CHAPTER 4

Derivatives

THE CONCEPT AND DEFINITION OF A DERIVATIVE

Concepts that shape the course of mathematics are few and far between. The derivative, the fundamental element of the differential calculus, is such a concept. That branch of mathematics called analysis, of which advanced calculus is a part, is the end result. There were two problems that led to the discovery of the derivative. The older one of defining and representing the tangent line to a curve at one of its points had concerned early Greek philosophers. The other problem of representing the instantaneous velocity of an object whose motion was not constant was much more a problem of the seventeenth century. At the end of that century, these problems and their relationship were resolved. As is usually the case, many mathematicians contributed, but it was Isaac Newton and Gottfried Wilhelm Leibniz who independently put together organized bodies of thought upon which others could build.

The tangent problem provides a visual interpretation of the derivative and can be brought to mind no matter what the complexity of a particular application. It leads to the definition of the derivative as the limit of a difference quotient in the following way. (See Fig. 4-1.)

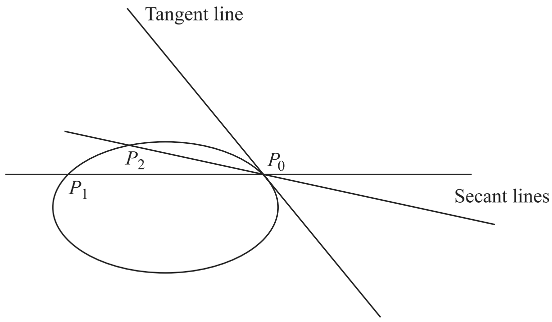

Fig. 4-1

Let $P_o(x_0)$ be a point on the graph of $y = f(x)$. Let $P(x)$ be a nearby point on this same graph of the function f. Then the line through these two points is called a *secant line*. Its slope, m_s, is the difference quotient

$$m_s = \frac{f(x) - f(x_0)}{x - x_0} = \frac{\Delta y}{\Delta x}$$

where Δx and Δy are called the increments in x and y, respectively. Also this slope may be written

$$m_s = \frac{f(x_0 + h) - f(x_0)}{h}$$

where $h = x - x_0 = \Delta x$. See Fig. 4-2.

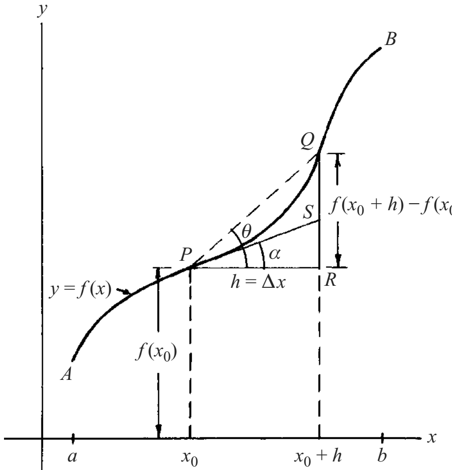

Fig. 4-2

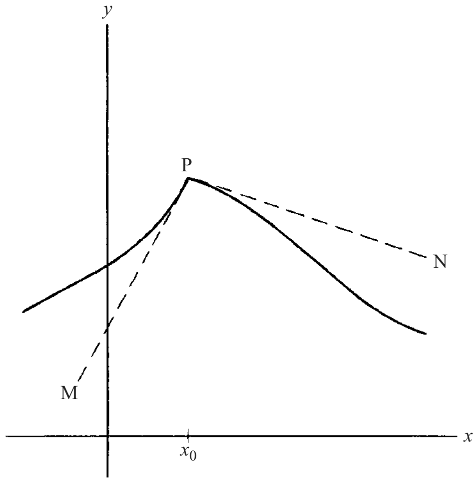

Fig. 4-3

We can imagine a sequence of lines formed as $h \to 0$. It is the limiting line of this sequence that is the natural one to be the tangent line to the graph at P_0.

To make this mode of reasoning precise, the limit (when it exists), is formed as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

As indicated, this limit is given the name $f'(x_0)$. It is called the *derivative* of the function f at its domain value x_0. If this limit can be formed at each point of a subdomain of the domain of f, then f is said to be *differentiable* on that subdomain and a new function f' has been constructed.

This limit concept was not understood until the middle of the nineteenth century. A simple example illustrates the conceptual problem that faced mathematicians from 1700 until that time. Let the graph of f be the parabola $y = x^2$, then a little algebraic manipulation yields

$$m_s = \frac{2x_0 h + h^2}{h} = 2x_0 + h$$

Newton, Leibniz, and their contemporaries simply let $h = 0$ and said that $2x_0$ was the slope of the tangent line at P_0. However, this raises the ghost of a $\frac{0}{0}$ form in the middle term. True understanding of the calculus is in the comprehension of how the introduction of something new (the derivative, i.e., the limit of a difference quotient) resolves this dilemma.

Note 1: The creation of new functions from difference quotients is not limited to f'. If, starting with f', the limit of the difference quotient exists, then f'' may be constructed and so on and so on.

Note 2: Since the continuity of a function is such a strong property, one might think that differentiability followed. This is not necessarily true, as is illustrated in Fig. 4-3.

The following theorem puts the matter in proper perspective:

Theorem: If f is differentiable at a domain value, then it is continuous at that value.

As indicated above, the converse of this theorem is not true.

RIGHT- AND LEFT-HAND DERIVATIVES

The status of the derivative at end points of the domain of f, and in other special circumstances, is clarified by the following definitions.

The *right-hand derivative* of $f(x)$ at $x = x_0$ is defined as

$$f'_+(x_0) = \lim_{h \to 0+} \frac{f(x_0 + h) - f(x_0)}{h} \tag{3}$$

if this limit exists. Note that in this case $h(= \Delta x)$ is restricted only to positive values as it approaches zero.

Similarly, the *left-hand derivative* of $f(x)$ at $x = x_0$ is defined as

$$f'_-(x_0) = \lim_{h \to 0-} \frac{f(x_0 + h) - f(x_0)}{h} \tag{4}$$

if this limit exists. In this case h is restricted to negative values as it approaches zero.

A function f has a derivative at $x = x_0$ if and only if $f'_+(x_0) = f'_-(x_0)$.

DIFFERENTIABILITY IN AN INTERVAL

If a function has a derivative at all points of an interval, it is said to be *differentiable in the interval*. In particular if f is defined in the closed interval $a \leqq x \leqq b$, i.e. $[a, b]$, then f is differentiable in the interval if and only if $f'(x_0)$ exists for each x_0 such that $a < x_0 < b$ and if $f'_+(a)$ and $f'_-(b)$ both exist.

If a function has a continuous derivative, it is sometimes called *continuously differentiable*.

PIECEWISE DIFFERENTIABILITY

A function is called *piecewise differentiable* or *piecewise smooth* in an interval $a \leqq x \leqq b$ if $f'(x)$ is piecewise continuous. An example of a piecewise continuous function is shown graphically on Page 48.

An equation for the tangent line to the curve $y = f(x)$ at the point where $x = x_0$ is given by

$$y - f(x_0) = f'(x_0)(x - x_0) \tag{7}$$

The fact that a function can be continuous at a point and yet not be differentiable there is shown graphically in Fig. 4-3. In this case there are two tangent lines at P represented by PM and PN. The slopes of these tangent lines are $f'_-(x_0)$ and $f'_+(x_0)$ respectively.

DIFFERENTIALS

Let $\Delta x = dx$ be an increment given to x. Then

$$\Delta y = f(x + \Delta x) - f(x) \tag{8}$$

is called the *increment* in $y = f(x)$. If $f(x)$ is continuous and has a continuous first derivative in an interval, then

$$\Delta y = f'(x)\Delta x + \epsilon \Delta x = f'(x)dx + dx \tag{9}$$

where $\epsilon \to 0$ as $\Delta x \to 0$. The expression

$$\mathrm{d}y = f'(x)\mathrm{d}x \tag{10}$$

is called the *differential of y or f(x)* or the *principal part of* Δy. Note that $\Delta y \neq dy$ in general. However if $\Delta x = dx$ is small, then dy is a close approximation of Δy (see Problem 11). The quantity dx, called the *differential of* x, and dy need not be small.

Because of the definitions *(8)* and *(10)*, we often write

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \qquad (11)$$

It is emphasized that dx and dy are *not* the limits of Δx and Δy as $\Delta x \to 0$, since these limits are zero whereas dx and dy are not necessarily zero. Instead, given dx we determine dy from *(10)*, i.e., dy is a dependent variable determined from the independent variable dx for a given x.

Geometrically, dy is represented in Fig. 4-1, for the particular value $x = x_0$, by the line segment SR, whereas Δy is represented by QR.

The geometric interpretation of the derivative as the slope of the tangent line to a curve at one of its points is fundamental to its application. Also of importance is its use as representative of instantaneous velocity in the construction of physical models. In particular, this physical viewpoint may be used to introduce the notion of differentials.

Newton's Second and First Laws of Motion imply that the path of an object is determined by the forces acting on it, and that if those forces suddenly disappear, the object takes on the tangential direction of the path at the point of release. Thus, the nature of the path in a small neighborhood of the point of release becomes of interest. With this thought in mind, consider the following idea.

Suppose the graph of a function f is represented by $y = f(x)$. Let $x = x_0$ be a domain value at which f' exists (i.e., the function is differentiable at that value). Construct a new linear function

$$dy = f'(x_0)\, dx$$

with dx as the (independent) domain variable and dy the range variable generated by this rule. This linear function has the graphical interpretation illustrated in Fig. 4-4.

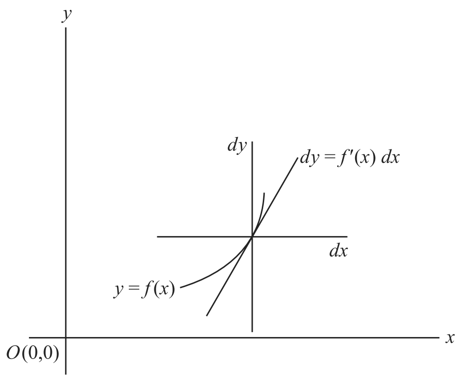

Fig. 4-4

That is, a coordinate system may be constructed with its origin at P_0 and the dx and dy axes parallel to the x and y axes, respectively. In this system our linear equation is the equation of the tangent line to the graph at P_0. It is representative of the path in a small neighborhood of the point; and if the path is that of an object, the linear equation represents its new path when all forces are released.

dx and dy are called differentials of x and y, respectively. Because the above linear equation is valid at every point in the domain of f at which the function has a derivative, the subscript may be dropped and we can write

$$dy = f'(x)\, dx$$

The following important observations should be made. $\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, thus $\frac{dy}{dx}$ is not the same thing as $\frac{\Delta y}{\Delta x}$.

