrt{x^4+x^2+1}}$ diverges since $\displaystyle\lim_{x\to\infty} x \cdot \frac{x}{\sqrt{x^4+x^2+1}} = 1$.

Similar test can be devised using $g(x) = e^{-tx}$.

3. **Series test** for integrals with non-negative integrands. $\int_a^\infty f(x)\,dx$ converges or diverges according as Σu_n, where $u_n = f(n)$, converges or diverges.

4. **Absolute and conditional convergence**. $\int_a^\infty f(x)\,dx$ is called *absolutely convergent* if $\int_a^\infty |f(x)|\,dx$ converges. If $\int_a^\infty f(x)\,dx$ converges but $\int_a^\infty |f(x)|\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ is called *conditionally convergent*.

Theorem 2. If $\int_a^\infty |f(x)|\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges. In words, an absolutely convergent integral converges.

EXAMPLE 1. $\displaystyle\int_a^\infty \frac{\cos x}{x^2+1}\,dx$ is absolutely convergent and thus convergent since

$$\int_0^\infty \left|\frac{\cos x}{x^2+1}\right| dx \leqq \int_0^\infty \frac{dx}{x^2+1} \quad \text{and} \quad \int_0^\infty \frac{dx}{x^2+1} \text{ converges.}$$

EXAMPLE 2. $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ converges (see Problem 12.11), but $\displaystyle\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ does not converge (see Problem 12.12). Thus, $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ is conditionally convergent.

Any of the tests used for integrals with non-negative integrands can be used to test for absolute convergence.

IMPROPER INTEGRALS OF THE SECOND KIND

If $f(x)$ becomes unbounded only at the end point $x = a$ of the interval $a \leqq x \leqq b$, then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0+} \int_{a+\epsilon}^b f(x)\,dx \tag{4}$$

and define it to be an improper integral of the second kind. If the limit on the right of (*4*) exists, we call the integral on the left *convergent*; otherwise, it is *divergent*.

Similarly if $f(x)$ becomes unbounded only at the end point $x = b$ of the interval $a \leqq x \leqq b$, then we extend the category of improper integrals of the second kind.

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0+} \int_a^{b-\epsilon} f(x)\,dx \tag{5}$$

Note: Be alert to the word *unbounded*. This is distinct from undefined. For example, $\int_0^1 \frac{\sin x}{x}\,dx = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{\sin x}{x}\,dx$ is a proper integral, since $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and hence is bounded as $x \to 0$ even though the function is undefined at $x = 0$. In such case the integral on the left of (*5*) is called convergent or divergent according as the limit on the right exists or does not exist.

Finally, the category of improper integrals of the second kind also includes the case where $f(x)$ becomes unbounded only at an interior point $x = x_0$ of the interval $a \leqq x \leqq b$, then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon_1 \to 0+} \int_a^{x_0-\epsilon_1} f(x)\,dx + \lim_{\epsilon_2 \to 0+} \int_{x_0+\epsilon_2}^b f(x)\,dx \tag{6}$$

The integral on the left of (*6*) converges or diverges according as the limits on the right exist or do not exist.

Extensions of these definitions can be made in case $f(x)$ becomes unbounded at two or more points of the interval $a \leqq x \leqq b$.

CAUCHY PRINCIPAL VALUE

It may happen that the limits on the right of (*6*) do not exist when ϵ_1 and ϵ_2 approach zero independently. In such case it is possible that by choosing $\epsilon_1 = \epsilon_2 = \epsilon$ in (*6*), i.e., writing

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0+} \left\{ \int_a^{x_0-\epsilon} f(x)\,dx + \int_{x_0+\epsilon}^b f(x)\,dx \right\} \tag{7}$$

the limit does exist. If the limit on the right of (*7*) does exist, we call this limiting value the *Cauchy principal value* of the integral on the left. See Problem 12.14.

EXAMPLE. The natural logarithm (i.e., base e) may be defined as follows:

$$\ln x = \int_1^x \frac{dt}{t}, \qquad 0 < x < \infty$$

Since $f(x) = \frac{1}{x}$ is unbounded as $x \to 0$, this is an improper integral of the second kind (see Fig. 12-2). Also, $\int_0^\infty \frac{dt}{t}$ is an integral of the third kind, since the interval to the right is unbounded.

Now $\lim_{\epsilon \to 0} \int_\epsilon^1 \frac{dt}{t} = \lim_{\epsilon \to 0} [\ln 1 - \ln \epsilon] \to -\infty$ as $\epsilon \to 0$; therefore, this improper integral of the second kind is divergent. Also, $\int_1^\infty \frac{dt}{t} = \lim_{x \to \infty} \int_1^x \frac{dt}{t} = \lim_{x \to \infty} [\ln x - \ln i] \to \infty$; this integral (which is of the first kind) also diverges.

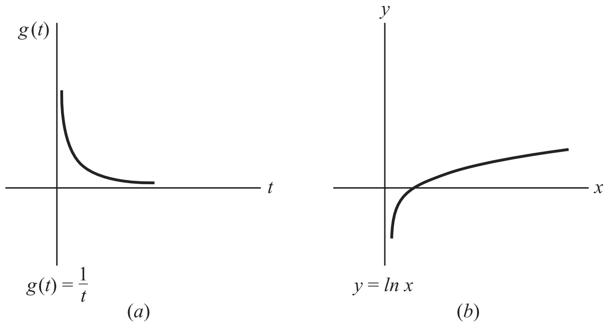

Fig. 12-2

SPECIAL IMPROPER INTEGRALS OF THE SECOND KIND

1. $\displaystyle\int_a^b \frac{dx}{(x-a)^p}$ converges if $p < 1$ and diverges if $p \geqq 1$.
2. $\displaystyle\int_a^b \frac{dx}{(b-x)^p}$ converges if $p < 1$ and diverges if $p \geqq 1$.

These can be called *p integrals of the second kind*. Note that when $p \leqq 0$ the integrals are proper.

CONVERGENCE TESTS FOR IMPROPER INTEGRALS OF THE SECOND KIND

The following tests are given for the case where $f(x)$ is unbounded only at $x = a$ in the interval $a \leqq x \leqq b$. Similar tests are available if $f(x)$ is unbounded at $x = b$ or at $x = x_0$ where $a < x_0 < b$.

1. **Comparison test** for integrals with non-negative integrands.
   (*a*) *Convergence*. Let $g(x) \geqq 0$ for $a < x \leqq b$, and suppose that $\int_a^b g(x)\,dx$ converges. Then if $0 \leqq f(x) \leqq g(x)$ for $a < x \leqq b$, $\int_a^b f(x)\,dx$ also converges.

EXAMPLE. $\dfrac{1}{\sqrt{x^4-1}} < \dfrac{1}{\sqrt{x-1}}$ for $x > 1$. Then since $\displaystyle\int_1^5 \frac{dx}{\sqrt{x-1}}$ converges (p integral with $a = 1$, $p = \frac{1}{2}$), $\displaystyle\int_1^5 \frac{dx}{\sqrt{x^4-1}}$ also converges.

   (*b*) *Divergence*. Let $g(x) \geqq 0$ for $a < x \leqq b$, and suppose that $\int_a^b g(x)\,dx$ diverges. Then if $f(x) \geqq g(x)$ for $a < x \leqq b$, $\int_a^b f(x)\,dx$ also diverges.

EXAMPLE. $\dfrac{\ln x}{(x-3)^4} > \dfrac{1}{(x-3)^4}$ for $x > 3$. Then since $\displaystyle\int_3^6 \frac{dx}{(x-3)^4}$ diverges (p integral with $a = 3$, $p = 4$), $\displaystyle\int_3^6 \frac{\ln x}{(x-3)^4}\,dx$ also diverges.

2. **Quotient test** for integrals with non-negative integrands.
   (*a*) If $f(x) \geqq 0$ and $g(x) \geqq 0$ for $a < x \leqq b$, and if $\lim_{x\to a} \dfrac{f(x)}{g(x)} = A \neq 0$ or ∞, then $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ either both converge or both diverge.

(*b*) If $A = 0$ in (*a*), then $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ converges.

(*c*) If $A = \infty$ in (*a*), and $\int_a^b g(x)\,dx$ diverges, then $\int_a^b f(x)\,dx$ diverges.

This test is related to the comparison test and is a very useful alternative to it. In particular taking $g(x) = 1/(x-a)^p$ we have from known facts about the p integral the following theorems.

Theorem 3. Let $\lim_{x\to a+} (x-a)^p f(x) = A$. Then

(i) $\int_a^b f(x)\,dx$ converges if $p < 1$ and A is finite

(ii) $\int_a^b f(x)\,dx$ diverges if $p \geqq 1$ and $A \neq 0$ (A may be infinite).

If $f(x)$ becomes unbounded only at the upper limit these conditions are replaced by those in

Theorem 4. Let $\lim_{x\to b-} (b-x)^p f(x) = B$. Then

(i) $\int_a^b f(x)\,dx$ converges if $p < 1$ and B is finite

(ii) $\int_a^b f(x)\,dx$ diverges if $p \geqq 1$ and $B \neq 0$ (B may be infinite).

EXAMPLE 1. $\int_1^5 \frac{dx}{\sqrt{x^4-1}}$ converges, since $\lim_{x\to 1+} (x-1)^{1/2} \cdot \frac{1}{(x^4-1)^{1/2}} = \lim_{x\to 1+} \sqrt{\frac{x-1}{x^4-1}} = \frac{1}{2}$.

EXAMPLE 2. $\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}}$ diverges, since $\lim_{x\to 3-} (3-x) \cdot \frac{1}{(3-x)\sqrt{x^2+1}} = \frac{1}{\sqrt{10}}$.

3. **Absolute and conditional convergence.** $\int_a^b f(x)\,dx$ is called *absolute convergent* if $\int_a^b |f(x)|\,dx$ converges. If $\int_a^b f(x)\,dx$ converges but $\int_a^b |f(x)|\,dx$ diverges, then $\int_a^b f(x)\,dx$ is called *conditionally convergent*.

Theorem 5. If $\int_a^b |f(x)|\,dx$ converges, then $\int_a^b f(x)\,dx$ converges. In words, an absolutely convergent integral converges.

EXAMPLE. Since $\left|\frac{\sin x}{\sqrt[3]{x-\pi}}\right| \leqq \frac{1}{\sqrt[3]{x-\pi}}$ and $\int_\pi^{4\pi} \frac{dx}{\sqrt[3]{x-\pi}}$ converges (p integral with $a = \pi, p = \frac{1}{3}$), it follows that $\int_\pi^{4\pi} \left|\frac{\sin x}{\sqrt[3]{x-\pi}}\right| dx$ converges and thus $\int_\pi^{4\pi} \frac{\sin x}{\sqrt[3]{x-\pi}}\,dx$ converges (absolutely).

Any of the tests used for integrals with non-negative integrands can be used to test for absolute convergence.

IMPROPER INTEGRALS OF THE THIRD KIND

Improper integrals of the third kind can be expressed in terms of improper integrals of the first and second kinds, and hence the question of their convergence or divergence is answered by using results already established.

IMPROPER INTEGRALS CONTAINING A PARAMETER, UNIFORM CONVERGENCE

Let

$$\phi(\alpha) = \int_a^\infty f(x, \alpha)\, dx \tag{8}$$

This integral is analogous to an infinite series of functions. In seeking conditions under which we may differentiate or integrate $\phi(\alpha)$ with respect to α, it is convenient to introduce the concept of *uniform convergence* for integrals by analogy with infinite series.