On the other hand, dy and Δy are related. In particular, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$ means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $-\varepsilon < \frac{\Delta y}{\Delta x} - \frac{dy}{dx} < \varepsilon$ whenever $|\Delta x| < \delta$. Now dx is an independent variable and the axes of x and dx are parallel; therefore, dx may be chosen equal to Δx. With this choice

$$-\varepsilon \Delta x < \Delta y - dy < \varepsilon \Delta x$$

or

$$dy - \varepsilon \Delta x < \Delta y < dy + \varepsilon \Delta x$$

From this relation we see that dy is an approximation to Δy in small neighborhoods of x. dy is called the *principal part* of Δy.

The representation of f' by $\frac{dy}{dx}$ has an algebraic suggestiveness that is very appealing and will appear in much of what follows. In fact, this notation was introduced by Leibniz (without the justification provided by knowledge of the limit idea) and was the primary reason his approach to the calculus, rather than Newton's was followed.

THE DIFFERENTIATION OF COMPOSITE FUNCTIONS

Many functions are a composition of simpler ones. For example, if f and g have the rules of correspondence $u = x^3$ and $y = \sin u$, respectively, then $y = \sin x^3$ is the rule for a composite function $F = g(f)$. The domain of F is that subset of the domain of F whose corresponding range values are in the domain of g. The rule of composite function differentiation is called the *chain rule* and is represented by $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ $[F'(x) = g'(u)f'(x)]$.

In the example

$$\frac{dy}{dx} \equiv \frac{d(\sin x^3)}{dx} = \cos x^3(3x^2 dx)$$

The importance of the chain rule cannot be too greatly stressed. Its proper application is essential in the differentiation of functions, and it plays a fundamental role in changing the variable of integration, as well as in changing variables in mathematical models involving differential equations.

IMPLICIT DIFFERENTIATION

The rule of correspondence for a function may not be explicit. For example, the rule $y = f(x)$ is *implicit* to the equation $x^2 + 4xy^5 + 7xy + 8 = 0$. Furthermore, there is no reason to believe that this equation can be solved for y in terms of x. However, assuming a common domain (described by the independent variable x) the left-hand member of the equation can be construed as a composition of functions and differentiated accordingly. (The rules of differentiation are listed below for your review.)

In this example, differentiation with respect to x yields

$$2x + 4\left(y^5 + 5xy^4\frac{dy}{dx}\right) + 7\left(y + x\frac{dy}{dx}\right) = 0$$

Observe that this equation can be solved for $\frac{dy}{dx}$ as a function of x and y (but not of x alone).

RULES FOR DIFFERENTIATION

If f, g, and h are differentiable functions, the following differentiation rules are valid.

1. $\dfrac{d}{dx}\{f(x)+g(x)\} = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x) = f'(x) + g'(x)$ (Addition Rule)

2. $\dfrac{d}{dx}\{f(x)-g(x)\} = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x) = f'(x) - g'(x)$

3. $\dfrac{d}{dx}\{Cf(x)\} = C\dfrac{d}{dx}f(x) = Cf'(x)$ where C is any constant

4. $\dfrac{d}{dx}\{f(x)g(x)\} = f(x)\dfrac{d}{dx}g(x) + g(x)\dfrac{d}{dx}f(x) = f(x)g'(x) + g(x)f'(x)$ (Product Rule)

5. $\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{g(x)\dfrac{d}{dx}f(x) - f(x)\dfrac{d}{dx}g(x)}{[g(x)]^2} = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ if $g(x) \neq 0$ (Quotient Rule)

6. If $y = f(u)$ where $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = f'(u)\frac{du}{dx} = f'\{g(x)\}g'(x) \tag{12}$$

Similarly if $y = f(u)$ where $u = g(v)$ and $v = h(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx} \tag{13}$$

The results (*12*) and (*13*) are often called *chain rules* for differentiation of composite functions.

7. If $y = f(x)$, and $x = f^{-1}(y)$; then dy/dx and dx/dy are related by

$$\frac{dy}{dx} = \frac{1}{dx/dy} \tag{14}$$

8. If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \tag{15}$$

Similar rules can be formulated for differentials. For example,

$$d\{f(x)+g(x)\} = df(x) + dg(x) = f'(x)dx + g'(x)dx = \{f'(x)+g'(x)\}dx$$

$$d\{f(x)g(x)\} = f(x)dg(x) + g(x)df(x) = \{f(x)g'(x) + g(x)f'(x)\}dx$$

DERIVATIVES OF ELEMENTARY FUNCTIONS

In the following we assume that u is a differentiable function of x; if $u = x$, $du/dx = 1$. The inverse functions are defined according to the principal values given in Chapter 3.

1. $\dfrac{d}{dx}(C) = 0$
2. $\dfrac{d}{dx}u^n = nu^{n-1}\dfrac{du}{dx}$
3. $\dfrac{d}{dx}\sin u = \cos u\dfrac{du}{dx}$
4. $\dfrac{d}{dx}\cos u = -\sin u\dfrac{du}{dx}$
5. $\dfrac{d}{dx}\tan u = \sec^2 u\dfrac{du}{dx}$
6. $\dfrac{d}{dx}\cot u = -\csc^2 u\dfrac{du}{dx}$
7. $\dfrac{d}{dx}\sec u = \sec u \tan u\dfrac{du}{dx}$
8. $\dfrac{d}{dx}\csc u = -\csc u \cot u\dfrac{du}{dx}$
9. $\dfrac{d}{dx}\log_a u = \dfrac{\log_a e}{u}\dfrac{du}{dx} \quad a > 0, a \neq 1$
10. $\dfrac{d}{dx}\log_e u = \dfrac{d}{dx}\ln u = \dfrac{1}{u}\dfrac{du}{dx}$
11. $\dfrac{d}{dx}a^u = a^u \ln a\dfrac{du}{dx}$
12. $\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}$
13. $\dfrac{d}{dx}\sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$
14. $\dfrac{d}{dx}\cos^{-1} u = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$
15. $\dfrac{d}{dx}\tan^{-1} u = \dfrac{1}{1+u^2}\dfrac{du}{dx}$
16. $\dfrac{d}{dx}\cot^{-1} u = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$
17. $\dfrac{d}{dx}\sec^{-1} u = \pm\dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx} \quad \begin{cases} + \text{ if } u > 1 \\ - \text{ if } u < -1 \end{cases}$
18. $\dfrac{d}{dx}\csc^{-1} u = \mp\dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx} \quad \begin{cases} - \text{ if } u > 1 \\ + \text{ if } u < -1 \end{cases}$
19. $\dfrac{d}{dx}\sinh u = \cosh u\dfrac{du}{dx}$
20. $\dfrac{d}{dx}\cosh u = \sinh u\dfrac{du}{dx}$
21. $\dfrac{d}{dx}\tanh u = \operatorname{sech}^2 u\dfrac{du}{dx}$
22. $\dfrac{d}{dx}\coth u = -\operatorname{csch}^2 u\dfrac{du}{dx}$
23. $\dfrac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u\dfrac{du}{dx}$
24. $\dfrac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u\dfrac{du}{dx}$
25. $\dfrac{d}{dx}\sinh^{-1} u = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$
26. $\dfrac{d}{dx}\cosh^{-1} u = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$
27. $\dfrac{d}{dx}\tanh^{-1} u = \dfrac{1}{1-u^2}\dfrac{du}{dx}, \quad |u| < 1$
28. $\dfrac{d}{dx}\coth^{-1} u = \dfrac{1}{1-u^2}\dfrac{du}{dx}, \quad |u| > 1$
29. $\dfrac{d}{dx}\operatorname{sech}^{-1} u = \dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$
30. $\dfrac{d}{dx}\operatorname{csch}^{-1} u = -\dfrac{1}{u\sqrt{u^2+1}}\dfrac{du}{dx}$

HIGHER ORDER DERIVATIVES

If $f(x)$ is differentiable in an interval, its derivative is given by $f'(x)$, y' or dy/dx, where $y = f(x)$. If $f'(x)$ is also differentiable in the interval, its derivative is denoted by $f''(x)$, y'' or $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$. Similarly, the nth derivative of $f(x)$, if it exists, is denoted by $f^{(n)}(x)$, $y^{(n)}$ or $\dfrac{d^ny}{dx^n}$, where n is called the order of the derivative. Thus derivatives of the first, second, third, . . . orders are given by $f'(x), f''(x), f'''(x), \ldots$.

Computation of higher order derivatives follows by repeated application of the differentiation rules given above.

MEAN VALUE THEOREMS

These theorems are fundamental to the rigorous establishment of numerous theorems and formulas. (See Fig. 4-5.)

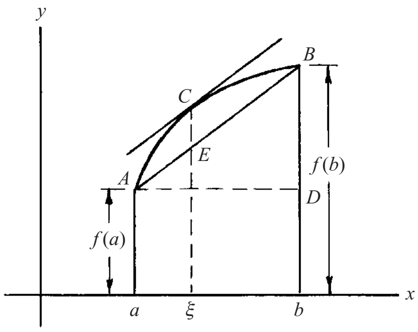

Fig. 4-5

1. **Rolle's theorem**. If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) and if $f(a) = f(b) = 0$, then there exists a point ξ in (a, b) such that $f'(\xi) = 0$.

   Rolle's theorem is employed in the proof of the mean value theorem. It then becomes a special case of that theorem.

2. **The mean value theorem**. If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then there exists a point ξ in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi) \qquad a < \xi < b \tag{16}$$

   Rolle's theorem is the special case of this where $f(a) = f(b) = 0$.

   The result (*16*) can be written in various alternative forms; for example, if x and x_0 are in (a, b), then

$$f(x) = f(x_0) + f'(\xi)(x - x_0) \qquad \xi \text{ between } x_0 \text{ and } x \tag{17}$$

   We can also write (*16*) with $b = a + h$, in which case $\xi = a + \theta h$, where $0 < \theta < 1$.

   The mean value theorem is also called the *law of the mean*.

3. **Cauchy's generalized mean value theorem**. If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b), then there exists a point ξ in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)} \qquad a < \xi < b \tag{18}$$

   where we assume $g(a) \neq g(b)$ and $f'(x)$, $g'(x)$ are not simultaneously zero. Note that the special case $g(x) = x$ yields (*16*).

L'HOSPITAL'S RULES

If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$, where A and B are either both zero or both infinite, $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ is often called an *indeterminate* of the form $0/0$ or ∞/∞, respectively, although such terminology is somewhat misleading since there is usually nothing indeterminate involved. The following theorems, called *L'Hospital's rules*, facilitate evaluation of such limits.

1. If $f(x)$ and $g(x)$ are differentiable in the interval (a, b) except possibly at a point x_0 in this interval, and if $g'(x) \neq 0$ for $x \neq x_0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \qquad (19)$$

whenever the limit on the right can be found. In case $f'(x)$ and $g'(x)$ satisfy the same conditions as $f(x)$ and $g(x)$ given above, the process can be repeated.

2. If $\lim_{x \to x_0} f(x) = \infty$ and $\lim_{x \to x_0} g(x) = \infty$, the result (*19*) is also valid.

These can be extended to cases where $x \to \infty$ or $-\infty$, and to cases where $x_0 = a$ or $x_0 = b$ in which only one sided limits, such as $x \to a+$ or $x \to b-$, are involved.

Limits represented by the so-called *indeterminate forms* $0 \cdot \infty$, ∞^0, 0^0, 1^∞, and $\infty - \infty$ can be evaluated on replacing them by equivalent limits for which the above rules are applicable (see Problem 4.29).

APPLICATIONS

1. Relative Extrema and Points of Inflection

See Chapter 3 where relative extrema and points of inflection were described and a diagram is presented. In this chapter such points are characterized by the variation of the tangent line, and then by the derivative, which represents the slope of that line.

Assume that f has a derivative at each point of an open interval and that P_1 is a point of the graph of f associated with this interval. Let a varying tangent line to the graph move from left to right through P_1. If the point is a relative minimum, then the tangent line rotates counterclockwise. The slope is negative to the left of P_1 and positive to the right. At P_1 the slope is zero. At a relative maximum a similar analysis can be made except that the rotation is clockwise and the slope varies from positive to negative. Because f'' designates the change of f', we can state the following theorem. (See Fig. 4-6.)

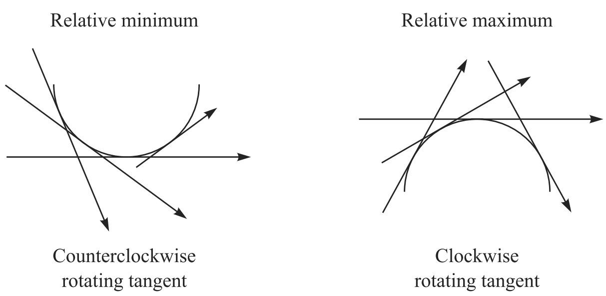

Fig. 4-6

Theorem. Assume that x_1 is a number in an open set of the domain of f at which f' is continuous and f'' is defined. If $f'(x_1) = 0$ and $f''(x_1) \neq 0$, then $f(x_1)$ is a relative extreme of f. Specifically:

(*a*) If $f''(x_1) > 0$, then $f(x_1)$ is a relative minimum,

(*b*) If $f''(x_1) < 0$, then $f(x_1)$ is a relative maximum.

(The domain value x_1 is called a *critical value*.)

This theorem may be generalized in the following way. Assume existence and continuity of derivatives as needed and suppose that $f'(x_1) = f''(x_1) = \cdots f^{(2p-1)}(x_1) = 0$ and $f^{(2p)}(x_1) \neq 0$ (p a positive integer). Then:

(*a*) f has a relative minimum at x_1 if $f^{(2p)}(x_1) > 0$,

(*b*) f has a relative maximum at x_1 if $f^{(2p)}(x_1) < 0$.

(Notice that the order of differentiation in each succeeding case is two greater. The nature of the intermediate possibilities is suggested in the next paragraph.)

It is possible that the slope of the tangent line to the graph of f is positive to the left of P_1, zero at the point, and again positive to the right. Then P_1 is called a *point of inflection*. In the simplest case this point of inflection is characterized by $f'(x_1) = 0$, $f''(x_1) = 0$, and $f'''(x_1) \neq 0$.

2. **Particle motion**

The fundamental theories of modern physics are relativity, electromagnetism, and quantum mechanics. Yet Newtonian physics must be studied because it is basic to many of the concepts in these other theories, and because it is most easily applied to many of the circumstances found in everyday life. The simplest aspect of Newtonian mechanics is called *kinematics*, or the *geometry of motion*. In this model of reality, objects are idealized as points and their paths are represented by curves. In the simplest (one-dimensional) case, the curve is a straight line, and it is the speeding up and slowing down of the object that is of importance. The calculus applies to the study in the following way.

If x represents the distance of a particle from the origin and t signifies time, then $x = f(t)$ designates the position of a particle at time t. Instantaneous velocity (or speed in the one-dimensional case) is represented by $\dfrac{dx}{dt} = \lim\limits_{\Delta t \to 0} \dfrac{f(t + \Delta t)}{\Delta t}$ (the limiting case of the formula $\dfrac{\text{change in distance}}{\text{change in time}}$ for speed when the motion is constant). Furthermore, the instantaneous change in velocity is called *acceleration* and represented by $\dfrac{d^2x}{dt^2}$.

Path, velocity, and acceleration of a particle will be represented in three dimensions in Chapter 7 on vectors.

3. **Newton's method**

It is difficult or impossible to solve algebraic equations of higher degree than two. In fact, it has been proved that there are no general formulas representing the roots of algebraic equations of degree five and higher in terms of radicals. However, the graph $y = f(x)$ of an algebraic equation $f(x) = 0$ crosses the x-axis at each single-valued real root. Thus, by trial and error, consecutive integers can be found between which a root lies. Newton's method is a systematic way of using tangents to obtain a better approximation of a specific real root. The procedure is as follows. (See Fig. 4-7.)

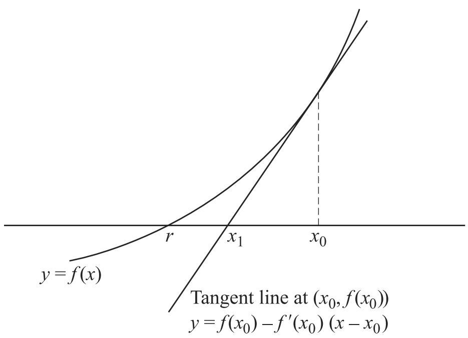

Fig. 4-7

Suppose that f has as many derivatives as required. Let r be a real root of $f(x) = 0$, i.e., $f(r) = 0$. Let x_0 be a value of x near r. For example, the integer preceding or following r. Let $f'(x_0)$ be the slope of the graph of $y = f(x)$ at $P_0[x_0, f(x_0)]$. Let $Q_1(x_1, 0)$ be the x-axis intercept of the tangent line at P_0 then

$$\frac{0 - f(x_0)}{x - x_0} = f'(x_0)$$

where the two representations of the slope of the tangent line have been equated. The solution of this relation for x_1 is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Starting with the tangent line to the graph at $P_1[x_1, f(x_1)]$ and repeating the process, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f(x_1)}{f'(x_1)}$$

and in general

$$x_n = x_0 - \sum_{k=0}^{n} \frac{f(x_k)}{f'(x_k)}$$

Under appropriate circumstances, the approximation x_n to the root r can be made as good as desired.

Note: Success with Newton's method depends on the shape of the function's graph in the neighborhood of the root. There are various cases which have not been explored here.

Solved Problems

DERIVATIVES

4.1. (*a*) Let $f(x) = \dfrac{3+x}{3-x}$, $x \neq 3$. Evaluate $f'(2)$ from the definition.

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{1}{h}\left(\frac{5+h}{1-h} - 5\right) = \lim_{h\to 0}\frac{1}{h}\cdot\frac{6h}{1-h} = \lim_{h\to 0}\frac{6}{1-h} = 6$$

Note: By using rules of differentiation we find

$$f'(x) = \frac{(3-x)\dfrac{d}{dx}(3+x) - (3+x)\dfrac{d}{dx}(3-x)}{(3-x)^2} = \frac{(3-x)(1) - (3+x)(-1)}{(3-x)^2} = \frac{6}{(3-x)^2}$$

at all points x where the derivative exists. Putting $x = 2$, we find $f'(2) = 6$. Although such rules are often useful, one must be careful not to apply them indiscriminately (see Problem 4.5).

(*b*) Let $f(x) = \sqrt{2x-1}$. Evaluate $f'(5)$ from the definition.

$$f'(5) = \lim_{h\to 0}\frac{f(5+h)-f(5)}{h} = \lim_{h\to 0}\frac{\sqrt{9+2h}-3}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{9+2h}-3}{h}\cdot\frac{\sqrt{9+2h}+3}{\sqrt{9+2h}+3} = \lim_{h\to 0}\frac{9+2h-9}{h(\sqrt{9+2h}+3)} = \lim_{h\to 0}\frac{2}{\sqrt{9+2h}+3} = \frac{1}{3}$$

By using rules of differentiation we find $f'(x) = \dfrac{d}{dx}(2x-1)^{1/2} = \frac{1}{2}(2x-1)^{-1/2}\dfrac{d}{dx}(2x-1) = (2x-1)^{-1/2}$. Then $f'(5) = 9^{-1/2} = \frac{1}{3}$.

4.2. (*a*) Show directly from definition that the derivative of $f(x) = x^3$ is $3x^2$.

(*b*) Show from definition that $\dfrac{d}{dx}\sqrt{x}) = \dfrac{1}{2\sqrt{x}}$.

(*a*) $\dfrac{f(x+h)-f(x)}{h} = \dfrac{1}{h}[(x+h)^3 - x^3]$

$$= \frac{1}{h}[x^3 + 3x^2h + 3xh^2 + h^3] - x^3] = 3x^2 + 3xh + h^2$$

Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 3x^2$$

(*b*) $\displaystyle\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

The result follows by multiplying numerator and denominator by $\sqrt{x+h}-\sqrt{x}$ and then letting $h \to 0$.

4.3. If $f(x)$ has a derivative at $x = x_0$, prove that $f(x)$ must be continuous at $x = x_0$.

$$f(x_0+h) - f(x_0) = \frac{f(x_0+h)-f(x_0)}{h} \cdot h, \qquad h \neq 0$$

Then $$\lim_{h\to 0} f(x_0+h) - f(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \cdot \lim_{h\to 0} h = f'(x_0)\cdot 0 = 0$$

since $f'(x_0)$ exists by hypothesis. Thus

$$\lim_{h\to 0} f(x_0+h) - f(x_0) = 0 \qquad \text{or} \qquad \lim_{h\to 0} f(x_0+h) = f(x_0)$$

showing that $f(x)$ is continuous at $x = x_0$.