We shall suppose that the integral (*8*) converges for $\alpha_1 \leqq \alpha \leqq \alpha_2$, or briefly $[\alpha_1, \alpha_2]$.

Definition.

The integral (*8*) is said to be *uniformly convergent* in $[\alpha_1, \alpha_2]$ if for each $\epsilon > 0$, we can find a number N depending on ϵ but not on α, such that

$$\left|\phi(\alpha) - \int_a^u f(x, \alpha)\, dx\right| < \epsilon \qquad \text{for all } u > N \text{ and all } \alpha \text{ in } [\alpha_1, \alpha_2]$$

This can be restated by nothing that $\left|\phi(\alpha) - \int_a^u f(x, \alpha)\, dx\right| = \left|\int_u^\infty f(x, \alpha)\, dx\right|$, which is analogous in an infinite series to the absolute value of the remainder after N terms.

The above definition and the properties of uniform convergence to be developed are formulated in terms of improper integrals of the first kind. However, analogous results can be given for improper integrals of the second and third kinds.

SPECIAL TESTS FOR UNIFORM CONVERGENCE OF INTEGRALS

1. **Weierstrass *M* test.** If we can find a function $M(x) \geqq 0$ such that

   (*a*) $|f(x, \alpha)| \leqq M(x) \qquad \alpha_1 \leqq \alpha \leqq \alpha_2, x > a$

   (*b*) $\int_a^\infty M(x)\, dx$ converges,

   then $\int_a^\infty f(x, \alpha)\, dx$ is uniformly and absolutely convergent in $\alpha_1 \leqq \alpha \leqq \alpha_2$.

   EXAMPLE. Since $\left|\dfrac{\cos \alpha x}{x^2+1}\right| \leqq \dfrac{1}{x^2+1}$ and $\int_0^\infty \dfrac{dx}{x^2+1}$ converges, it follows that $\int_0^\infty \dfrac{\cos \alpha x}{x^2+1}\, dx$ is uniformly and absolutely convergent for all real values of α.

   As in the case of infinite series, it is possible for integrals to be uniformly convergent without being absolutely convergent, and conversely.

2. **Dirichlet's test.** Suppose that
   (*a*) $\psi(x)$ is a positive monotonic decreasing function which approaches zero as $x \to \infty$.
   (*b*) $\left| \int_a^u f(x, \alpha)\, dx \right| < P$ for all $u > a$ and $\alpha_1 \leqq \alpha \leqq \alpha_2$.
   Then the integral $\int_a^\infty f(x, \alpha)\psi(x)\, dx$ is uniformly convergent for $\alpha_1 \leqq \alpha \leqq \alpha_2$.

THEOREMS ON UNIFORMLY CONVERGENT INTEGRALS

Theorem 6. If $f(x, \alpha)$ is continuous for $x \geqq a$ and $\alpha_1 \leqq \alpha \leqq \alpha_2$, and if $\int_a^\infty f(x, \alpha)\, dx$ is uniformly convergent for $\alpha_1 \leqq \alpha \leqq \alpha_2$, then $\phi(\alpha) = \int_a^\infty f(x, \alpha)\, dx$ is continous in $\alpha_1 \leqq \alpha \leqq \alpha_2$. In particular, if α_0 is any point of $\alpha_1 \leqq \alpha \leqq \alpha_2$, we can write

$$\lim_{\alpha \to \alpha_0} \phi(\alpha) = \lim_{\alpha \to \alpha_0} \int_a^\infty f(x, \alpha)\, dx = \int_a^\infty \lim_{\alpha \to \alpha_0} f(x, \alpha)\, dx \tag{9}$$

If α_0 is one of the end points, we use right or left hand limits.

Theorem 7. Under the conditions of Theorem 6, we can integrate $\phi(\alpha)$ with respect to α from α_1 to α_2 to obtain

$$\int_{\alpha_1}^{\alpha_2} \phi(\alpha)\, d\alpha = \int_{\alpha_1}^{\alpha_2} \left\{ \int_a^\infty f(x, \alpha)\, dx \right\} d\alpha = \int_a^\infty \left\{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha)\, d\alpha \right\} dx \tag{10}$$

which corresponds to a change of the order of integration.

Theorem 8. If $f(x, \alpha)$ is continuous and has a continuous partial derivative with respect to α for $x \geqq a$ and $\alpha_1 \leqq \alpha \leqq \alpha_2$, and if $\int_a^\infty \frac{\partial f}{\partial \alpha}\, dx$ converges uniformly in $\alpha_1 \leqq \alpha \leqq \alpha_2$, then if a does not depend on α,

$$\frac{d\phi}{d\alpha} = \int_a^\infty \frac{\partial f}{\partial \alpha}\, dx \tag{11}$$

If a depends on α, this result is easily modified (see Leibnitz's rule, Page 186).

EVALUATION OF DEFINITE INTEGRALS

Evaluation of definite integrals which are improper can be achieved by a variety of techniques. One useful device consists of introducing an appropriately placed parameter in the integral and then differentiating or integrating with respect to the parameter, employing the above properties of uniform convergence.

LAPLACE TRANSFORMS

Operators that transform one set of objects into another are common in mathematics. The derivative and the indefinite integral both are examples. Logarithms provide an immediate arithmetic advantage by replacing multiplication, division, and powers, respectively, by the relatively simpler processes of addition, subtraction, and multiplication. After obtaining a result with logarithms an anti-logarithm procedure is necessary to find its image in the original framework. The Laplace transform has a role similar to that of logarithms but in the more sophisticated world of differential equations. (See Problems 12.34 and 12.36.)

The Laplace transform of a function $F(x)$ is defined as

$$f(s) = \mathscr{L}\{F(x)\} = \int_0^\infty e^{-sx} F(x)\,dx \qquad (12)$$

and is analogous to power series as seen by replacing e^{-s} by t so that $e^{-sx} = t^x$. Many properties of power series also apply to Laplace transforms. The adjacent short table of Laplace transforms is useful. In each case a is a real constant.

$F(x)$	$\mathscr{L}\{F(x)\}$	
a	$\dfrac{a}{8}$	$8 > 0$
e^{ax}	$\dfrac{1}{8 - a}$	$8 > a$
$\sin\ ax$	$\dfrac{a}{8^2 + a^2}$	$8 > 0$
$\cos\ ax$	$\dfrac{8}{8^2 + a^2}$	$8 > 0$
$x^n\ n = 1, 2, 3, \ldots$	$\dfrac{n!}{8^{n+1}}$	$8 > 0$
$Y'(x)$	$8\mathscr{L}\{Y(x)\} - Y(0)$	
$Y''(x)$	$8^2\mathscr{L}\{Y(x)\} - 8Y(0) - Y'(0)$	

LINEARITY

The Laplace transform is a linear operator, i.e.,

$$\zeta\{F(x) + G(x)\} = \zeta\{F(x)\} + \zeta\{G(x)\}.$$

This property is essential for returning to the solution after having calculated in the setting of the transforms. (See the following example and the previously cited problems.)

CONVERGENCE

The exponential e^{-st} contributes to the convergence of the improper integral. What is required is that $F(x)$ does not approach infinity too rapidly as $x \to \infty$. This is formally stated as follows:

If there is some constant a such that $|F(x)| \le e^{ax}$ for all sufficiently large values of x, then $f(s) = \int_0^\infty e^{-sx} F(x)\,dx$ converges when $s > a$ and f has derivatives of all orders. (The differentiations of f can occur under the integral sign $>$.)

APPLICATION

The feature of the Laplace transform that (when combined with linearity) establishes it as a tool for solving differential equations is revealed by applying integration by parts to $f(s) = \int_0^x e^{-st} F(t)\,dt$. By letting $u = F(t)$ and $dv = e^{-st}\,dt$, we obtain after letting $x \to \infty$

$$\int_0^x e^{-st} F(t)\,dt = \frac{1}{s} F(0) + \frac{1}{s} \int_0^\infty e^{-st} F'(t)\,dt.$$

Conditions must be satisfied that guarantee the convergence of the integrals (for example, $e^{-st}F(t) \to 0$ as $t \to \infty$).

This result of integration by parts may be put in the form

(*a*) $\zeta\{F'(t)\} = s\zeta\{F(t)\} + F'(0)$.
Repetition of the procedure combined with a little algebra yields

(*b*) $\zeta\{F''(t)\} = s^2\zeta\{F(t)\} - sF(0) - F'(0)$.
The Laplace representation of derivatives of the order needed can be obtained by repeating the process.

To illustrate application, consider the differential equation

$$\frac{d^2y}{dt^2} + 4y = 3\sin t,$$

where $y = F(t)$ and $F(0) = 1$, $F'(0) = 0$. We use

$$\zeta\{\sin at\} = \frac{a}{s^2 + a^2},\ \zeta\{\cos at\} = \frac{s}{s^2 + a^2}$$

and recall that

$$f(s) = \zeta\{F(t)\}\zeta\{F''(t)\} + 4\zeta\{F(t)\} = 3\zeta\{\sin t\}$$

Using (*b*) we obtain

$$s^2 f(s) - s + 4f(s) = \frac{3}{s^2 + 1}.$$

Solving for $f(s)$ yields

$$f(s) = \frac{3}{(s^2 + 4)(s^2 + 1)} + \frac{s}{s^2 + 4} = \frac{1}{s^2 + 1} - \frac{1}{s^2 + 4} + \frac{s}{s^2 + 4}.$$

(Partial fractions were employed.)

Referring to the table of Laplace transforms, we see that this last expression may be written

$$f(s) = \zeta\{\sin t\} - \tfrac{1}{2}\zeta\{\sin 2t\} + \zeta\{\cos 2t\}$$

then using the linearity of the Laplace transform

$$f(s) = \zeta\{\sin t - \tfrac{1}{2}\sin 2t + \cos 2t\}.$$

We find that

$$F(t) = \sin t - \tfrac{1}{2}\sin 2t + \cos 2t$$

satisfies the differential equation.

IMPROPER MULTIPLE INTEGRALS

The definitions and results for improper single integrals can be extended to improper multiple integrals.

Solved Problems

IMPROPER INTEGRALS

12.1. Classify according to the type of improper integral.

(*a*) $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}(x+1)}$ (*c*) $\displaystyle\int_{3}^{10} \frac{x\,dx}{(x-2)^2}$ (*e*) $\displaystyle\int_{0}^{\pi} \frac{1-\cos x}{x^2}\,dx$

(*b*) $\displaystyle\int_{0}^{\infty} \frac{dx}{1+\tan x}$ (*d*) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4+x^2+1}$

(*a*) *Second kind* (integrand is unbounded at $x = 0$ and $x = -1$).

(*b*) *Third kind* (integration limit is infinite and integrand is unbounded where $\tan x = -1$).

(*c*) This is a *proper* integral (integrand becomes unbounded at $x = 2$, but this is *outside* the range of integration $3 \leqq x \leqq 10$).

(*d*) *First kind* (integration limits are infinite but integrand is bounded).

(*e*) This is a *proper* integral $\left(\text{since } \lim_{x\to 0+} \frac{1-\cos x}{x^2} = \frac{1}{2} \text{ by applying L'Hospital's rule}\right)$.

12.2. Show how to transform the improper integral of the second kind, $\int_1^2 \frac{dx}{\sqrt{x(2-x)}}$, into (*a*) an improper integral of the first kind, (*b*) a proper integral.

(*a*) Consider $\int_1^{2-\epsilon} \frac{dx}{\sqrt{x(2-x)}}$ where $0 < \epsilon < 1$, say. Let $2 - x = \frac{1}{y}$. Then the integral becomes $\int_1^{1/\epsilon} \frac{dy}{y\sqrt{2y-1}}$. As $\epsilon \to 0+$, we see that consideration of the given integral is equivalent to consideration of $\int_1^{\infty} \frac{dy}{y\sqrt{2y-1}}$, which is an improper integral of the first kind.

(*b*) Letting $2 - x = v^2$ in the integral of (*a*), it becomes $2\int_{\sqrt{\epsilon}}^1 \frac{dv}{\sqrt{v^2+2}}$. We are thus led to consideration of $2\int_0^1 \frac{dv}{\sqrt{v^2+1}}$, which is a proper integral.

From the above we see that an improper integral of the first kind *may* be transformed into an improper integral of the second kind, and conversely (actually this can *always* be done).

We also see that an improper integral may be transformed into a proper integral (this can only *sometimes* be done).

IMPROPER INTEGRALS OF THE FIRST KIND

12.3. Prove the comparison test (Page 308) for convergence of improper integrals of the first kind.

Since $0 \leqq f(x) \leqq g(x)$ for $x \geqq a$, we have using Property 7, Page 92,

$$0 \leqq \int_a^b f(x)\,dx \leqq \int_a^b g(x)\,dx \leqq \int_a^{\infty} g(x)\,dx$$

But by hypothesis the last integral exists. Thus

$$\lim_{b\to\infty} \int_a^b f(x)\,dx \text{ exists, and hence } \int_a^{\infty} f(x)\,dx \text{ converges}$$

12.4. Prove the quotient test (*a*) on Page 309.

By hypothesis, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = A > 0$. Then given any $\epsilon > 0$, we can find N such that $\left|\frac{f(x)}{g(x)} - A\right| < \epsilon$ when $x \geqq N$. Thus for $x \geqq N$, we have

$$A - \epsilon \leqq \frac{f(x)}{g(x)} \leqq A + \epsilon \qquad \text{or} \qquad (A-\epsilon)g(x) \leqq f(x) \leqq (A+\epsilon)g(x)$$

Then

$$(A-\epsilon)\int_N^b g(x)\,dx \leqq \int_N^b f(x)\,dx \leqq (A+\epsilon)\int_N^b g(x)\,dx \tag{1}$$

There is no loss of generality in choosing $A - \epsilon > 0$.

If $\int_a^{\infty} g(x)\,dx$ converges, then by the inequality on the right of (*1*),

$$\lim_{b\to\infty} \int_N^b f(x)\,dx \text{ exists, and so } \int_a^{\infty} f(x)\,dx \text{ converges}$$

If $\int_a^{\infty} g(x)\,dx$ diverges, then by the inequality on the left of (*1*),

$$\lim_{b\to\infty}\int_N^b f(x)\,dx = \infty \quad \text{and so} \quad \int_a^\infty f(x)\,dx \quad \text{diverges}$$

For the cases where $A = 0$ and $A = \infty$, see Problem 12.41.

As seen in this and the preceding problem, there is in general a marked similarity between proofs for infinite series and improper integrals.

12.5. Test for convergence: (*a*) $\displaystyle\int_1^\infty \frac{x\,dx}{3x^4+5x^2+1}$, (*b*) $\displaystyle\int_2^\infty \frac{x^2-1}{\sqrt{x^6+16}}\,dx$.

(*a*) **Method 1:** For large x, the integrand is approximately $x/3x^4 = 1/3x^3$.

Since $\dfrac{x}{3x^4+5x^2+1} \leqq \dfrac{1}{3x^3}$, and $\dfrac{1}{3}\displaystyle\int_1^\infty \frac{dx}{x^3}$ converges (p integral with $p = 3$), it follows by the comparison test that $\displaystyle\int_1^\infty \frac{x\,dx}{3x^4+5x^2+1}$ also converges.

Note that the purpose of examining the integrand for large x is to obtain a suitable comparison integral.

Method 2: Let $f(x) = \dfrac{x}{3x^4+5x^2+1}$, $g(x) = \dfrac{1}{x^3}$. Since $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{1}{3}$, and $\displaystyle\int_1^\infty g(x)\,dx$ converges, $\displaystyle\int_1^\infty f(x)\,dx$ also converges by the quotient test.

Note that in the comparison function $g(x)$, we have discarded the factor $\frac{1}{3}$. It could, however, just as well have been included.

Method 3: $\displaystyle\lim_{x\to\infty} x^3\left(\frac{x}{3x^4+5x^2+1}\right) = \frac{1}{3}$. Hence, by Theorem 1, Page 309, the required integral converges.

(*b*) **Method 1:** For large x, the integrand is approximately $x^2/\sqrt{x^6} = 1/x$.

For $x \geqq 2$, $\dfrac{x^2-1}{\sqrt{x^6+1}} \geqq \dfrac{1}{2}\cdot\dfrac{1}{x}$. Since $\dfrac{1}{2}\displaystyle\int_2^\infty \frac{dx}{x}$ diverges, $\displaystyle\int_2^\infty \frac{x^2-1}{\sqrt{x^6+16}}\,dx$ also diverges.

Method 2: Let $f(x) = \dfrac{x^2-1}{\sqrt{x^6-16}}$, $g(x) = \dfrac{1}{x}$. Then since $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$, and $\displaystyle\int_2^\infty g(x)\,dx$ diverges, $\displaystyle\int_2^\infty f(x)\,dx$ also diverges.

Method 3: Since $\displaystyle\lim_{x\to\infty} x\left(\frac{x^2-1}{\sqrt{x^6+16}}\right) = 1$, the required integral diverges by Theorem 1, Page 309.

Note that Method 1 may (and often does) require one to obtain a suitable inequality factor (in this case $\frac{1}{2}$, or any positive constant less than $\frac{1}{2}$) before the comparison test can be applied. Methods 2 and 3, however, do not require this.

12.6. Prove that $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges.

$\displaystyle\lim_{x\to\infty} x^2 e^{-x^2} = 0$ (by L'Hospital's rule or otherwise). Then by Theorem 1, with $A = 0, p = 2$ the given integral converges. Compare Problem 11.10(*a*), Chapter 11.

12.7. Examine for convergence:

(*a*) $\displaystyle\int_1^\infty \frac{\ln x}{x+a}\,dx$, where a is a positive constant; (*b*) $\displaystyle\int_0^\infty \frac{1-\cos x}{x^2}\,dx$.

(*a*) $\displaystyle\lim_{x\to\infty} x\cdot\frac{\ln x}{x+a} = \infty$. Hence by Theorem 1, Page 309, with $A = \infty, p = 1$, the given integral diverges.

(*b*) $\displaystyle\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \int_0^\pi \frac{1-\cos x}{x^2}\,dx + \int_\pi^\infty \frac{1-\cos x}{x^2}\,dx$

The first integral on the right converges [see Problem 12.1(*e*)].

Since $\displaystyle\lim_{x\to\infty} x^{3/2}\left(\frac{1-\cos x}{x^2}\right) = 0$, the second integral on the right converges by Theorem 1, Page 309, with $A = 0$ and $p = 3/2$.

Thus, the given integral converges.

12.8. Test for convergence: (*a*) $\displaystyle\int_{-\infty}^{-1} \frac{e^x}{x}\,dx$, (*b*) $\displaystyle\int_{-\infty}^{\infty} \frac{x^3+x^2}{x^6+1}\,dx$.

(*a*) Let $x = -y$. Then the integral becomes $\displaystyle -\int_1^\infty \frac{e^{-y}}{y}\,dy$.

Method 1: $\dfrac{e^{-y}}{y} \leqq e^{-y}$ for $y \leqq 1$. Then since $\displaystyle\int_1^\infty e^{-y}\,dy$ converges, $\displaystyle\int_1^\infty \frac{e^{-y}}{y}\,dy$ converges; hence the given integral converges.

Method 2: $\displaystyle\lim_{y\to\infty} y^2\left(\frac{e^{-y}}{y}\right) = \lim_{y\to\infty} ye^{-y} = 0$. Then the given integral converges by Theorem 1, Page 309, with $A = 0$ and $p = 2$.

(*b*) Write the given integral as $\displaystyle\int_{-\infty}^{0} \frac{x^3+x^2}{x^6+1}\,dx + \int_0^\infty \frac{x^3+x^2}{x^6+1}\,dx$. Letting $x = -y$ in the first integral, it becomes $\displaystyle -\int_0^\infty \frac{y^3-y^2}{y^6+1}\,dy$. Since $\displaystyle\lim_{y\to\infty} y^3\left(\frac{y^3-y^2}{y^6+1}\right) = 1$, this integral converges.

Since $\displaystyle\lim_{x\to\infty} x^3\left(\frac{x^3+x^2}{x^6+1}\right) = 1$, the second integral converges.

Thus the given integral converges.

ABSOLUTE AND CONDITIONAL CONVERGENCE FOR IMPROPER INTEGRALS OF THE FIRST KIND

12.9. Prove that $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty |f(x)|\,dx$ converges, i.e., an absolutely convergent integral is convergent.

We have $-|f(x)| \leqq f(x) \leqq |f(x)|$, i.e., $0 \leqq f(x) + |f(x)| \leqq 2|f(x)|$. Then

$$0 \leqq \int_a^b [f(x) + |f(x)|]\,dx \leqq 2\int_a^b |f(x)|\,dx$$

If $\displaystyle\int_a^\infty |f(x)|\,dx$ converges, it follows that $\displaystyle\int_a^\infty [f(x) + |f(x)|]\,dx$ converges. Hence, by subtracting $\displaystyle\int_a^\infty |f(x)|\,dx$, which converges, we see that $\displaystyle\int_a^\infty f(x)\,dx$ converges.

12.10. Prove that $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\,dx$ converges.

Method 1:

$\left|\dfrac{\cos x}{x^2}\right| \leqq \dfrac{1}{x^2}$ for $x \geqq 1$. Then by the comparison test, since $\displaystyle\int_1^\infty \frac{dx}{x^2}$ converges, it follows that $\displaystyle\int_1^\infty \left|\frac{\cos x}{x^2}\right| dx$ converges, i.e., $\displaystyle\int_1^\infty \frac{\cos x}{x^2}\,dx$ converges absolutely, and so converges by Problem 12.9.

Method 2:

Since $\lim_{x\to\infty} x^{3/2}\left|\frac{\cos x}{x^2}\right| = \lim_{x\to\infty}\left|\frac{\cos x}{x^{1/2}}\right| = 0$, it follows from Theorem 1, Page 309, with $A = 0$ and $p = 3/2$, that $\int_1^\infty \left|\frac{\cos x}{x^2}\right| dx$ converges, and hence $\int_1^\infty \frac{\cos x}{x^2}\, dx$ converges (absolutely).

12.11. Prove that $\int_0^\infty \frac{\sin x}{x}\, dx$ converges.

Since $\int_0^1 \frac{\sin x}{x}\, dx$ converges $\left(\text{because } \frac{\sin x}{x} \text{ is continuous in } 0 < x \leqq 1 \text{ and } \lim_{x\to 0+} \frac{\sin x}{x} = 1\right)$ we need only show that $\int_1^\infty \frac{\sin x}{x}\, dx$ converges.