4.4. Let $f(x) = \begin{cases} x\sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

(*a*) Is $f(x)$ continuous at $x = 0$? (*b*) Does $f(x)$ have a derivative at $x = 0$?

(*a*) By Problem 3.22(*b*) of Chapter 3, $f(x)$ is continuous at $x = 0$.

(*b*) $\displaystyle f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{h\sin 1/h - 0}{h} = \lim_{h\to 0} \sin\frac{1}{h}$

which does not exist.

This example shows that even though a function is continuous at a point, it need not have a derivative at the point, i.e., the converse of the theorem in Problem 4.3 is not necessarily true.

It is possible to construct a function which is continuous at every point of an interval but has a derivative nowhere.

4.5. Let $f(x) = \begin{cases} x^2\sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

(*a*) Is $f(x)$ differentiable at $x = 0$? (*b*) Is $f'(x)$ continuous at $x = 0$?

(*a*) $\displaystyle f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2\sin 1/h - 0}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0$

by Problem 3.13, Chapter 3. Then $f(x)$ has a derivative (is differentiable) at $x = 0$ and its value is 0.

(*b*) From elementary calculus differentiation rules, if $x \neq 0$,

$$f'(x) = \frac{d}{dx}\left(x^2\sin\frac{1}{x}\right) = x^2\frac{d}{dx}\left(\sin\frac{1}{x}\right) + \left(\sin\frac{1}{x}\right)\frac{d}{dx}(x^2)$$

$$= x^2\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + \left(\sin\frac{1}{x}\right)(2x) = -\cos\frac{1}{x} + 2x\sin\frac{1}{x}$$

Since $\lim_{x \to 0} f'(x) = \lim_{x \to 0}\left(-\cos\frac{1}{x} + 2x \sin\frac{1}{x}\right)$ does not exist (because $\lim_{x \to 0} \cos 1/x$ does not exist), $f'(x)$ cannot be continuous at $x = 0$ in spite of the fact that $f'(0)$ exists.

This shows that we cannot calculate $f'(0)$ in this case by simply calculating $f'(x)$ and putting $x = 0$, as is frequently supposed in elementary calculus. It is only when the derivative of a function is *continuous* at a point that this procedure gives the right answer. This happens to be true for most functions arising in elementary calculus.

4.6. Present an "ϵ, δ" definition of the derivative of $f(x)$ at $x = x_0$.

$f(x)$ has a derivative $f'(x_0)$ at $x = x_0$ if, given any $\epsilon > 0$, we can find $\delta > 0$ such that

$$\left|\frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0)\right| < \epsilon \qquad \text{when} \qquad 0 < |h| < \delta$$

RIGHT- AND LEFT-HAND DERIVATIVES

4.7. Let $f(x) = |x|$. (*a*) Calculate the right-hand derivatives of $f(x)$ at $x = 0$. (*b*) Calculate the left-hand derivative of $f(x)$ at $x = 0$. (*c*) Does $f(x)$ have a derivative at $x = 0$? (*d*) Illustrate the conclusions in (*a*), (*b*), and (*c*) from a graph.

(*a*) $f'_+(0) = \lim_{h \to 0+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0+} \frac{|h| - 0}{h} = \lim_{h \to 0+} \frac{h}{h} = 1$

since $|h| = h$ for $h > 0$.

(*b*) $f'_-(0) = \lim_{h \to 0-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0-} \frac{|h| - 0}{h} = \lim_{h \to 0-} \frac{-h}{h} = -1$

since $|h| = -h$ for $h < 0$.

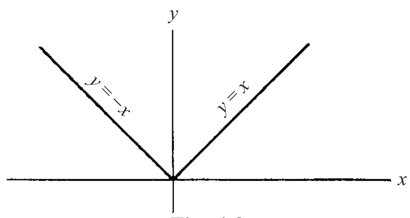

Fig. 4-8

(*c*) No. The derivative at 0 does not exist if the right and left hand derivatives are unequal.

(*d*) The required graph is shown in the adjoining Fig. 4-8. Note that the slopes of the lines $y = x$ and $y = -x$ are 1 and -1 respectively, representing the right and left hand derivatives at $x = 0$. However, the derivative at $x = 0$ does not exist.

4.8. Prove that $f(x) = x^2$ is differentiable in $0 \leqq x \leqq 1$.

Let x_0 be any value such that $0 < x_0 < 1$. Then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(x_0 + h)^2 - x_0^2}{h} = \lim_{h \to 0}(2x_0 + h) = 2x_0$$

At the end point $x = 0$,

$$f'_+(0) = \lim_{h \to 0+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0+} \frac{h^2 - 0}{h} = \lim_{h \to 0+} h = 0$$

At the end point $x = 1$,

$$f'_-(1) = \lim_{h \to 0-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0-} \frac{(1 + h)^2 - 1}{h} = \lim_{h \to 0-} (2 + h) = 2$$

Then $f(x)$ is differentiable in $0 \leqq x \leqq 1$. We may write $f'(x) = 2x$ for any x in this interval. It is customary to write $f'_+(0) = f'(0)$ and $f'_-(1) = f'(1)$ in this case.

4.9. Find an equation for the tangent line to $y = x^2$ at the point where (*a*) $x = 1/3$, (*b*) $x = 1$.

(*a*) From Problem 4.8, $f'(x_0) = 2x_0$ so that $f'(1/3) = 2/3$. Then the equation of the tangent line is

$$y - f(x_0) = f'(x_0)(x - x_0) \quad \text{or} \quad y - \tfrac{1}{9} = \tfrac{2}{3}(x - \tfrac{1}{3}), \qquad \text{i.e., } y = \tfrac{2}{3}x - \tfrac{1}{9}$$

(*b*) As in part (*a*), $y - f(1) = f'(1)(x - 1)$ or $y - 1 = 2(x - 1)$, i.e., $y = 2x - 1$.

DIFFERENTIALS

4.10. If $y = f(x) = x^3 - 6x$, find (*a*) Δy, (*b*) dy, (*c*) $\Delta y - dy$.

(*a*)
$$\begin{aligned}\Delta y = f(x + \Delta x) - f(x) &= \{(x + \Delta x)^3 - 6(x + \Delta x)\} - \{x^3 - 6x\}\\ &= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 6x - 6\Delta x - x^3 + 6x\\ &= (3x^2 - 6)\Delta x + 3x(\Delta x)^2 + (\Delta x)^3\end{aligned}$$

(*b*) dy = principal part of $\Delta y = (3x^2 - 6)\Delta x = (3x^2 - 6)dx$, since by definition $\Delta x = dx$.

Note that $f'(x) = 3x^2 - 6$ and $dy = (3x^2 - 6)dx$, i.e., $dy/dx = 3x^2 - 6$. It must be emphasized that dy and dx are not necessarily small.

(*c*) From (*a*) and (*b*), $\Delta y - dy = 3x(\Delta x)^2 + (\Delta x)^3 = \epsilon\Delta x$, where $\epsilon = 3x\Delta x + (\Delta x)^2$.

Note that $\epsilon \to 0$ as $\Delta x \to 0$, i.e., $\dfrac{\Delta y - dy}{\Delta x} \to 0$ as $\Delta x \to 0$. Hence $\Delta y - dy$ is an infinitesimal of higher order than Δx (see Problem 4.83).

In case Δx is small, dy and Δy are approximately equal.

4.11. Evaluate $\sqrt[3]{25}$ approximately by use of differentials.

If Δx is small, $\Delta y = f(x + \Delta x) - f(x) = f'(x)\Delta x$ approximately.
Let $f(x) = \sqrt[3]{x}$. Then $\sqrt[3]{x + \Delta x} - \sqrt[3]{x} \approx \frac{1}{3}x^{-2/3}\Delta x$ (where $\approx$ denotes *approximately equal to*).
If $x = 27$ and $\Delta x = -2$, we have

$$\sqrt[3]{27 - 2} - \sqrt[3]{27} \approx \tfrac{1}{3}(27)^{-2/3}(-2), \qquad \text{i.e., } \sqrt[3]{25} - 3 \approx -2/27$$

Then $\sqrt[3]{25} \approx 3 - 2/27$ or 2.926.

If is interesting to observe that $(2.926)^3 = 25.05$, so that the approximation is fairly good.

DIFFERENTIATION RULES: DIFFERENTIATION OF ELEMENTARY FUNCTIONS

4.12. Prove the formula $\dfrac{d}{dx}\{f(x)g(x)\} = f(x)\dfrac{d}{dx}g(x) + g(x)\dfrac{d}{dx}f(x)$, assuming f and g are differentiable.

By definition,

$$\begin{aligned}\frac{d}{dx}\{f(x)g(x)\} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x)\{g(x + \Delta x) - g(x)\} + g(x)\{f(x + \Delta x) - f(x)\}}{\Delta x}\\ &= \lim_{\Delta x \to 0} f(x + \Delta x)\left\{\frac{g(x + \Delta x) - g(x)}{\Delta x}\right\} + \lim_{\Delta x \to 0} g(x)\left\{\frac{f(x + \Delta x) - f(x)}{\Delta x}\right\}\\ &= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)\end{aligned}$$

Another method:

Let $u = f(x)$, $v = g(x)$. Then $\Delta u = f(x + \Delta x) - f(x)$ and $\Delta v = g(x + \Delta x) - g(x)$, i.e., $f(x + \Delta x) = u + \Delta u$, $g(x + \Delta x) = v + \Delta v$. Thus

$$\frac{d}{dx}uv = \lim_{\Delta x \to 0} \frac{(u+\Delta u)(v+\Delta v) - uv}{\Delta x} = \lim_{\Delta x \to 0} \frac{u\Delta v + v\Delta u + \Delta u \Delta v}{\Delta x}$$
$$= \lim_{\Delta x \to 0}\left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\Delta v\right) = u\frac{dv}{dx} + v\frac{du}{dx}$$

where it is noted that $\Delta v \to 0$ as $\Delta x \to 0$, since v is supposed differentiable and thus continuous.

4.13. If $y = f(u)$ where $u = g(x)$, prove that $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ assuming that f and g are differentiable.

Let x be given an increment $\Delta x \neq 0$. Then as a consequence u and y take on increments Δu and Δy respectively, where

$$\Delta y = f(u+\Delta u) - f(u), \qquad \Delta u = g(x+\Delta x) - g(x) \tag{1}$$

Note that as $\Delta x \to 0$, $\Delta y \to 0$ and $\Delta u \to 0$.