Method 1: Integration by parts yields

$$\int_1^M \frac{\sin x}{x}\, dx = -\frac{\cos x}{x}\bigg|_1^M + \int_1^M \frac{\cos x}{x^2}\, dx = \cos 1 - \frac{\cos M}{M} + \int_1^M \frac{\cos x}{x^2}\, dx \tag{1}$$

or on taking the limit on both sides of (*1*) as $M \to \infty$ and using the fact that $\lim_{M\to\infty} \frac{\cos M}{M} = 0$,

$$\int_1^\infty \frac{\sin x}{x}\, dx = \cos 1 + \int_1^\infty \frac{\cos x}{x^2}\, dx \tag{2}$$

Since the integral on the right of (*2*) converges by Problem 12.10, the required results follows.

The technique of integration by parts to establish convergence is often useful in practice.

Method 2:

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x}\, dx &= \int_0^\pi \frac{\sin x}{x}\, dx + \int_\pi^{2\pi} \frac{\sin x}{x}\, dx + \cdots + \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\, dx + \cdots \\ &= \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\, dx\end{aligned}$$

Letting $x = v + n\pi$, the summation becomes

$$\sum_{n=0}^\infty (-1)^n \int_0^\pi \frac{\sin v}{n + n\pi}\, dv = \int_0^\pi \frac{\sin v}{v}\, dv - \int_0^\pi \frac{\sin v}{v+\pi}\, dv + \int_0^\pi \frac{\sin v}{v + 2\pi}\, dv - \cdots$$

This is an alternating series. Since $\frac{1}{v + n\pi} \leqq \frac{1}{v + (n+1)\pi}$ and $\sin v \geqq 0$ in $[0, \pi]$, it follows that

$$\int_0^\pi \frac{\sin v}{v + n\pi}\, dv \leqq \int_0^\pi \frac{\sin v}{v + (n+1)\pi}\, dv$$

Also,

$$\lim_{n\to\infty} \int_0^\pi \frac{\sin v}{v + n\pi}\, dv \leqq \lim_{n\to\infty} \int_0^\pi \frac{dv}{n\pi} = 0$$

Thus, each term of the alternating series is in absolute value less than or equal to the preceding term, and the nth term approaches zero as $n \to \infty$. Hence, by the alternating series test (Page 267) the series and thus the integral converges.

12.12. Prove that $\int_0^\infty \frac{\sin x}{x}\, dx$ converges conditionally.

Since by Problem 12.11 the given integral converges, we must show that it is not absolutely convergent, i.e., $\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ diverges.

As in Problem 12.11, Method 2, we have

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{n=0}^\infty \int_0^\pi \frac{\sin v}{v + n\pi}\, dv \tag{1}$$

Now $\dfrac{1}{v+n\pi} \geqq \dfrac{1}{(n+1)\pi}$ for $0 \leqq v \leqq \pi$. Hence,

$$\int_0^\pi \frac{\sin v}{v+n\pi}\, dv \geqq \frac{1}{(n+1)\pi}\int_9^\pi \sin v\, dv = \frac{2}{(n+1)\pi} \tag{2}$$

Since $\sum_{n=0}^\infty \dfrac{2}{(n+1)\pi}$ diverges, the series on the right of (*1*) diverges by the comparison test. Hence, $\int_0^\infty \left|\dfrac{\sin x}{x}\right| dx$ diverges and the required result follows.

IMPROPER INTEGRALS OF THE SECOND KIND, CAUCHY PRINCIPAL VALUE

12.13. (*a*) Prove that $\int_{-1}^7 \dfrac{dx}{\sqrt[3]{x+1}}$ converges and (*b*) find its value.

The integrand is unbounded at $x = -1$. Then we define the integral as

$$\lim_{\epsilon\to 0+}\int_{-1+\epsilon}^7 \frac{dx}{\sqrt[3]{x+1}} = \lim_{\epsilon\to 0+} \left.\frac{(x+1)^{2/3}}{2/3}\right|_{-1+\epsilon}^7 = \lim_{\epsilon\to 0+}\left(6 - \frac{3}{2}\epsilon^{2/3}\right) = 6$$

This shows that the integral converges to 6.

12.14. Determine whether $\int_{-1}^5 \dfrac{dx}{(x-1)^3}$ converges (*a*) in the usual sense, (*b*) in the Cauchy principal value sense.

(*a*) By definition,

$$\begin{aligned}\int_{-1}^5 \frac{dx}{(x-1)^3} &= \lim_{\epsilon_1\to 0+}\int_{-1}^{1-\epsilon_1}\frac{dx}{(x-1)^3} + \lim_{\epsilon_2\to 0+}\int_{1+\epsilon_2}^5 \frac{dx}{(x-1)^3}\\ &= \lim_{\epsilon_1\to 0+}\left(\frac{1}{8} - \frac{1}{2\epsilon_1^2}\right) + \lim_{\epsilon_2\to 0+}\left(\frac{1}{2\epsilon_2^2} - \frac{1}{32}\right)\end{aligned}$$

and since the limits do not exist, the integral does not converge in the usual sense.

(*b*) Since

$$\lim_{\epsilon\to 0+}\left\{\int_{-1}^{1-\epsilon}\frac{dx}{(x-1)^3} + \int_{1+\epsilon}^5 \frac{dx}{(x-1)^3}\right\} = \lim_{\epsilon\to 0+}\left\{\frac{1}{8} - \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon^2} - \frac{1}{32}\right\} = \frac{3}{32}$$

the integral exists in the Cauchy principal value sense. The principal value is 3/32.

12.15. Investigate the convergence of:

(*a*) $\displaystyle\int_2^3 \frac{dx}{x^2(x^3-8)^{2/3}}$ (*c*) $\displaystyle\int_1^5 \frac{dx}{\sqrt{(5-x)(x-1)}}$ (*e*) $\displaystyle\int_0^{\pi/2}\frac{dx}{(\cos x)^{1/n}}, n > 1.$

(*b* $\displaystyle\int_0^\pi \frac{\sin x}{x^3}\, dx$ (*d*) $\displaystyle\int_{-1}^1 \frac{2^{\sin^{-1} x}}{1-x}\, dx$

(*a*) $\displaystyle\lim_{x\to 2+}(x-2)^{2/3}\cdot\frac{1}{x^2(x^3-8)^{2/3}} = \lim_{x\to 2+}\frac{1}{x^2}\left(\frac{1}{x^2+2x+4}\right)^{2/3} = \frac{1}{8\sqrt[3]{18}}.$ Hence, the integral converges by Theorem 3(*i*), Page 312.

(*b*) $\lim_{x\to 0+} x^2 \cdot \dfrac{\sin x}{x^3} = 1$. Hence, the integral diverges by Theorem 3(*ii*) on Page 312.

(*c*) Write the integral as $\displaystyle\int_1^3 \frac{dx}{\sqrt{(5-x)(x-1)}} + \int_3^5 \frac{dx}{\sqrt{(5-x)(x-1)}}$.

Since $\displaystyle\lim_{x\to 1+} (x-1)^{1/2} \cdot \frac{1}{\sqrt{(5-x)(x-1)}} = \frac{1}{2}$, the first integral converges.

Since $\displaystyle\lim_{x\to 5-} (5-x)^{1/2} \cdot \frac{1}{\sqrt{(5-x)(x-1)}} = \frac{1}{2}$, the second integral converges.

Thus, the given integral converges.

(*d*) $\displaystyle\lim_{x\to 1-} (1-x) \cdot \frac{2^{\sin^{-1} x}}{1-x} = 2^{\pi/2}$. Hence, the integral diverges.

Another method:

$$\frac{2^{\sin^{-1} x}}{1-x} \geqq \frac{2^{-\pi/2}}{1-x}, \text{ and } \int_{-1}^{1} \frac{dx}{1-x} \text{ diverges. Hence, the given integral diverges.}$$

(*e*) $\displaystyle\lim_{x\to 1/2\pi-} (\pi/2 - x)^{1/n} \cdot \frac{1}{(\cos x)^{1/n}} = \lim_{x\to 1/2\pi-} \left(\frac{\pi/2 - x}{\cos x}\right)^{1/n} = 1$. Hence the integral converges.

12.16. If m and n are real numbers, prove that $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ (*a*) converges if $m > 0$ and $n > 0$ simultaneously and (*b*) diverges otherwise.

(*a*) For $m \geqq 1$ and $n \geqq 1$ simultaneously, the integral converges, since the integrand is continuous in $0 \leqq x \leqq 1$. Write the integral as

$$\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx + \int_{1/2}^{1} x^{m-1}(1-x)^{n-1}\,dx \tag{1}$$

If $0 < m < 1$ and $0 < n < 1$, the first integral converges, since $\lim_{x\to 0+} x^{1-m} \cdot x^{m-1}(1-x)^{n-1} = 1$, using Theorem 3(*i*), Page 312, with $p = 1 - m$ and $a = 0$.

Similarly, the second integral converges since $\lim_{x\to 1-} (1-x)^{1-n} \cdot x^{m-1}(1-x)^{n-1} = 1$, using Theorem 4(*i*), Page 312, with $p = 1 - n$ and $b = 1$.

Thus, the given integral converges if $m > 0$ and $n > 0$ simultaneously.

(*b*) If $m \leqq 0$, $\lim_{x\to 0+} x \cdot x^{m-1}(1-x)^{n-1} = \infty$. Hence, the first integral in (*1*) diverges, regardless of the value of n, by Theorem 3(*ii*), Page 312, with $p = 1$ and $a = 0$.

Similarly, the second integral diverges if $n \leqq 0$ regardless of the value of m, and the required result follows.

Some interesting properties of the given integral, called the *beta integral* or *beta function*, are considered in Chapter 15.

12.17. Prove that $\displaystyle\int_0^{\pi} \frac{1}{x} \sin\frac{1}{x}\,dx$ converges conditionally.

Letting $x = 1/y$, the integral becomes $\displaystyle\int_{1/\pi}^{\infty} \frac{\sin y}{y}\,dy$ and the required result follows from Problem 12.12.

IMPROPER INTEGRALS OF THE THIRD KIND

12.18. If n is a real number, prove that $\displaystyle\int_0^{\infty} x^{n-1}e^{-x}\,dx$ (*a*) converges if $n > 0$ and (*b*) diverges if $n \leqq 0$.

Write the integral as

$$\int_0^1 x^{n-1}e^{-x}\,dx + \int_1^\infty x^{n-1}e^{-x}\,dx \tag{1}$$

(*a*) If $n \geqq 1$, the first integral in (*1*) converges since the integrand is continuous in $0 \leqq x \leqq 1$.

If $0 < n < 1$, the first integral in (*1*) is an improper integral of the second kind at $x = 0$. Since $\lim_{x\to 0+} x^{1-n} \cdot x^{n-1}e^{-x} = 1$, the integral converges by Theorem 3(*i*), Page 312, with $p = 1 - n$ and $a = 0$.

Thus, the first integral converges for $n > 0$.

If $n > 0$, the second integral in (*1*) is an improper integral of the first kind. Since $\lim_{x\to\infty} x^2 \cdot x^{n-1}e^{-x} = 0$ (by L'Hospital's rule or otherwise), this integral converges by Theorem 1(*i*), Page 309, with $p = 2$.

Thus, the second integral also converges for $n > 0$, and so the given integral converges for $n > 0$.

(*b*) If $n \leqq 0$, the first integral of (*1*) diverges since $\lim_{x\to 0+} x \cdot x^{n-1}e^{-x} = \infty$ [Theorem 3(*ii*), Page 312].

If $n \leqq 0$, the second integral of (*1*) converges since $\lim_{x\to\infty} x \cdot x^{n-1}e^{-x} = 0$ [Theorem 1(*i*), Page 309].

Since the first integral in (*1*) diverges while the second integral converges, their sum also diverges, i.e., the given integral diverges if $n \leqq 0$.

Some interesting properties of the given integral, called the *gamma function*, are considered in Chapter 15.

UNIFORM CONVERGENCE OF IMPROPER INTEGRALS

12.19. (*a*) Evaluate $\phi(\alpha) = \int_0^\infty \alpha e^{-\alpha x}\,dx$ for $\alpha > 0$.

(*b*) Prove that the integral in (*a*) converges uniformly to 1 for $\alpha \geqq \alpha_1 > 0$.

(*c*) Explain why the integral does not converge uniformly to 1 for $\alpha > 0$.

(*a*) $\phi(\alpha) = \lim_{b\to\infty} \int_a^b \alpha e^{-\alpha x}\,dx = \lim_{b\to\infty} -e^{-\alpha x}\Big|_{x=0}^{b} = \lim_{b\to\infty} 1 - e^{-\alpha b} = 1 \quad \text{if } \alpha > 0$

.

Thus, the integral converges to 1 for all $\alpha > 0$.

(*b*) **Method 1**, using definition:

The integral converges uniformly to 1 in $\alpha \geqq \alpha_1 > 0$ if for each $\epsilon > 0$ we can find N, depending on ϵ but not on α, such that $\left|1 - \int_0^u \alpha e^{-\alpha x}\,dx\right| < \epsilon$ for all $u > N$.

Since $\left|1 - \int_0^u \alpha e^{-\alpha x}\,dx\right| = |1 - (1 - e^{-\alpha u})| = e^{-\alpha u} \leqq e^{-\alpha_1 u} < \epsilon$ for $u > \dfrac{1}{\alpha_1}\ln\dfrac{1}{\epsilon} = N$, the result follows.

Method 2, using the Weierstrass M test:

Since $\lim_{x\to\infty} x^2 \cdot \alpha e^{-\alpha x} = 0$ for $\alpha \geqq \alpha_1 > 0$, we can choose $|\alpha e^{-\alpha x}| < \dfrac{1}{x^2}$ for sufficiently large x, say $x \geqq x_0$. Taking $M(x) = \dfrac{1}{x^2}$ and noting that $\int_{x_0}^\infty \dfrac{dx}{x^2}$ converges, it follows that the given integral is uniformly convergent to 1 for $\alpha \geqq \alpha_1 > 0$.

(*c*) As $\alpha_1 \to 0$, the number N in the first method of (*b*) increases without limit, so that the integral cannot be uniformly convergent for $\alpha > 0$.

12.20. If $\phi(\alpha) = \int_0^\infty f(x, \alpha)\,dx$ is uniformly convergent for $\alpha_1 \leqq \alpha \leqq \alpha_2$, prove that $\phi(\alpha)$ is continuous in this interval.

Let $\phi(\alpha) = \int_a^u f(x, \alpha)\,dx + R(u, \alpha)$, where $R(u, \alpha) = \int_u^\infty f(x, \alpha)\,dx$.

Then $\phi(\alpha + h) = \int_a^u f(x, \alpha + h)\,dx + R(u, \alpha + h)$ and so

$$\phi(\alpha + h) - \phi(\alpha) = \int_a^u \{f(x, \alpha + h) - f(x, \alpha)\}\,dx + R(u, \alpha + h) - R(u, \alpha)$$

Thus

$$|\phi(\alpha + h) - \phi(\alpha)| \leqq \int_a^u |f(x, \alpha + h) - f(x, \alpha)| dx + |R(u, \alpha + h)| + |R(u, \alpha)| \tag{1}$$

Since the integral is uniformly convergent in $\alpha_1 \leqq \alpha \leqq \alpha_2$, we can, for each $\epsilon > 0$, find N independent of α such that for $u > N$,

$$|R(u, \alpha + h)| < \epsilon/3, \qquad |R(u, \alpha)| < \epsilon/3 \tag{2}$$

Since $f(x, \alpha)$ is continuous, we can find $\delta > 0$ corresponding to each $\epsilon > 0$ such that

$$\int_a^u |f(x, \alpha + h) - f(x, \alpha)|\,dx < \epsilon/3 \qquad \text{for } |h| < \delta \tag{3}$$

Using (*2*) and (*3*) in (*1*), we see that $|\phi(\alpha + h) - \phi(\alpha)| < \epsilon$ for $|h| < \delta$, so that $\phi(\alpha)$ is continuous.

Note that in this proof we assume that α and $\alpha + h$ are both in the interval $\alpha_1 \leqq \alpha \leqq \alpha_2$. Thus, if $\alpha = \alpha_1$, for example, $h > 0$ and right-hand continuity is assumed.

Also note the analogy of this proof with that for infinite series.

Other properties of uniformly convergent integrals can be proved similarly.

12.21. (*a*) Show that $\lim_{\alpha \to 0+} \int_0^\infty \alpha e^{-\alpha x}\,dx \neq \int_0^\infty \left(\lim_{\alpha \to 0+} \alpha e^{-\alpha x} \right) dx$. (*b*) Explain the result in (*a*).

(*a*) $\lim_{\alpha \to 0+} \int_0^\infty \alpha e^{-\alpha x}\,dx = \lim_{\alpha \to 0+} = 1$ by Problem 12.19(*a*).

$$\int_0^\infty \left(\lim_{\alpha \to 0+} \alpha e^{-\alpha x} \right) dx = \int_0^\infty 0\,dx = 0. \quad \text{Thus the required result follows.}$$

(*b*) Since $\phi(\alpha) = \int_0^\infty \alpha e^{-\alpha x}\,dx$ is not uniformly convergent for $\alpha \geqq 0$ (see Problem 12.19), there is no guarantee that $\phi(\alpha)$ will be continuous for $\alpha \geqq 0$. Thus $\lim_{\alpha \to 0+} \phi(\alpha)$ may not be equal to $\phi(0)$.

12.22. (*a*) Prove that $\int_0^\infty e^{-\alpha x} \cos rx\,dx = \dfrac{\alpha}{\alpha^2 + r^2}$ for $\alpha > 0$ and any real value of r.

(*b*) Prove that the integral in (*a*) converges uniformly and absolutely for $a \leqq \alpha \leqq b$, where $0 < a < b$ and any r.

(*a*) From integration formula 34, Page 96, we have

$$\lim_{M \to \infty} \int_0^M e^{-\alpha x} \cos rx\,dx = \lim_{M \to \infty} \left. \frac{e^{-\alpha x}(r \sin rx - \alpha \cos rx)}{\alpha^2 + r^2} \right|_0^M = \frac{\alpha}{\alpha^2 + r^2}$$

(*b*) This follows at once from the Weierstrass M test for integrals, by noting that $|e^{-\alpha x} \cos rx| \leqq e^{-\alpha x}$ and $\int_0^\infty e^{-\alpha x}\,dx$ converges.

EVALUATION OF DEFINITE INTEGRALS

12.23. Prove that $\int_0^{\pi/2} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2$.

The given integral converges [Problem 12.42(*f*)]. Letting $x = \pi/2 - y$,

$$I = \int_0^{\pi/2} \ln \sin x \, dx = \int_0^{\pi/2} \ln \cos y \, dy = \int_0^{\pi/2} \ln \cos x \, dx$$

Then

$$\begin{aligned} 2I &= \int_0^{\pi/2} (\ln \sin x + \ln \cos x) \, dx = \int_0^{\pi/2} \ln\left(\frac{\sin 2x}{2}\right) dx \\ &= \int_0^{\pi/2} \ln \sin 2x \, dx - \int_0^{\pi/2} \ln 2 \, dx = \int_0^{\pi/2} \ln \sin 2x \, dx - \frac{\pi}{2} \ln 2 \end{aligned} \tag{1}$$

Letting $2x = v$,

$$\begin{aligned} \int_0^{\pi/2} \ln \sin 2x \, dx &= \frac{1}{2}\int_0^{\pi} \ln \sin v \, dv = \frac{1}{2}\left\{\int_0^{\pi/2} \ln \sin v \, dv + \int_{\pi/2}^{\pi} \ln \sin v \, dv\right\} \\ &= \frac{1}{2}(I + I) = I \quad \text{(letting } v = \pi - u \text{ in the last integral)} \end{aligned}$$

Hence, (*1*) becomes $2I = I - \frac{\pi}{2} \ln 2$ or $I = -\frac{\pi}{2} \ln 2$.

12.24. Prove that $\int_0^{\pi} x \ln \sin x \, dx = -\frac{\pi^2}{2} \ln 2$.

Let $x = \pi - y$. Then, using the results in the preceding problem,

$$\begin{aligned} J = \int_0^{\pi} x \ln \sin x \, dx &= \int_0^{\pi} (\pi - u) \ln \sin u \, du = \int_0^{\pi} (\pi - x) \ln \sin x \, dx \\ &= \pi \int_0^{\pi} \ln \sin x \, dx - \int_0^{\pi} x \ln \sin x \, dx \\ &= -\pi^2 \ln 2 - J \end{aligned}$$

or $J = -\frac{\pi^2}{2} \ln 2$.

12.25. (*a*) Prove that $\phi(\alpha) = \int_0^{\infty} \frac{dx}{x^2 + \alpha}$ is uniformly convergent for $\alpha \geqq 1$.

(*b*) Show that $\phi(\alpha) = \frac{\pi}{2\sqrt{\alpha}}$. (*c*) Evaluate $\int_0^{\infty} \frac{dx}{(x^2 + 1)^2}$.

(*d*) Prove that $\int_0^{\infty} \frac{dx}{(x^2 + 1)^{n+1}} = \int_0^{\pi/2} \cos^{2n} \theta \, d\theta = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2}$.

(*a*) The result follows from the Weierestrass test, since $\frac{1}{x^2 + \alpha} \leqq \frac{1}{x^2 + 1}$ for $a \geqq 1$ and $\int_0^{\infty} \frac{dx}{x^2 + 1}$ converges.

(*b*) $\phi(\alpha) = \lim_{b \to \infty} \int_0^b \frac{dx}{x^2 + \alpha} = \lim_{b \to \infty} \frac{1}{\sqrt{\alpha}} \tan^{-1} \frac{x}{\sqrt{\alpha}} \Big|_0^b = \lim_{b \to \infty} \frac{1}{\sqrt{\alpha}} \tan^{-1} \frac{b}{\sqrt{\alpha}} = \frac{\pi}{2\sqrt{\alpha}}$.

(*c*) From (*b*), $\int_0^\infty \frac{dx}{x^2+\alpha} = \frac{\pi}{2\sqrt{\alpha}}$. Differentiating both sides with respect to α, we have

$$\int_0^\infty \frac{\partial}{\partial\alpha}\left(\frac{1}{x^2+\alpha}\right)dx = -\int_0^\infty \frac{dx}{(x^2+\alpha)^2} = -\frac{\pi}{4}\alpha^{-3/2}$$

the result being justified by Theorem 8, Page 314, since $\int_0^\infty \frac{dx}{(x^2+\alpha)^2}$ is uniformly convergent for $\alpha \geqq 1$ $\left(\text{because } \frac{1}{(x^2+\alpha)^2} \leqq \frac{1}{(x^2+1)^2} \text{ and } \int_0^\infty \frac{dx}{(x^2+1)^2} \text{ converges}\right)$.

Taking the limit as $\alpha \to 1+$, using Theorem 6, Page 314, we find $\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$.