If $\Delta u \neq 0$, let us write $\epsilon = \dfrac{\Delta y}{\Delta u} - \dfrac{dy}{du}$ so that $\epsilon \to 0$ as $\Delta u \to 0$ and

$$\Delta y = \frac{dy}{du}\Delta u + \epsilon \Delta u \tag{2}$$

If $\Delta u = 0$ for values of Δx, then (*1*) shows that $\Delta y = 0$ for these values of Δx. For such cases, we define $\epsilon = 0$.

It follows that in both cases, $\Delta u \neq 0$ or $\Delta u = 0$, (*2*) holds. Dividing (*2*) by $\Delta x \neq 0$ and taking the limit as $\Delta x \to 0$, we have

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{dy}{du}\frac{\Delta u}{\Delta x} + \epsilon\frac{\Delta u}{\Delta x}\right) = \frac{dy}{du}\cdot \lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0}\epsilon \cdot \lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x}$$
$$= \frac{dy}{du}\frac{du}{dx} + 0 \cdot \frac{du}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} \tag{3}$$

4.14. Given $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$, derive the formulas

(*a*) $\dfrac{d}{dx}(\tan x) = \sec^2 x$, (*b*) $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

(*a*) $$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \dfrac{d}{dx}(\sin x) - \sin x \dfrac{d}{dx}(\cos x)}{\cos^2 x}$$
$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = {}^2 x$$

(*b*) If $y = \sin^{-1} x$, then $x = \sin y$. Taking the derivative with respect to x,

$$1 = \cos y \frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

We have supposed here that the principal value $-\pi/2 \leqq \sin^{-1} x \leqq \pi/2$, is chosen so that $\cos y$ is positive, thus accounting for our writing $\cos y = \sqrt{1-\sin^2 y}$ rather than $\cos y = \pm\sqrt{1-\sin^2 y}$.

4.15. Derive the formula $\dfrac{d}{dx}(\log_a u) = \dfrac{\log_a e}{u}\dfrac{du}{dx}$ $(a > 0, a \neq 1)$, where u is a differentiable function of x.

Consider $y = f(u) = \log_a u$. By definition,

$$\frac{dy}{du} = \lim_{\Delta u \to 0}\frac{f(u+\Delta u) - f(u)}{\Delta u} = \lim_{\Delta u \to 0}\frac{\log_a(u+\Delta u) - \log_a u}{\Delta u}$$
$$= \lim_{\Delta u \to 0}\frac{1}{\Delta u}\log_a\left(\frac{u+\Delta u}{u}\right) = \lim_{\Delta u \to 0}\frac{1}{u}\log_a\left(1+\frac{\Delta u}{u}\right)^{u/\Delta u}$$

Since the logarithm is a continuous function, this can be written

$$\frac{1}{u}\log_a\left\{\lim_{\Delta u\to 0}\left(1+\frac{\Delta u}{u}\right)^{u/\Delta u}\right\} = \frac{1}{u}\log_a e$$

by Problem 2.19, Chapter 2, with $x = u/\Delta u$.

Then by Problem 4.13, $\dfrac{d}{dx}(\log_a u) = \dfrac{\log_a e}{u}\dfrac{du}{dx}$.

4.16. Calculate dy/dx if (*a*) $xy^3 - 3x^2 = xy + 5$, (*b*) $e^{xy} + y\ln x = \cos 2x$.

(*a*) Differentiate with respect to x, considering y as a function of x. (We sometimes say that y is an *implicit function of* x, since we cannot solve explicitly for y in terms of x.) Then

$$\frac{d}{dx}(xy^3) - \frac{d}{dx}(3x^2) = \frac{d}{dx}(xy) + \frac{d}{dx}(5) \qquad \text{or} \qquad (x)(3y^2y') + (y^3)(1) - 6x = (x)(y') + (y)(1) + 0$$

where $y' = dy/dx$. Solving, $y' = (6x - y^3 + y)/(3xy^2 - x)$.

(*b*) $$\frac{d}{dx}(e^{xy}) + \frac{d}{dx}(y\ln x) = \frac{d}{dx}(\cos 2x), \qquad e^{xy}(xy' + y) + \frac{y}{x} + (\ln x)y' = -2\sin 2x.$$

Solving, $$y' = -\frac{2x\sin 2x + xye^{xy} + y}{x^2e^{xy} + x\ln x}$$

4.17. If $y = \cosh(x^2 - 3x + 1)$, find (*a*) dy/dx, (*b*) d^2y/dx^2.

(*a*) Let $y = \cosh u$, where $u = x^2 - 3x + 1$. Then $dy/du = \sinh u$, $du/dx = 2x - 3$, and

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (\sinh u)(2x-3) = (2x-3)\sinh(x^2 - 3x + 1)$$

(*b*) $$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\sinh u\frac{du}{dx}\right) = \sinh u\frac{d^2u}{dx^2} + \cosh u\left(\frac{du}{dx}\right)^2 \\ &= (\sinh u)(2) + (\cosh u)(2x-3)^2 = 2\sinh(x^2-3x+1) + (2x-3)^2\cosh(x^2-3x+1)\end{aligned}$$

4.18. If $x^2y + y^3 = 2$, find (*a*) y', (*b*) y'' at the point (1, 1).

(*a*) Differentiating with respect to x, $x^2y' + 2xy + 3y^2y' = 0$ and

$$y' = \frac{-2xy}{x^2 + 3xy^2} = -\frac{1}{2} \text{ at } (1, 1)$$

(*b*) $$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{-2xy}{x^2+3y^2}\right) = -\frac{(x^2+3y^2)(2xy'+2y) - (2xy)(2x+6yy')}{(x^2+3y^2)^2}$$

Substituting $x = 1$, $y = 1$, and $y' = -\frac{1}{2}$, we find $y'' = -\frac{3}{8}$.

MEAN VALUE THEOREMS

4.19. Prove Rolle's theorem.

Case 1: $f(x) \equiv 0$ in $[a, b]$. Then $f'(x) = 0$ for all x in (a, b).

Case 2: $f(x) \not\equiv 0$ in $[a, b]$. Since $f(x)$ is continuous there are points at which $f(x)$ attains its maximum and minimum values, denoted by M and m respectively (see Problem 3.34, Chapter 3).

Since $f(x) \not\equiv 0$, at least one of the values M, m is not zero. Suppose, for example, $M \neq 0$ and that $f(\xi) = M$ (see Fig. 4-9). For this case, $f(\xi + h) \leqq f(\xi)$.

If $h > 0$, then $\dfrac{f(\xi + h) - f(\xi)}{h} \leqq 0$ and

$$\lim_{h \to 0+} \frac{f(\xi + h) - f(\xi)}{h} \leqq 0 \qquad (1)$$

If $h < 0$, then $\dfrac{f(\xi + h) - f(\xi)}{h} \geqq 0$ and

$$\lim_{h \to 0-} \frac{f(\xi + h) - f(\xi)}{h} \geqq 0 \qquad (2)$$

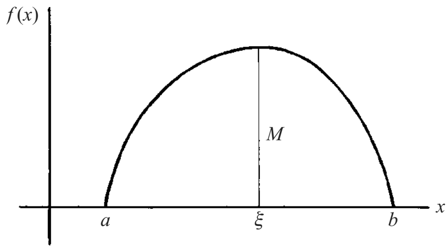

Fig. 4-9

But by hypothesis $f(x)$ has a derivative at all points in (a, b). Then the right-hand derivative (*1*) must be equal to the left-hand derivative (*2*). This can happen only if they are both equal to zero, in which case $f'(\xi) = 0$ as required.

A similar argument can be used in case $M = 0$ and $m \neq 0$.

4.20. Prove the mean value theorem.

Define $F(x) = f(x) - f(a) - (x - a)\dfrac{f(b) - f(a)}{b - a}$.

Then $F(a) = 0$ and $F(b) = 0$.

Also, if $f(x)$ satisfies the conditions on continuity and differentiability specified in Rolle's theorem, then $F(x)$ satisfies them also.

Then applying Rolle's theorem to the function $F(x)$, we obtain

$$F'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0, \quad a < \xi < b \qquad \text{or} \qquad f'(\xi) = \frac{f(b) - f(a)}{b - a}, \quad a < \xi < b$$

4.21. Verify the mean value theorem for $f(x) = 2x^2 - 7x + 10$, $a = 2$, $b = 5$.

$f(2) = 4$, $f(5) = 25$, $f'(\xi) = 4\xi - 7$. Then the mean value theorem states that $4\xi - 7 = (25 - 4)/(5 - 2)$ or $\xi = 3.5$. Since $2 < \xi < 5$, the theorem is verified.

4.22. If $f'(x) = 0$ at all points of the interval (a, b), prove that $f(x)$ must be a constant in the interval.

Let $x_1 < x_2$ be any two different points in (a, b). By the mean value theorem for $x_1 < \xi < x_2$,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi) = 0$$

Thus, $f(x_1) = f(x_2) =$ constant. From this it follows that if two functions have the same derivative at all points of (a, b), the functions can only differ by a constant.

4.23. If $f'(x) > 0$ at all points of the interval (a, b), prove that $f(x)$ is strictly increasing.

Let $x_1 < x_2$ be any two different points in (a, b). By the mean value theorem for $x_1 < \xi < x_2$,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi) > 0$$

Then $f(x_2) > f(x_1)$ for $x_2 > x_1$, and so $f(x)$ is strictly increasing.

4.24. (*a*) Prove that $\dfrac{b - a}{1 + b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b - a}{1 + a^2}$ if $a < b$.

(*b*) Show that $\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$.

(*a*) Let $f(x) = \tan^{-1} x$. Since $f'(x) = 1/(1 + x^2)$ and $f'(\xi) = 1/(1 + \xi^2)$, we have by the mean value theorem

$$\frac{\tan^{-1} b - \tan^{-1} a}{b - a} = \frac{1}{1 + \xi^2} \qquad a < \xi < b$$

Since $\xi > a$, $1/(1 + \xi^2) < 1/(1 + a^2)$. Since $\xi < b$, $1/(1 + \xi^2) > 1/(1 + b^2)$. Then

$$\frac{1}{1 + b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b - a} < \frac{1}{1 + a^2}$$

and the required result follows on multiplying by $b - a$.