(*d*) Differentiating both sides of $\int_0^\infty \frac{dx}{x^2+\alpha} = \frac{\pi}{2}\alpha^{-1/2}$ n times, we find

$$(-1)(-2)\cdots(-n)\int_0^\infty \frac{dx}{(x^2+\alpha)^{n+1}} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)\frac{\pi}{2}\alpha^{-(2n+1)/2}$$

where justification proceeds as in part (*c*). Letting $\alpha \to 1+$, we find

$$\int_0^\infty \frac{dx}{(x^2+1)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}\frac{\pi}{2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\frac{\pi}{2}$$

Substituting $x = \tan\theta$, the integral becomes $\int_0^{\pi/2} \cos^{2n}\theta\, d\theta$ and the required result is obtained.

12.26. Prove that $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x\sec rx}\,dx = \frac{1}{2}\ln\frac{b^2+r^2}{a^2+r^2}$ where $a, b > 0$.

From Problem 12.22 and Theorem 7, Page 314, we have

$$\int_{x=0}^\infty\left\{\int_{\alpha=a}^b e^{-\alpha x}\cos rx\,d\alpha\right\}dx = \int_{\alpha=a}^b\left\{\int_{x=0}^\infty e^{-\alpha x}\cos rx\,dx\right\}d\alpha$$

or

$$\int_{x=0}^\infty \left.\frac{e^{-\alpha x}\cos rx}{-x}\right|_{\alpha=a}^b dx = \int_{\alpha=a}^b \frac{\alpha}{\alpha^2+r^2}\,d\alpha$$

i.e.,

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x\sec rx}\,dx = \frac{1}{2}\ln\frac{b^2+r^2}{a^2+r^2}$$

12.27. Prove that $\int_0^\infty e^{-\alpha x}\frac{1-\cos x}{x^2}\,dx = \tan^{-1}\frac{1}{\alpha} - \frac{\alpha}{2}\ln(\alpha^2+1)$, $\alpha > 0$.

By Problem 12.22 and Theorem 7, Page 314, we have

$$\int_0^r\left\{\int_0^\infty e^{-\alpha x}\cos rx\,dx\right\}dr = \int_0^\infty\left\{\int_0^r e^{-\alpha x}\cos rx\,dr\right\}dx$$

or

$$\int_0^\infty e^{-\alpha x}\frac{\sin rx}{x}\,dx = \int_0^r \frac{a}{\alpha^2+r^2} = \tan^{-1}\frac{r}{\alpha}$$

Integrating again with respect to r from 0 to r yields

$$\int_0^\infty e^{-\alpha x}\frac{1-\cos rx}{x^2}\,dx = \int_0^r \tan^{-1}\frac{r}{\alpha}\,dr = r\tan^{-1}\frac{r}{\alpha} - \frac{\alpha}{2}\ln(\alpha^2+r^2)$$

using integration by parts. The required result follows on letting $r = 1$.

12.28. Prove that $\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}$.

Since $e^{-\alpha x}\frac{1-\cos x}{x^2} \leqq \frac{1-\cos x}{x^2}$ for $\alpha \geqq 0, x \geqq 0$ and $\int_0^\infty \frac{1-\cos x}{x^2}\,dx$ converges [see Problem 12.7(*b*)], it follows by the Weierstrass test that $\int_0^\infty e^{-\alpha x}\frac{1-\cos x}{x^2}\,dx$ is uniformly convergent and represents a continuous function of α for $\alpha \geqq 0$ (Theorem 6, Page 314). Then letting $\alpha \to 0+$, using Problem 12.27, we have

$$\lim_{\alpha\to 0+}\int_0^\infty e^{-\alpha x}\frac{1-\cos x}{x^2}\,dx = \int_0^\infty \frac{1-\cos x}{x^2}\,dx = \lim_{\alpha\to 0}\left\{\tan^{-1}\frac{1}{\alpha} - \frac{\alpha}{2}\ln(\alpha^2+1)\right\} = \frac{\pi}{2}$$

12.29. Prove that $\int_0^\infty \frac{\sin x}{x} = \int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$.

Integrating by parts, we have

$$\int_\epsilon^M \frac{1-\cos x}{x^2}\,dx = \left(-\frac{1}{x}\right)(1-\cos x)\Big|_\epsilon^M + \int_\epsilon^M \frac{\sin x}{x}\,dx = \frac{1-\cos\epsilon}{\epsilon} - \frac{1-\cos M}{M} + \int_\epsilon^M \frac{\sin x}{x}\,dx$$

Taking the limit as $\epsilon \to 0+$ and $M \to \infty$ shows that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \frac{1-\cos x}{x}\,dx = \frac{\pi}{2}$$

Since $\int_0^\infty \frac{1-\cos x}{x^2}\,dx = 2\int_0^\infty \frac{\sin^2(x/2)}{x^2}\,dx = \int_0^\infty \frac{\sin^2 u}{u^2}\,du$ on letting $u = x/2$, we also have $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$.

12.30. Prove that $\int_0^\infty \frac{\sin^3 x}{x}\,dx = \frac{\pi}{4}$.

$$\sin^3 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 = \frac{(e^{ix})^3 - 3(e^{ix})^2(e^{-ix}) + 3(e^{ix})(e^{-ix})^2 - (e^{-ix})^3}{(2i)^3}$$
$$= -\frac{1}{4}\left(\frac{e^{-3ix}-e^{-3ix}}{2i}\right) + \frac{3}{4}\left(\frac{e^{ix}-e^{-ix}}{2i}\right) = -\frac{1}{4}\sin 3x + \frac{3}{4}\sin x$$

Then

$$\int_0^\infty \frac{\sin^3 x}{x}\,dx = \frac{3}{4}\int_0^\infty \frac{\sin x}{x}\,dx - \frac{1}{4}\int_0^\infty \frac{\sin 3x}{x}\,dx = \frac{3}{4}\int_0^\infty \frac{\sin x}{x}\,dx - \frac{1}{4}\int_0^\infty \frac{\sin u}{u}\,du$$
$$= \frac{3}{4}\left(\frac{\pi}{2}\right) - \frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

MISCELLANEOUS PROBLEMS

12.31. Prove that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$.

By Problem 12.6, the integral converges. Let $I_M = \int_0^M e^{-x^2}\,dx = \int_0^M e^{-y^2}\,dy$ and let $\lim_{M\to\infty} I_M = I$, the required value of the integral. Then

$$
\begin{aligned}
I_M^2 &= \left(\int_0^M e^{-x^2}\,dx\right)\left(\int_0^M e^{-y^2}\,dy\right) \\
&= \int_0^M \int_0^M e^{-(x^2+y^2)}\,dx\,dy \\
&= \iint_{\mathscr{R}_M} e^{-(x^2+y^2)}\,dx\,dy
\end{aligned}
$$

where $\mathscr{R}_M$ is the square $OACE$ of side M (see Fig. 12-3). Since integrand is positive, we have

$$\iint_{\mathscr{R}_1} e^{-(x^2+y^2)}\,dx\,dy \leqq I_M^2 \leqq \iint_{\mathscr{R}_2} e^{-(x^2+y^2)}\,dx\,dy \tag{1}$$

where $\mathscr{R}_1$ and $\mathscr{R}_2$ are the regions in the first quadrant bounded by the circles having radii M and $M\sqrt{2}$, respectively.

Using polar coordinates, we have from (*1*),

$$\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{M} e^{-\rho^2}\rho\,d\rho\,d\phi \leqq I_M^2 \leqq \int_{\phi=0}^{\pi/2}\int_{\rho=0}^{M\sqrt{2}} e^{-\rho^2}\rho\,d\rho\,d\phi \tag{2}$$

or

$$\frac{\pi}{4}(1-e^{-M^2}) \leqq I_M^2 \leqq \frac{\pi}{4}(1-e^{-2M^2}) \tag{3}$$

Then taking the limit as $M \to \infty$ in (*3*), we find $\lim_{M\to\infty} I_M^2 = I^2 = \pi/4$ and $I = \sqrt{\pi}/2$.

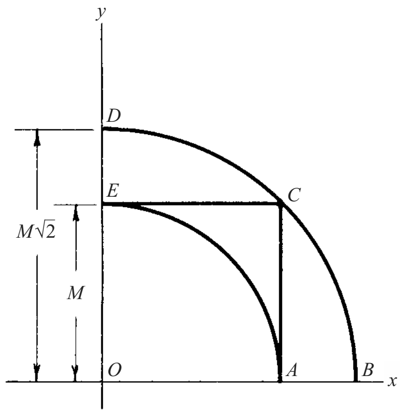

Fig. 12-3

12.32. Evaluate $\int_0^\infty e^{-x^2}\cos\alpha x\,dx$.

Let $I(\alpha) = \int_0^\infty e^{-x^2}\cos\alpha x\,dx$. Then using integration by parts and appropriate limiting procedures,

$$\frac{dI}{d\alpha} = \int_0^\infty -xe^{-x^2}\sin\alpha x\,dx = \frac{1}{2}e^{-x^2}\sin\alpha x\Big|_0^\infty - \frac{1}{2}\alpha\int_0^\infty e^{-x^2}\cos\alpha x\,dx = -\frac{\alpha}{2}I$$

The differentiation under the integral sign is justified by Theorem 8, Page 314, and the fact that $\int_0^\infty xe^{-x^2}\sin\alpha x\,dx$ is uniformly convergent for all α (since by the Weierstrass test, $|xe^{-x^2}\sin\alpha x| \leqq xe^{-x^2}$ and $\int_0^\infty xe^{-x^2}\,dx$ converges).

From Problem 12.31 and the uniform convergence, and thus continuity, of the given integral (since $|e^{-x^2}\cos\alpha x| \leqq e^{-x^2}$ and $\int_0^\infty e^{-x^2}dx$ converges, so that that Weierstrass test applies), we have $I(0) = \lim_{\alpha\to 0} I(\alpha) = \frac{1}{2}\sqrt{\pi}$.

Solving $\frac{dI}{d\alpha} = -\frac{\alpha}{2}I$ subject to $I(0) = \frac{\sqrt{\pi}}{2}$, we find $I(\alpha) = \frac{\sqrt{\pi}}{2}e^{-\alpha^2/4}$.

12.33. (*a*) Prove that $I(\alpha) = \int_0^\infty e^{-(x-\alpha/x)^2}\,dx = \frac{\sqrt{\pi}}{2}$. (*b*) Evaluate $\int_0^\infty e^{-(x^2+x^{-2})}\,dx$.

(*a*) We have $I'(\alpha) = 2\int_0^\infty e^{-(x-\alpha/x)^2}(1-\alpha/x^2)\,dx$.

The differentiation is proved valid by observing that the integrand remains bounded as $x \to 0+$ and that for sufficiently large x,

$$e^{-(x-\alpha/x)^2}(1-\alpha/x^2) = e^{-x^2+2\alpha-\alpha^2/x^2}(1-\alpha/x^2) \leqq e^{2\alpha}e^{-x^2}$$

so that $I'(\alpha)$ converges uniformly for $\alpha \geqq 0$ by the Weierstrass test, since $\int_0^\infty e^{-x^2}\,dx$ converges. Now

$$I'(\alpha) = 2\int_0^\infty e^{-(x-\alpha/x)^2}\,dx - 2\alpha\int_0^\infty \frac{e^{-(x-\alpha/x)^2}}{x^2}\,dx = 0$$

as seen by letting $\alpha/x = y$ in the second integral. Thus $I(\alpha) = c$, a constant. To determine c, let $\alpha \to 0+$ in the required integral and use Problem 12.31 to obtain $c = \sqrt{\pi}/2$.