(*b*) Let $b = 4/3$ and $a = 1$ in the result of part (*a*). Then since $\tan^{-1} 1 = \pi/4$, we have

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{1}{6} \qquad \text{or} \qquad \frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

4.25. Prove Cauchy's generalized mean value theorem.

Consider $G(x) = f(x) - f(a) - \alpha\{g(x) - g(a)\}$, where α is a constant. Then $G(x)$ satisfies the conditions of Rolle's theorem, provided $f(x)$ and $g(x)$ satisfy the continuity and differentiability conditions of Rolle's theorem and if $G(a) = G(b) = 0$. Both latter conditions are satisfied if the constant $\alpha = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

Applying Rolle's theorem, $G'(\xi) = 0$ for $a < \xi < b$, we have

$$f'(\xi) - \alpha g'(\xi) = 0 \qquad \text{or} \qquad \frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}, \qquad a < \xi < b$$

as required.

L'HOSPITAL'S RULE

4.26. Prove L'Hospital's rule for the case of the "indeterminate forms" (*a*) 0/0, (*b*) ∞/∞.

(*a*) We shall suppose that $f(x)$ and $g(x)$ are differentiable in $a < x < b$ and $f(x_0) = 0$, $g(x_0) = 0$, where $a < x_0 < b$.

By Cauchy's generalized mean value theorem (Problem 25),

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)} \qquad x_0 < \xi < x$$

Then

$$\lim_{x \to x_0+} \frac{f(x)}{g(x)} = \lim_{x \to x_0+} \frac{f'(\xi)}{g'(\xi)} = \lim_{x \to x_0+} \frac{f'(x)}{g'(x)} = L$$

since as $x \to x_0+$, $\xi \to x_0+$.

Modification of the above procedure can be used to establish the result if $x \to x_0-$, $x \to x_0$, $x \to \infty$, $x \to -\infty$.

(*b*) We suppose that $f(x)$ and $g(x)$ are differentiable in $a < x < b$, and $\lim\limits_{x \to x_0+} f(x) = \infty$, $\lim\limits_{x \to x_0+} g(x) = \infty$ where $a < x_0 < b$.

Assume x_1 is such that $a < x_0 < x < x_1 < b$. By Cauchy's generalized mean value theorem,

$$\frac{f(x) - f(x_1)}{g(x) - g(x_1)} = \frac{f'(\xi)}{g'(\xi)} \qquad x < \xi < x_1$$

Hence

$$\frac{f(x) - f(x_1)}{g(x) - g(x_1)} = \frac{f(x)}{g(x)} \cdot \frac{1 - f(x_1)/f(x)}{1 - g(x_1)/g(x)} = \frac{f'(\xi)}{g'(\xi)}$$

from which we see that

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \cdot \frac{1 - g(x_1)/g(x)}{1 - f(x_1)/f(x)} \tag{1}$$

Let us now suppose that $\lim_{x \to x_0+} \frac{f'(x)}{g'(x)} = L$ and write (*1*) as

$$\frac{f(x)}{g(x)} = \left(\frac{f'(\xi)}{g'(\xi)} - L\right)\left(\frac{1 - g(x_1)/g(x)}{1 - f(x_1)/f(x)}\right) + L\left(\frac{1 - g(x_1)/g(x)}{1 - f(x_1)/f(x)}\right) \tag{2}$$

We can choose x_1 so close to x_0 that $|f'(\xi)/g'(\xi) - L| < \epsilon$. Keeping x_1 fixed, we see that

$$\lim_{x \to x_0+} \left(\frac{1 - g(x_1)/g(x)}{1 - f(x_1)/f(x)}\right) = 1 \qquad \text{since} \lim_{x \to x_0+} f(x) = \infty \text{ and } \lim_{x \to x_0} g(x) = \infty$$

Then taking the limit as $x \to x_0+$ on both sides of (*2*), we see that, as required,

$$\lim_{x \to x_0+} \frac{f(x)}{g(x)} = L = \lim_{x \to x_0+} \frac{f'(x)}{g'(x)}$$

Appropriate modifications of the above procedure establish the result if $x \to x_0-$, $x \to x_0$, $x \to \infty$, $x \to -\infty$.

4.27. Evaluate (*a*) $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$ (*b*) $\lim_{x \to 1} \frac{1 + \cos \pi x}{x^2 - 2x + 1}$

All of these have the "indeterminate form" 0/0.

(*a*) $\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{2e^{2x}}{1} = 2$

(*b*) $\lim_{x \to 1} \frac{1 + \cos \pi x}{x^2 - 2x + 1} = \lim_{x \to 1} \frac{-\pi \sin \pi x}{2x - 2} = \lim_{x \to 1} \frac{-\pi^2 \cos \pi x}{2} = \frac{\pi^2}{2}$

Note: Here L'Hospital's rule is applied twice, since the first application again yields the "indeterminate form" 0/0 and the conditions for L'Hospital's rule are satisfied once more.

4.28. Evaluate (*a*) $\lim_{x \to \infty} \frac{3x^2 - x + 5}{5x^2 + 6x - 3}$ (*b*) $\lim_{x \to \infty} x^2 e^{-x}$

All of these have or can be arranged to have the "indeterminate form" ∞/∞.

(*a*) $\lim_{x \to \infty} \frac{3x^2 - x + 5}{5x^2 + 6x - 3} = \lim_{x \to \infty} \frac{6x - 1}{10x + 6} = \lim_{x \to \infty} \frac{6}{10} = \frac{3}{5}$

(*b*) $\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0$

4.29. Evaluate $\lim_{x \to 0+} x^2 \ln x$.

$$\lim_{x \to 0+} x^2 \ln x = \lim_{x \to 0+} \frac{\ln x}{1/x^2} = \lim_{x \to 0+} \frac{1/x}{-2/x^3} = \lim_{x \to 0+} \frac{-x^2}{2} = 0$$

The given limit has the "indeterminate form" $0 \cdot \infty$. In the second step the form is altered so as to give the indeterminate form ∞/∞ and L'Hospital's rule is then applied.

4.30. Find $\lim_{x \to 0} (\cos x)^{1/x^2}$.

Since $\lim_{x \to 0} \cos x = 1$ and $\lim_{x \to 0} 1/x^2 = \infty$, the limit takes the "indeterminate form" 1^∞.

Let $F(x) = (\cos x)^{1/x^2}$. Then $\ln F(x) = (\ln \cos x)/x^2$ to which L'Hospital's rule can be applied. We have

$$\lim_{x\to 0} \frac{\ln \cos x}{x^2} = \lim_{x\to 0} \frac{(-\sin x)/(\cos x)}{2x} = \lim_{x\to 0} \frac{-\sin x}{2x \cos x} = \lim_{x\to 0} \frac{-\cos x}{-2x \sin x + 2\cos x} = -\frac{1}{2}$$

Thus, $\lim_{x\to 0} \ln F(x) = -\frac{1}{2}$. But since the logarithm is a continuous function, $\lim_{x\to 0} \ln F(x) = \ln(\lim_{x\to 0} F(x))$. Then

$$\ln(\lim_{x\to 0} F(x)) = -\tfrac{1}{2} \qquad \text{or} \qquad \lim_{x\to 0} F(x) = \lim_{x\to 0} (\cos x)^{1/x^2} = e^{-1/2}$$

4.31. If $F(x) = (e^{3x} - 5x)^{1/x}$, find (*a*) $\lim_{x\to 0} F(x)$ and (*b*) $\lim_{x\to 0} F(x)$.

The respective indeterminate forms in (*a*) and (*b*) are ∞^0 and 1^∞.

Let $G(x) = \ln F(x) = \dfrac{\ln(e^{3x} - 5x)}{x}$. Then $\lim_{x\to\infty} G(x)$ and $\lim_{x\to 0} G(x)$ assume the indeterminate forms ∞/∞ and 0/0 respectively, and L'Hospital's rule applies. We have

(*a*) $$\lim_{x\to\infty} \frac{\ln(e^{3x} - 5x)}{x} = \lim_{x\to\infty} \frac{3e^{3x} - 5}{e^{3x} - 5x} = \lim_{x\to 0} \frac{9e^{3x}}{3e^{3x} - 5} = \lim_{x\to\infty} \frac{27e^{3x}}{9e^{3x}} = 3$$

Then, as in Problem 4.30, $\lim_{x\to\infty} (e^{3x} - 5x)^{1/x} = e^3$.

(*b*) $$\lim_{x\to 0} \frac{\ln(e^{3x} - 5x)}{x} = \lim_{x\to 0} \frac{3e^{3x} - 5}{e^{3x} - 5x} = -2 \qquad \text{and} \qquad \lim_{x\to 0} (e^{3x} - 5x)^{1/x} = e^{-2}$$

4.32. Suppose the equation of motion of a particle is $x = \sin(c_1 t + c_2)$, where c_1 and c_2 are constants. (Simple harmonic motion.) (*a*) Show that the acceleration of the particle is proportional to its distance from the origin. (*b*) If $c_1 = 1$, $c_2 = \pi$, and $t \geq 0$, determine the velocity and acceleration at the end points and at the midpoint of the motion.

(*a*) $$\frac{dx}{dt} = c_1 \cos(c_1 t + c_2), \qquad \frac{d^2x}{dt^2} = -c_1^2 \sin(c_1 t + c_2) = -c_1^2 x.$$

This relation demonstrates the proportionality of acceleration and distance.

(*b*) The motion starts at 0 and moves to -1. Then it oscillates between this value and 1. The absolute value of the velocity is zero at the end points, and that of the acceleration is maximum there. The particle coasts through the origin (zero acceleration), while the absolute value of the velocity is maximum there.

4.33. Use Newton's method to determine $\sqrt{3}$ to three decimal points of accuracy.

$\sqrt{3}$ is a solution of $x^2 - 3 = 0$, which lies between 1 and 2. Consider $f(x) = x^2 - 3$ then $f'(x) = 2x$. The graph of f crosses the x-axis between 1 and 2. Let $x_0 = 2$. Then $f(x_0) = 1$ and $f'(x_0) = 1.75$. According to the Newton formula, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2 - .25 = 1.75$.

Then $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 1.732$. To verify the three decimal point accuracy, note that $(1.732)^2 = 2.9998$ and $(1.7333)^2 = 3.0033$.