(*b*) From (*a*), $\int_0^\infty e^{-(x-\alpha/x)^2}\,dx = \int_0^\infty e^{-(x^2-2\alpha+\alpha^2x^{-2})}\,dx = e^{2\alpha}\int_0^\infty e^{-(x^2+\alpha^2x^{-2})}\,dx = \frac{\sqrt{\pi}}{2}$.

Then $\int_0^\infty e^{-(x^2+\alpha^2x^{-2})}\,dx = \frac{\sqrt{\pi}}{2}e^{-2\alpha}$. Putting $\alpha = 1$, $\int_0^\infty e^{-(x^2+x^{-2})}\,dx = \frac{\sqrt{\pi}}{2}e^{-2}$.

12.34. Verify the results: (*a*) $\mathscr{L}\{e^{ax}\} = \frac{1}{s-a}$, $s > a$; (*b*) $\mathscr{L}\{\cos ax\} = \frac{s}{s^2+a^2}$, $s > 0$.

(*a*)
$$\begin{aligned}\mathscr{L}\{e^{ax}\} &= \int_0^\infty e^{-sx}e^{ax}\,dx = \lim_{M\to\infty}\int_0^M e^{-(s-a)x}\,dx \\ &= \lim_{M\to\infty}\frac{1-e^{-(s-a)M}}{s-a} = \frac{1}{s-a} \qquad \text{if } s > a\end{aligned}$$

(*b*) $\mathscr{L}\{\cos ax\} = \int_0^\infty e^{-sx}\cos ax\,dx = \frac{s}{s^2+a^2}$ by Problem 12.22 with $\alpha = s$, $r = a$.

Another method, using complex numbers.

From part (*a*), $\mathscr{L}\{e^{ax}\} = \frac{1}{s-a}$. Replace a by ai. Then

$$\begin{aligned}\mathscr{L}\{e^{aix}\} &= \mathscr{L}\{\cos ax + i\sin ax\} = \mathscr{L}\{\cos ax\} + i\mathscr{L}\{\sin ax\} \\ &= \frac{1}{s-ai} = \frac{s+ai}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}\end{aligned}$$

Equating real and imaginary parts: $\mathscr{L}\{\cos ax\} = \frac{s}{s^2+a^2}$, $\mathscr{L}\{\sin ax\} = \frac{a}{s^2+a^2}$.

The above *formal* method can be justified using methods of Chapter 16.

12.35. Prove that (*a*) $\mathscr{L}\{Y'(x)\} = s\mathscr{L}\{Y(x)\} - Y(0)$, (*b*) $\mathscr{L}\{Y''(x)\} = s^2\mathscr{L}\{Y(x)\} - sY(0) - Y'(0)$ under suitable conditions on $Y(x)$.

(*a*) By definition (and with the aid of integration by parts)

$$\begin{aligned}\mathscr{L}\{Y'(x)\} &= \int_0^\infty e^{-sx}Y'(x)\,dx = \lim_{M\to 0}\int_0^M e^{-sx}Y'(x)\,dx \\ &= \lim_{M\to\infty}\left\{e^{-sx}Y(x)\Big|_0^M + s\int_0^M e^{-sx}Y(x)\,dx\right\} \\ &= s\int_0^\infty e^{-sx}Y(x)\,dx - Y(0) = s\mathscr{L}\{Y(x)\} - Y(0)\end{aligned}$$

assuming that s is such that $\lim_{M\to\infty} e^{-sM}Y(M) = 0$.

(*b*) Let $U(x) = Y'(x)$. Then by part (*a*), $\mathscr{L}\{U'(x)\} = s\mathscr{L}\{U(x)\} - U(0)$. Thus

$$\begin{aligned}\mathscr{L}\{Y''(x)\} &= s\mathscr{L}\{Y'(x)\} - Y'(0) = s[s\mathscr{L}\{Y(x)\} - Y(0)] - Y'(0) \\ &= s^2\mathscr{L}\{Y(x)\} - sY(0) - Y'(0)\end{aligned}$$

12.36. Solve the differential equation $Y''(x) + Y(x) = x$, $Y(0) = 0$, $Y'(0) = 2$.

Take the Laplace transform of both sides of the given differential equation. Then by Problem 12.35,

$$\mathscr{L}\{Y''(x) + Y(x)\} = \mathscr{L}\{x\}, \qquad \mathscr{L}\{Y''(x)\} + \mathscr{L}\{Y(x)\} = 1/s^2$$

and so
$$s^2\mathscr{L}\{Y(x)\} - sY(0) - Y'(0) + \mathscr{L}\{Y(x)\} = 1/s^2$$

Solving for $\mathscr{L}\{Y(x)\}$ using the given conditions, we find

$$\mathscr{L}\{Y(x)\} = \frac{2s^2}{s^2(s^2+1)} = \frac{1}{s^2} + \frac{1}{s^2+1} \tag{1}$$

by methods of partial fractions.

Since $\dfrac{1}{s^2} = \mathscr{L}\{x\}$ and $\dfrac{1}{s^2+1} = \mathscr{L}\{\sin x\}$, it follows that $\dfrac{1}{s^2} + \dfrac{1}{s^2+1} = \mathscr{L}\{x + \sin x\}$.

Hence, from (*1*), $\mathscr{L}\{Y(x)\} = \mathscr{L}\{x + \sin x\}$, from which we can conclude that $Y(x) = x + \sin x$ which is, in fact, found to be a solution.

Another method:

If $\mathscr{L}\{F(x)\} = f(s)$, we call $f(s)$ the *inverse* Laplace transform of $F(x)$ and write $f(s) = \mathscr{L}^{-1}\{F(x)\}$. By Problem 12.78, $\mathscr{L}^{-1}\{f(s) + g(s)\} = \mathscr{L}^{-1}\{f(s)\} + \mathscr{L}^{-1}\{g(s)\}$. Then from (*1*),

$$Y(x) = \mathscr{L}^{-1}\left\{\frac{1}{s^2} + \frac{1}{s^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = x + \sin x$$

Inverse Laplace transforms can be read from the table on Page 315.

Supplementary Problems

IMPROPER INTEGRALS OF THE FIRST KIND

12.37. Test for convergence:

(*a*) $\displaystyle\int_0^\infty \frac{x^2+1}{x^4+1}\,dx$ (*d*) $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^4+4}$ (*g*) $\displaystyle\int_{-\infty}^\infty \frac{x^2\,dx}{(x^2+x+1)^{5/2}}$

(*b*) $\displaystyle\int_2^\infty \frac{x\,dx}{\sqrt{x^3-1}}$ (*e*) $\displaystyle\int_{-\infty}^\infty \frac{2+\sin x}{x^2+1}\,dx$ (*h*) $\displaystyle\int_1^\infty \frac{\ln x\,dx}{x+e^{-x}}$

(*c*) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{3x+2}}$ (*f*) $\displaystyle\int_2^\infty \frac{x\,dx}{(\ln x)^3}$ (*i*) $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

Ans. (*a*) conv., (*b*) div., (*c*) conv., (*d*) conv., (*e*) conv., (*f*) div., (*g*) conv., (*h*) div., (*i*) conv.

12.38. Prove that $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+2ax+b^2} = \frac{\pi}{\sqrt{b^2-a^2}}$ if $b > |a|$.

12.39. Test for convergence: (*a*) $\displaystyle\int_1^\infty e^{-x}\ln x\,dx$, (*b*) $\displaystyle\int_0^\infty e^{-x}\ln(1+e^x)\,dx$, (*c*) $\displaystyle\int_0^\infty e^{-x}\cosh x^2\,dx$.

Ans. (*a*) conv., (*b*) conv., (*c*) div.

12.40. Test for convergence, indicating absolute or conditional convergence where possible: (*a*) $\int_0^\infty \frac{\sin 2x}{x^3+1}\,dx$;

(*b*) $\int_{-\infty}^\infty e^{-ax^2}\cos bx\,dx$, where a, b are positive constants; (*c*) $\int_0^\infty \frac{\cos x}{\sqrt{x^2+1}}\,dx$; (*d*) $\int_0^\infty \frac{x\sin x}{\sqrt{x^2+a^2}}\,dx$;

(*e*) $\int_0^\infty \frac{\cos x}{\cosh x}\,dx$.

Ans. (*a*) abs. conv., (*b*) abs. conv., (*c*) cond. conv., (*d*) div., (*e*) abs. conv.

12.41. Prove the quotient tests (*b*) and (*c*) on Page 309.

IMPROPER INTEGRALS OF THE SECOND KIND

12.42. Test for convergence:

(*a*) $\int_0^1 \frac{dx}{(x+1)\sqrt{1-x^2}}$ (*d*) $\int_1^2 \frac{\ln x}{\sqrt[3]{8-x^3}}\,dx$ (*g*) $\int_0^3 \frac{x^2}{(3-x)^2}\,dx$ (*j*) $\int_0^1 \frac{dx}{x^x}$

(*b*) $\int_0^1 \frac{\cos x}{x^2}\,dx$ (*e*) $\int_0^1 \frac{dx}{\sqrt{\ln(1/x)}}$ (*h*) $\int_0^{\pi/2} \frac{e^{-x}\cos x}{x}\,dx$

(*c*) $\int_{-1}^1 \frac{e^{\tan^{-1} x}}{x}\,dx$ (*f*) $\int_0^{\pi/2} \ln\sin x\,dx$ (*i*) $\int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$, $|k| < 1$

Ans. (*a*) conv., (*b*) div., (*c*) div., (*d*) conv., (*e*) conv., (*f*) conv., (*g*) div., (*h*) div., (*i*) conv., (*j*) conv.

12.43. (*a*) Prove that $\int_0^5 \frac{dx}{4-x}$ diverges in the usual sense but converges in the Cauchy principal value senses. (*b*) Find the Cauchy principal value of the integral in (*a*) and give a geometric interpretation.
Ans. (*b*) $\ln 4$

12.44. Test for convergence, indicating absolute or conditional convergence where possible:

(*a*) $\int_0^1 \cos\left(\frac{1}{x}\right)dx$, (*b*) $\int_0^1 \frac{1}{x}\cos\left(\frac{1}{x}\right)dx$, (*c*) $\int_0^1 \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$.

Ans. (*a*) abs. conv., (*b*) cond. conv., (*c*) div.

12.45. Prove that $\int_0^{4\pi} \left(3x^2\sin\frac{1}{x} - x\cos\frac{1}{x}\right)dx = \frac{32\sqrt{2}}{\pi^3}$.

IMPROPER INTEGRALS OF THE THIRD KIND

12.46. Test for convergence: (*a*) $\int_0^\infty e^{-x}\ln x\,dx$, (*b*) $\int_0^\infty \frac{e^{-x}\,dx}{\sqrt{x\ln(x+1)}}$, (*c*) $\int_0^\infty \frac{e^{-x}\,dx}{\sqrt[3]{x}\,(3+2\sin x)}$.

Ans. (*a*) conv., (*b*) div., (*c*) conv.

12.47. Test for convergence: (*a*) $\int_0^\infty \frac{dx}{\sqrt[3]{x^4+x^2}}$; (*b*) $\int_0^\infty \frac{e^x\,dx}{\sqrt{\sinh(ax)}}$, $a > 0$.

Ans. (*a*) conv., (*b*) conv. if $a > 2$, div. if $0 < a \leqq 2$.

12.48. Prove that $\int_0^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\,dx$ converges if $0 \leqq |a| < \pi$ and diverges if $|a| \leqq \pi$.

12.49. Test for convergence, indicating absolute or conditional convergence where possible:

(a) $\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx$, (b) $\int_0^\infty \frac{\sin\sqrt{x}}{\sinh\sqrt{x}}\,dx$. *Ans.* (a) cond. conv., (b) abs. conv.