MISCELLANEOUS PROBLEMS

4.34. If $x = g(t)$ and $y = f(t)$ are twice differentiable, find (*a*) dy/dx, (*b*) d^2y/dx^2.

(*a*) Letting primes denote derivatives with respect to t, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)} \qquad \text{if } g'(t) \neq 0$$

(*b*)
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{f'(t)}{g'(t)}\right) = \frac{\frac{d}{dt}\left(\frac{f'(t)}{g'(t)}\right)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{f'(t)}{g'(t)}\right)}{g'(t)}$$
$$= \frac{1}{g'(t)}\left\{\frac{g'(t)f''(t) - f'(t)g''(t)}{[g'(t)]^2}\right\} = \frac{g'(t)f''(t) - f'(t)g''(t)}{[g'(t)]^3} \qquad \text{if } g'(t) \neq 0$$

4.35. Let $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Prove that (*a*) $f'(0) = 0$, (*b*) $f''(0) = 0$.

(*a*) $$f'_+(0) = \lim_{h\to 0+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0+} \frac{e^{-1/h^2} - 0}{h} = \lim_{h\to 0+} \frac{e^{-1/h^2}}{h}$$

If $h = 1/u$, using L'Hospital's rule this limit equals

$$\lim_{u\to\infty} ue^{-u^2} = \lim_{u\to\infty} u/e^{u^2} = \lim_{u\to\infty} 1/2ue^{u^2} = 0$$

Similarly, replacing $h \to 0+$ by $h \to 0-$ and $u \to \infty$ by $u \to -\infty$, we find $f'_-(0) = 0$. Thus $f'_+(0) = f'_-(0) = 0$, and so $f'(0) = 0$.

(*b*) $$f''_+(0) = \lim_{h\to 0+} \frac{f'(h) - f'(0)}{h} = \lim_{h\to 0+} \frac{e^{-1/h^2}\cdot 2h^{-3} - 0}{h} = \lim_{h\to 0+} \frac{2e^{-1/h^2}}{h^4} = \lim_{u\to\infty} \frac{2u^4}{e^{u^2}} = 0$$

by successive applications of L'Hospital's rule.

Similarly, $f''_-(0) = 0$ and so $f''(0) = 0$.

In general, $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$

4.36. Find the length of the longest ladder which can be carried around the corner of a corridor, whose dimensions are indicated in the figure below, if it is assumed that the ladder is carried parallel to the floor.

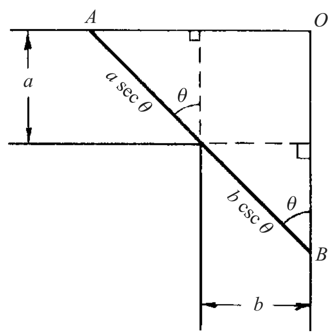

Fig. 4-10

The length of the *longest* ladder is the same as the *shortest* straight line segment AB [Fig. 4-10], which touches both outer walls and the corner formed by the inner walls.

As seen from Fig. 4-10, the length of the ladder AB is

$$L = a\sec\theta + b\csc\theta$$

L is a minimum when

$$dL/d\theta = a\sec\theta\tan\theta - b\csc\theta\cot\theta = 0$$

i.e., $\quad a\sin^3\theta = b\cos^3\theta \quad$ or $\quad \tan\theta = \sqrt[3]{b/a}$

Then $\quad \sec\theta = \dfrac{\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}}, \quad \csc\theta = \dfrac{\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}$

so that $\quad L = a\sec\theta + b\csc\theta = (a^{2/3} + b^{2/3})^{3/2}$

Although it is geometrically evident that this gives the minimum length, we can prove this analytically by showing that $d^2L/d\theta^2$ for $\theta = \tan^{-1}\sqrt[3]{b/a}$ is positive (see Problem 4.78).

Supplementary Problems

DERIVATIVES

4.37. Use the definition to compute the derivatives of each of the following functions at the indicated point: (*a*) $(3x-4)/(2x+3), x=1$; (*b*) $x^3-3x^2+2x-5, x=2$; (*c*) $\sqrt{x}, x=4$; (*d*) $\sqrt[3]{6x-4}$, $x=2$.
Ans. (*a*) 17/25, (*b*) 2, (*c*) $\frac{1}{4}$, (*d*) $\frac{1}{2}$

4.38. Show from definition that (*a*) $\dfrac{d}{dx}x^4 = 4x^3$, (*b*) $\dfrac{d}{dx}\dfrac{3+x}{3-x} = \dfrac{6}{(3-x)^2}$, $x \neq 3$

4.39. Let $f(x) = \begin{cases} x^3 \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Prove that (*a*) $f(x)$ is continuous at $x=0$, (*b*) $f(x)$ has a derivative at $x=0$, (*c*) $f'(x)$ is continuous at $x=0$.

4.40. Let $f(x) = \begin{cases} xe^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Determine whether $f(x)$ (*a*) is continuous at $x=0$, (*b*) has a derivative at $x=0$.

Ans. (*a*) Yes; (*b*) Yes, 0

4.41. Give an alternative proof of the theorem in Problem 4.3, Page 76, using "ϵ, δ definitions".

4.42. If $f(x) = e^x$, show that $f'(x_0) = e^{x_0}$ depends on the result $\lim\limits_{h\to 0}(e^h-1)/h = 1$.

4.43. Use the results $\lim\limits_{h\to 0}(\sin h)/h = 1$, $\lim\limits_{h\to 0}(1-\cos h)/h = 0$ to prove that if $f(x) = \sin x$, $f'(x_0) = \cos x_0$.

RIGHT- AND LEFT-HAND DERIVATIVES

4.44. Let $f(x) = x|x|$. (*a*) Calculate the right-hand derivative of $f(x)$ at $x=0$. (*b*) Calculate the left-hand derivative of $f(x)$ at $x=0$. (*c*) Does $f(x)$ have a derivative at $x=0$? (*d*) Illustrate the conclusions in (*a*), (*b*), and (*c*) from a graph.
Ans. (*a*) 0; (*b*) 0; (*c*) Yes, 0

4.45. Discuss the (*a*) continuity and (*b*) differentiability of $f(x) = x^p \sin 1/x$, $f(0) = 0$, where p is any positive number. What happens in case p is any real number?

4.46. Let $f(x) = \begin{cases} 2x-3, & 0 \leqq x \leqq 2 \\ x^2-3, & 2 < x \leqq 4 \end{cases}$. Discuss the (*a*) continuity and (*b*) differentiability of $f(x)$ in $0 \leqq x \leqq 4$.

4.47. Prove that the derivative of $f(x)$ at $x = x_0$ exists if and only if $f'_+(x_0) = f'_-(x_0)$.

4.48. (*a*) Prove that $f(x) = x^3 - x^2 + 5x - 6$ is differentiable in $a \leqq x \leqq b$, where a and b are any constants. (*b*) Find equations for the tangent lines to the curve $y = x^3 - x^2 + 5x - 6$ at $x=0$ and $x=1$. Illustrate by means of a graph. (*c*) Determine the point of intersection of the tangent lines in (*b*). (*d*) Find $f'(x), f''(x), f'''(x), f^{(\mathrm{IV})}(x), \ldots$.
Ans. (*b*) $y = 5x-6, y = 6x-7$; (*c*) $(1,-1)$; (*d*) $3x^2-2x+5, 6x-2, 6, 0, 0, 0, \ldots$

4.49. If $f(x) = x^2|x|$, discuss the existence of successive derivatives of $f(x)$ at $x=0$.

DIFFERENTIALS

4.50. If $y = f(x) = x + 1/x$, find (*a*) Δy, (*b*) dy, (*c*) $\Delta y - dy$, (*d*) $(\Delta y - dy)/\Delta x$, (*e*) dy/dx.

Ans. (*a*) $\Delta x - \dfrac{\Delta x}{x(x+\Delta x)}$, (*b*) $\left(1-\dfrac{1}{x^2}\right)\Delta x$, (*c*) $\dfrac{(\Delta x)^2}{x^2(x+\Delta x)}$, (*d*) $\dfrac{\Delta x}{x^2(x+\Delta x)}$, (*e*) $1-\dfrac{1}{x^2}$.

Note: $\Delta x = dx$.

4.51. If $f(x) = x^2 + 3x$, find (*a*) Δy, (*b*) dy, (*c*) $\Delta y/\Delta x$, (*d*) dy/dx, and (*e*) $(\Delta y - dy)/\Delta x$, if $x = 1$ and $\Delta x = .01$.
Ans. (*a*) .0501, (*b*) .05, (*c*) 5.01, (*d*) 5, (*e*) .01

4.52. Using differentials, compute approximate values for each of the following: (*a*) $\sin 31^\circ$, (*b*) $\ln(1.12)$, (*c*) $\sqrt[5]{36}$.
Ans. (*a*) 0.515, (*b*) 0.12, (*c*) 2.0125

4.53. If $y = \sin x$, evaluate (*a*) Δy, (*b*) dy. (*c*) Prove that $(\Delta y - dy)/\Delta x \to 0$ as $\Delta x \to 0$.

DIFFERENTIATION RULES AND ELEMENTARY FUNCTIONS

4.54. Prove: (*a*) $\dfrac{d}{dx}\{f(x) + g(x)\} = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$, (*b*) $\dfrac{d}{dx}\{f(x) - g(x)\} = \dfrac{d}{dx}f(x) - \dfrac{d}{dx}g(x)$,
(*c*) $\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0.$

4.55. Evaluate (*a*) $\dfrac{d}{dx}\{x^3 \ln(x^2 - 2x + 5)\}$ at $x = 1$, (*b*) $\dfrac{d}{dx}\{\sin^2(3x + \pi/6)\}$ at $x = 0$.
Ans. (*a*) $3\ln 4$, (*b*) $\frac{3}{2}\sqrt{3}$

4.56. Derive the formulas: (*a*) $\dfrac{d}{dx}a^u = a^u \ln a \dfrac{du}{dx}$, $a > 0, a \neq 1$; (*b*) $\dfrac{d}{dx}\csc u = -\csc u \cot u \dfrac{du}{dx}$;
(*c*) $\dfrac{d}{dx}\tanh u = \operatorname{sech}^2 u \dfrac{du}{dx}$ where u is a differentiable function of x.

4.57. Compute (*a*) $\dfrac{d}{dx}\tan^{-1} x$, (*b*) $\dfrac{d}{dx}\csc^{-1} x$, (*c*) $\dfrac{d}{dx}\sinh^{-1} x$, (*d*) $\dfrac{d}{dx}\coth^{-1} x$, paying attention to the use of principal values.

4.58. If $y = x^x$, computer dy/dx. [Hint: Take logarithms before differentiating.]
Ans. $x^x(1 + \ln x)$

4.59. If $y = \{\ln(3x + 2)\}^{\sin^{-1}(2x+.5)}$, find dy/dx at $x = 0$.

Ans. $\left(\dfrac{\pi}{4\ln 2} + \dfrac{2\ln\ln 2}{\sqrt{3}}\right)(\ln 2)^{\pi/6}$

4.60. If $y = f(u)$, where $u = g(v)$ and $v = h(x)$, prove that $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dv}\cdot\dfrac{dv}{dx}$ assuming f, g, and h are differentiable.