UNIFORM CONVERGENCE OF IMPROPER INTEGRALS

12.50. (a) Prove that $\phi(\alpha) = \int_0^\infty \frac{\cos\alpha x}{1+x^2}\,dx$ is uniformly convergent for all α.

(b) Prove that $\phi(\alpha)$ is continuous for all α. (c) Find $\lim_{\alpha\to 0}\phi(\alpha)$. *Ans.* (c) $\pi/2$.

12.51. Let $\phi(\alpha) = \int_0^\infty F(x,\alpha)\,dx$, where $F(x,\alpha) = \alpha^2 x e^{-\alpha x^2}$. (a) Show that $\phi(\alpha)$ is not continuous at $\alpha = 0$, i.e., $\lim_{\alpha\to 0}\int_0^\infty F(x,\alpha)\,dx \neq \int_0^\infty \lim_{\alpha\to 0} F(x,\alpha)\,dx$. (b) Explain the result in (a).

12.52. Work Problem 12.51 if $F(x,\alpha) = \alpha^2 x e^{-\alpha x}$.

12.53. If $F(x)$ is bounded and continuous for $-\infty < x < \infty$ and

$$V(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yF(\lambda)\,d\lambda}{y^2+(\lambda-x)^2}$$

prove that $\lim_{y\to 0} V(x,y) = F(x)$.

12.54. Prove (a) Theorem 7 and (b) Theorem 8 on Page 314.

12.55. Prove the Weierstrass M test for uniform convergence of integrals.

12.56. Prove that if $\int_0^\infty F(x)\,dx$ converges, then $\int_0^\infty e^{-\alpha x}F(x)\,dx$ converges uniformly for $\alpha \geqq 0$.

12.57. Prove that (a) $\phi(a) = \int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx$ converges uniformly for $a \geqq 0$, (b) $\phi(a) = \frac{\pi}{2} - \tan^{-1} a$, (c) $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ (compare Problems 12.27 through 12.29).

12.58. State the definition of uniform convergence for improper integrals of the second kind.

12.59. State and prove a theorem corresponding to Theorem 8, Page 314, if a is a differentiable function of α.

EVALUATION OF DEFINITE INTEGRALS

Establish each of the following results. Justify all steps in each case.

12.60. $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln(b/a)$, $a,b > 0$

12.61. $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x\csc rx}\,dx = \tan^{-1}(b/r) - \tan^{-1}(a/r)$, $a,b,r > 0$

12.62. $\int_0^\infty \frac{\sin rx}{x(1+x^2)}\,dx = \frac{\pi}{2}(1-e^{-r})$, $r \geqq 0$

12.63. $\int_0^\infty \frac{1-\cos rx}{x^2}\,dx = \frac{\pi}{2}|r|$

12.64. $\int_0^\infty \frac{x\sin rx}{a^2+x^2}\,dx = \frac{\pi}{2}e^{-ar}$, $a,r \geqq 0$

12.65. (*a*) Prove that $\int_0^\infty e^{-\alpha x}\left(\frac{\cos ax - \cos bx}{x}\right)dx = \frac{1}{2}\ln\left(\frac{\alpha^2+b^2}{\alpha^2+a^2}\right), \quad \alpha \geqq 0.$

(*b*) Use (*a*) to prove that $\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \ln\left(\frac{b}{a}\right)$.

[The results of (*b*) and Problem 12.60 are special cases of *Frullani's integral*, $\int_0^\infty \frac{F(ax)-F(bx)}{x}\,dx = F(0)\ln\left(\frac{b}{a}\right)$, where $F(t)$ is continuous for $t > 0$, $F'(0)$ exists and $\int_1^\infty \frac{F(t)}{t}\,dt$ converges.]

12.66. Given $\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\pi/\alpha}$, $\alpha > 0$. Prove that for $p = 1, 2, 3, \ldots$,

$$\int_0^\infty x^{2p}e^{-\alpha x^2}\,dx = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{(2p-1)}{2}\,\frac{\sqrt{\pi}}{2\alpha^{(2p+1)/2}}$$

12.67. If $a > 0, b > 0$, prove that $\int_0^\infty (e^{-a/x^2} - e^{-b/x^2})\,dx = \sqrt{\pi b} - \sqrt{\pi a}$.

12.68. Prove that $\int_0^\infty \frac{\tan^{-1}(x/a) - \tan^{-1}(x/b)}{x}\,dx = \frac{\pi}{2}\ln\left(\frac{b}{a}\right)$ where $a > 0, b > 0$.

12.69. Prove that $\int_{-\infty}^\infty \frac{dx}{(x^2+x+1)^3} = \frac{4\pi}{3\sqrt{3}}$. [Hint: Use Problem 12.38.]

MISCELLANEOUS PROBLEMS

12.70. Prove that $\int_0^\infty \left\{\frac{\ln(1+x)}{x}\right\}^2 dx$ converges.

12.71. Prove that $\int_0^\infty \frac{dx}{1+x^3\sin^2 x}$ converges. $\left[\text{Hint: Consider } \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{dx}{1+x^3\sin^2 x} \text{ and use the fact that } \int_{n\pi}^{(n+1)\pi} \frac{dx}{1+x^3\sin^2 x} \leqq \int_{n\pi}^{(n+1)\pi} \frac{dx}{1+(n\pi)^3\sin^2 x}.\right]$

12.72. Prove that $\int_0^\infty \frac{x\,dx}{1+x^3\sin^2 x}$ diverges.

12.73. (*a*) Prove that $\int_0^\infty \frac{\ln(1+\alpha^2x^2)}{1+x^2}\,dx = \pi\ln(1+\alpha), \quad \alpha \geqq 0.$

(*b*) Use (*a*) to show that $\int_0^{\pi/2} \ln\sin\theta\,d\theta = -\frac{\pi}{2}\ln 2.$

12.74. Prove that $\int_0^\infty \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}.$

12.75. Evaluate (*a*) $\mathscr{L}\{1/\sqrt{x}\}$, (*b*) $\mathscr{L}\{\cosh ax\}$, (*c*) $\mathscr{L}\{(\sin x)/x\}$.

Ans. (*a*) $\sqrt{\pi/s}$, $s > 0$ (*b*) $\frac{s}{s^2-a^2}$, $s > |a|$ (*c*) $\tan^{-1}\left(\frac{1}{s}\right)$, $s > 0$.

12.76. (*a*) If $\mathscr{L}\{F(x)\} = f(s)$, prove that $\mathscr{L}\{e^{ax}F(x)\} = f(s-a)$, (*b*) Evaluate $\mathscr{L}\{e^{ax}\sin bx\}$.

Ans. (*b*) $\frac{b}{(s-a)^2+b^2}$, $s > a$

12.77. (*a*) If $\mathscr{L}\{F(x)\} = f(s)$, prove that $\mathscr{L}\{x^n F(x)\} = (-1)^n f^{(n)}(s)$, giving suitable restrictions on $F(x)$.

(*b*) Evaluate $\mathscr{L}\{x \cos x\}$. *Ans.* (*b*) $\dfrac{s^2 - 1}{(s^2+1)^2}$, $s > 0$

12.78. Prove that $\mathscr{L}^{-1}\{f(s) + g(s)\} = \mathscr{L}^{-1}\{f(s)\} + \mathscr{L}^{-1}\{g(s)\}$, stating any restrictions.

12.79. Solve using Laplace transforms, the following differential equations subject to the given conditions.
(*a*) $Y''(x) + 3Y'(x) + 2Y(x) = 0$; $Y(0) = 3$, $Y'(0) = 0$
(*b*) $Y''(x) - Y'(x) = x$; $Y(0) = 2$, $Y'(0) = -3$
(*c*) $Y''(x) + 2Y'(x) + 2Y(x) = 4$; $Y(0) = 0$, $Y'(0) = 0$
Ans. (*a*) $Y(x) = 6e^{-x} - 3e^{-2x}$, (*b*) $Y(x) = 4 - 2e^x - \frac{1}{2}x^2 - x$, (*c*) $Y(x) = 1 - e^{-x}(\sin x + \cos x)$

12.80. Prove that $\mathscr{L}\{F(x)\}$ exists if $F(x)$ is piecewise continuous in every finite interval $[0, b]$ where $b > 0$ and if $F(x)$ is of *exponential order* as $x \to \infty$, i.e., there exists a constant α such that $|e^{-\alpha x}F(x)| < P$ (a constant) for all $x > b$.

12.81. If $f(s) = \mathscr{L}\{F(x)\}$ and $g(s) = \mathscr{L}\{G(x)\}$, prove that $f(s)g(s) = \mathscr{L}\{H(x)\}$ where

$$H(x) = \int_0^x F(u)G(x-u)\,du$$

is called the *convolution* of F and G, written F^*G.

$$\left[\text{Hint: Write } f(s)g(s) = \lim_{M\to\infty}\left\{\int_0^M e^{-su}F(u)\,du\right\}\left\{\int_0^M e^{-sv}G(v)\,dv\right\}\right.$$
$$\left. = \lim_{M\to\infty}\int_0^M\int_0^M e^{-s(u+v)}F(u)\,G(v)\,du\,dv \text{ and then let } u+v=t.\right]$$

12.82. (*a*) Find $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right|$. (*b*) Solve $Y''(x) + Y(x) = R(x)$, $Y(0) = Y'(0) = 0$.

(*c*) Solve the integral equation $Y(x) = x + \displaystyle\int_0^x Y(u)\sin(x-u)\,du$. [Hint: Use Problem 12.81.]

Ans. (*a*) $\frac{1}{2}(\sin x - x\cos x)$, (*b*) $Y(x) = \displaystyle\int_0^x R(u)\sin(x-u)\,du$, (*c*) $Y(x) = x + x^3/6$

12.83. Let $f(x), g(x)$, and $g'(x)$ be continuous in every finite interval $a \leqq x \leqq b$ and suppose that $g'(x) \leqq 0$. Suppose also that $h(x) = \displaystyle\int_a^x f(x)\,dx$ is bounded for all $x \geqq a$ and $\displaystyle\lim_{x\to 0} g(x) = 0$.

(*a*) Prove that $\displaystyle\int_a^\infty f(x)\,g(x)\,dx = -\int_a^\infty g'(x)\,h(x)\,dx$.

(*b*) Prove that the integral on the right, and hence the integral on the left, is convergent. The result is that under the give conditions on $f(x)$ and $g(x)$, $\displaystyle\int_a^\infty f(x)\,g(x)\,dx$ converges and is sometimes called *Abel's integral test*.

$\Bigg[$Hint: For (*a*), consider $\displaystyle\lim_{b\to\infty}\int_a^b f(x)\,g(x)\,dx$ after replacing $f(x)$ by $h'(x)$ and integrating by parts. For (*b*), first prove that if $|h(x)| < H$ (a constant), then $\left|\displaystyle\int_a^b g'(x)\,h(x)\,dx\right| \leqq H\{g(a) - g(b)\}$; and then let $b \to \infty$.$\Bigg]$

12.84. Use Problem 12.83 to prove that (*a*) $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ and (*b*) $\displaystyle\int_0^\infty \sin x^p\,dx, p > 1$, converge.

12.85. (*a*) Given that $\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ [see Problems 15.27 and 15.68(*a*), Chapter 15], evaluate

$$\int_0^\infty \int_0^\infty \sin(x^2 + y^2)\,dx\,dy$$

(*b*) Explain why the method of Problem 12.31 cannot be used to evaluate the multiple integral in (*a*).

Ans. $\pi/4$