4.61. Calculate (*a*) dy/dx and (*b*) d^2y/dx^2 if $xy - \ln y = 1$.
Ans. (*a*) $y^2/(1 - xy)$, (*b*) $(3y^3 - 2xy^4)/(1 - xy)^3$ provided $xy \neq 1$

4.62. If $y = \tan x$, prove that $y''' = 2(1 + y^2)(1 + 3y^2)$.

4.63. If $x = \sec t$ and $y = \tan t$, evaluate (*a*) dy/dx, (*b*) d^2y/dx^2, (*c*) d^3y/dx^3, at $t = \pi/4$.
Ans. (*a*) $\sqrt{2}$, (*b*) -1, (*c*) $3\sqrt{2}$

4.64. Prove that $\dfrac{d^2y}{dx^2} = -\dfrac{d^2x}{dy^2}\bigg/\left(\dfrac{dx}{dy}\right)^3$, stating precise conditions under which it holds.

4.65. Establish formulas (*a*) 7, (*b*) 18, and (*c*) 27, on Page 71.

MEAN VALUE THEOREMS

4.66. Let $f(x) = 1 - (x - 1)^{2/3}$, $0 \leqq x \leqq 2$. (*a*) Construct the graph of $f(x)$. (*b*) Explain why Rolle's theorem is not applicable to this function, i.e., there is no value ξ for which $f'(\xi) = 0$, $0 < \xi < 2$.

4.67. Verify Rolle's theorem for $f(x) = x^2(1-x)^2$, $0 \leqq x \leqq 1$.

4.68. Prove that between any two real roots of $e^x \sin x = 1$ there is at least one real root of $e^x \cos x = -1$. [Hint: Apply Rolle's theorem to the function $e^{-x} - \sin x$.]

4.69. (*a*) If $0 < a < b$, prove that $(1 - a/b) < \ln b/a < (b/a - 1)$
(*b*) Use the result of (*a*) to show that $\frac{1}{6} < \ln 1.2 < \frac{1}{5}$.

4.70. Prove that $(\pi/6 + \sqrt{3}/15) < \sin^{-1} .6 < (\pi/6 + 1/8)$ by using the mean value theorem.

4.71. Show that the function $F(x)$ in Problem 4.20(*a*) represents the difference in ordinants of curve ACB and line AB at any point x in (a, b).

4.72. (*a*) If $f'(x) \leqq 0$ at all points of (a, b), prove that $f(x)$ is monotonic decreasing in (a, b).
(*b*) Under what conditions is $f(x)$ strictly decreasing in (a, b)?

4.73. (*a*) Prove that $(\sin x)/x$ is strictly decreasing in $(0, \pi/2)$. (*b*) Prove that $0 \leqq \sin x \leqq 2x/\pi$ for $0 \leqq x \leqq \pi/2$.

4.74. (*a*) Prove that $\dfrac{\sin b - \sin a}{\cos a - \cos b} = \cot \xi$, where ξ is between a and b.
(*b*) By placing $a = 0$ and $b = x$ in (*a*), show that $\xi = x/2$. Does the result hold if $x < 0$?

L'HOSPITAL'S RULE

4.75. Evaluate each of the following limits.

(*a*) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$ (*e*) $\displaystyle\lim_{x\to 0+} x^3 \ln x$ (*i*) $\displaystyle\lim_{x\to 0}(1/x - \csc x)$ (*m*) $\displaystyle\lim_{x\to\infty} x \ln\left(\frac{x+3}{x-3}\right)$

(*b*) $\displaystyle\lim_{x\to 0} \frac{e^{2x} - 2e^x + 1}{\cos 3x - 2\cos 2x + \cos x}$ (*f*) $\displaystyle\lim_{x\to 0}(3^x - 2^x)/x$ (*j*) $\displaystyle\lim_{x\to 0} x^{\sin x}$ (*n*) $\displaystyle\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

(*c*) $\displaystyle\lim_{x\to 1+} (x^2 - 1)\tan \pi x/2$ (*g*) $\displaystyle\lim_{x\to\infty} (1 - 3/x)^{2x}$ (*k*) $\displaystyle\lim_{x\to 0}(1/x^2 - \cot^2 x)$ (*o*) $\displaystyle\lim_{x\to\infty} (x + e^x + e^{2x})^{1/x}$

(*d*) $\displaystyle\lim_{x\to\infty} x^3 e^{-2x}$ (*h*) $\displaystyle\lim_{x\to\infty} (1 + 2x)^{1/3x}$ (*l*) $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} x - \sin^{-1} x}{x(1 - \cos x)}$ (*p*) $\displaystyle\lim_{x\to 0+} (\sin x)^{1/\ln x}$

Ans. (*a*) $\frac{1}{6}$, (*b*) -1, (*c*) $-4/\pi$, (*d*) 0, (*e*) 0, (*f*) $\ln 3/2$, (*g*) e^{-6}, (*h*) 1, (*i*) 0, (*j*) 1, (*k*) $\frac{2}{3}$, (*l*) $\frac{1}{3}$, (*m*) 6, (*n*) $e^{-1/6}$, (*o*) e^2, (*p*) e

MISCELLANEOUS PROBLEMS

4.76. Prove that $\sqrt{\dfrac{1-x}{1+x}} < \dfrac{\ln(1+x)}{\sin^{-1} x} < 1$ if $0 < x < 1$.

4.77. If $\Delta f(x) = f(x + \Delta x) - f(x)$, (*a*) Prove that $\Delta\{\Delta f(x)\} = \Delta^2 f(x) = f(x + 2\Delta x) - 2f(x + \Delta x) + f(x)$, (*b*) derive an expression for $\Delta^n f(x)$ where n is any positive integer, (*c*) show that $\displaystyle\lim_{\Delta x\to 0} \frac{\Delta^n f(x)}{(\Delta x)^n} = f^{(n)}(x)$ if this limit exists.

4.78. Complete the analytic proof mentioned at the end of Problem 4.36.

4.79. Find the relative maximum and minima of $f(x) = x^2$, $x > 0$.
Ans. $f(x)$ has a relative minimum when $x = e^{-1}$.

4.80. A train moves according to the rule $x = 5t^3 + 30t$, where t and x are measured in hours and miles, respectively. (*a*) What is the acceleration after 1 minute? (*b*) What is the speed after 2 hours?

4.81. A stone thrown vertically upward has the law of motion $x = -16t^2 + 96t$. (Assume that the stone is at ground level at $t = 0$, that t is measured in seconds, and that x is measured in feet.) (*a*) What is the height of the stone at $t = 2$ seconds? (*b*) To what height does the stone rise? (*c*) What is the initial velocity, and what is the maximum speed attained?

4.82. A particle travels with constant velocities v_1 and v_2 in mediums I and II, respectively (see adjoining Fig. 4-11). Show that in order to go from point P to point Q in the least time, it must follow path PAQ where A is such that

$$(\sin\theta_1)/(\sin\theta_2) = v_1/v_2$$

Note: This is Snell's Law; a fundamental law of optics first discovered experimentally and then derived mathematically.

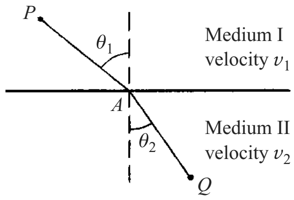

Fig. 4-11

4.83. A variable α is called an *infinitesimal* if it has zero as a limit. Given two infinitesimals α and β, we say that α is an infinitesimal of *higher order* (or the *same order*) if $\lim \alpha/\beta = 0$ (or $\lim \alpha/\beta = l \neq 0$). Prove that as $x \to 0$, (*a*) $\sin^2 2x$ and $(1 - \cos 3x)$ are infinitesimals of the same order, (*b*) $(x^3 - \sin^3 x)$ is an infinitesimal of higher order than $\{x - \ln(1+x) - 1 + \cos x\}$.

4.84. Why can we not use L'Hospital's rule to prove that $\lim\limits_{x\to 0} \dfrac{x^2 \sin 1/x}{\sin x} = 0$ (see Problem 3.91, Chap. 3)?

4.85. Can we use L'Hospital's rule to evaluate the limit of the sequence $u_n = n^3 e^{-n^2}$, $n = 1, 2, 3, \ldots$? Explain.

4.86 (1) Determine decimal approximations with at least three places of accuracy for each of the following irrational numbers. (*a*) $\sqrt{2}$, (*b*) $\sqrt{5}$, (*c*) $7^{1/3}$
(2) The cubic equation $x^3 - 3x^2 + x - 4 = 0$ has a root between 3 and 4. Use Newton's Method to determine it to at least three places of accuracy.

4.87. Using successive applications of Newton's method obtain the positive root of (*a*) $x^3 - 2x^2 - 2x - 7 = 0$, (*b*) $5 \sin x = 4x$ to 3 decimal places.
Ans. (*a*) 3.268, (*b*) 1.131

4.88. If D denotes the operator d/dx so that $Dy \equiv dy/dx$ while $D^k y \equiv d^k y/dx^k$, prove *Leibnitz's formula*

$$D^n(uv) = (D^n u)v + {}_nC_1(D^{n-1}u)(Dv) + {}_nC_2(D^{n-2}u)(D^2 v) + \cdots + {}_nC_r(D^{n-r}u)(D^r v) + \cdots + uD^n v$$

where ${}_nC_r = \binom{n}{r}$ are the binomial coefficients (see Problem 1.95, Chapter 1).

4.89. Prove that $\dfrac{d^n}{dx^n}(x^2 \sin x) = \{x^2 - n(n-1)\}\sin(x + n\pi/2) - 2nx\cos(x + n\pi/2)$.

4.90. If $f'(x_0) = f''(x_0) = \cdots = f^{(2n)}(x_0) = 0$ but $f^{(2n+1)}(x_0) \neq 0$, discuss the behavior of $f(x)$ in the neighborhood of $x = x_0$. The point x_0 in such case is often called a *point of inflection*. This is a generalization of the previously discussed case corresponding to $n = 1$.

4.91. Let $f(x)$ be twice differentiable in (a, b) and suppose that $f'(a) = f'(b) = 0$. Prove that there exists at least one point ξ in (a, b) such that $|f''(\xi)| \geqq \dfrac{4}{(b-a)^2}\{f(b) - f(a)\}$. Give a physical interpretation involving velocity and acceration of a particle.