Chapter 12

The Derivative

The expression for the slope of the tangent line

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

determines a number which depends on x. Thus, the expression defines a function, called the *derivative* of f.

Definition: The derivative f' of f is the function defined by the formula

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

NOTATION There are other notations traditionally used for the derivative:

$$D_x f(x) \quad \text{and} \quad \frac{dy}{dx}$$

When a variable y represents $f(x)$, the derivative is denoted by y', $D_x y$, or $\frac{dy}{dx}$. We shall use whichever notation is most convenient or customary in a given case.

The derivative is so important in all parts of pure and applied mathematics that we must devote a great deal of effort to finding formulas for the derivatives of various kinds of functions. If the limit in the above definition exists, the function f is said to be *differentiable* at x, and the process of calculating f' is called *differentiation* of f.

EXAMPLES

(*a*) Let $f(x) = 3x + 5$ for all x. Then,

$$f(x+h) = 3(x+h)+5 = 3x+3h+5$$
$$f(x+h)-f(x) = (3x+3h+5)-(3x+5) = 3x+3h+5-3x-5 = 3h$$
$$\frac{f(x+h)-f(x)}{h} = \frac{3h}{h} = 3$$

Hence,
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0} 3 = 3$$

or, in another notation, $D_x(3x+5) = 3$. In this case, the derivative is independent of x.

(*b*) Let us generalize to the case of the function $f(x) = Ax + B$, where A and B are constants. Then,

$$\frac{f(x+h)-f(x)}{h}=\frac{[A(x+h)+B]-(Ax+B)}{h}=\frac{Ax+Ah+B-Ax-B}{h}=\frac{Ah}{h}=A$$

So,
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0} A = A$$

Thus, we have proved:

Theorem 12.1: $D_x(Ax + B) = A$

By letting $A = 0$ in Theorem 12.1, we obtain:

Corollary 12.2: $D_x(B) = 0$; that is, *the derivative of a constant function is 0.*

Letting $A = 1$ and $B = 0$ in Theorem 12.1, we obtain:

Corollary 12.3: $D_x(x) = 1$

By the computations in Problems 11.1, 11.2, and 11.3(*a*), we have:

Theorem 12.4:
(i) $D_x(x^2) = 2x$
(ii) $D_x(x^3) = 3x^2$
(iii) $D_x\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

We shall need to know how to differentiate functions built up by arithmetic operations on simpler functions. For this purpose, several rules of differentiation will be proved.

RULE 1.
(i) $D_x(f(x) + g(x)) = D_x f(x) + D_x g(x)$
The derivative of a sum is the sum of the derivatives.
(ii) $D_x(f(x) - g(x)) = D_x f(x) - D_x g(x)$
The derivative of a difference is the difference of the derivatives.

For proofs of (i) and (ii), see Problem 12.1(*a*).

EXAMPLES

(*a*) $D_x(x^3 + x^2) = D_x(x^3) + D_x(x^2) = 3x^2 + 2x$

(*b*) $D_x\left(x^2 - \frac{1}{x}\right) = D_x(x^2) - D_x\left(\frac{1}{x}\right) = 2x - \left(-\frac{1}{x^2}\right) = 2x + \frac{1}{x^2}$

RULE 2. $D_x(c \cdot f(x)) = c \cdot D_x f(x)$
where c is a constant.

For a proof, see Problem 12.1(*b*).

EXAMPLES

(*a*) $D_x(7x^2) = 7 \cdot D_x(x^2) = 7 \cdot 2x = 14x$

(*b*) $D_x(12x^3) = 12 \cdot D_x(x^3) = 12(3x^2) = 36x^2$

(*c*) $D_x\left(-\frac{4}{x}\right) = D_x\left((-4)\frac{1}{x}\right) = -4 \cdot D_x\left(\frac{1}{x}\right) = -4\left(-\frac{1}{x^2}\right) = \frac{4}{x^2}$

(*d*)
$$\begin{aligned} D_x(3x^3 + 5x^2 + 2x + 4) &= D_x(3x^3) + D_x(5x^2) + D_x(2x) + D_x(4) \\ &= 3 \cdot D_x(x^3) + 5 \cdot D_x(x^2) + 2 \cdot D_x(x) + 0 \\ &= 3(3x^2) + 5(2x) + 2(1) = 9x^2 + 10x + 2 \end{aligned}$$

RULE 3 (Product Rule). $D_x(f(x) \cdot g(x)) = f(x) \cdot D_x g(x) + g(x) \cdot D_x f(x)$

For a proof, see Problem 13.1.

EXAMPLES

(*a*) $D_x(x^4) = D_x(x^3 \cdot x)$

ALGEBRA	$u^a \cdot u^b = u^{a+b}$ and	$\dfrac{u^a}{u^b} = u^{a-b}$

$= x^3 \cdot D_x(x) + x \cdot D_x(x^3)$ [by the product rule]

$= x^3(1) + x(3x^2) = x^3 + 3x^3 = 4x^3$

(*b*) $D_x(x^5) = D_x(x^4 \cdot x)$

$= x^4 \cdot D_x(x) + x \cdot D_x(x^4)$ [by the product rule]

$= x^4(1) + x(4x^3)$ [by example (*a*)]

$= x^4 + 4x^4 = 5x^4$

(*c*) $D_x((x^3 + x)(x^2 - x + 2)) = (x^3 + x) \cdot D_x(x^2 - x + 2) + (x^2 - x + 2) \cdot D_x(x^3 + x)$

$= (x^3 + x)(2x - 1) + (x^2 - x + 2)(3x^2 + 1)$

The reader may have noticed a pattern in the derivatives of the powers of x:

$$D_x(x) = 1 = 1 \cdot x^0 \qquad D_x(x^2) = 2x \qquad D_x(x^3) = 3x^2 \qquad D_x(x^4) = 4x^3 \qquad D_x(x^5) = 5x^4$$

This pattern does in fact hold for all powers of x.

RULE 4. $D_x(x^n) = nx^{n-1}$
where n is any positive integer.

For a proof, see Problem 12.2.

EXAMPLES

(*a*) $D_x(x^9) = 9x^8$

(*b*) $D_x(5x^{11}) = 5 \cdot D_x(x^{11}) = 5(11x^{10}) = 55x^{10}$

Using Rules 1, 2, and 4, we have an easy method for differentiating any polynomial.

EXAMPLE

$$D_x\left(\frac{3}{5}x^3 - 4x^2 + 2x - \frac{1}{2}\right) = D_x\left(\frac{3}{5}x^3\right) - D_x(4x^2) + D_x(2x) - D_x\left(\frac{1}{2}\right) \quad \text{[by Rule 1]}$$

$$= \frac{3}{5} \cdot D_x(x^3) - 4 \cdot D_x(x^2) + 2 \cdot D_x(x) - 0 \quad \text{[by Rule 2 and Corollary 12.2]}$$

$$= \frac{3}{5} \cdot (3x^2) - 4 \cdot (2x) + 2 \cdot (1) \quad \text{[by Rule 4]}$$

$$= \frac{9}{5}x^2 - 8x + 2$$

More concisely, we have:

RULE 5. To differentiate a polynomial, change each nonconstant term $a_k x^k$ to $ka_k x^{k-1}$ and drop the constant term (if any).

EXAMPLES

(*a*) $D_x(8x^5 - 2x^4 + 3x^2 + 5x + 7) = 40x^4 - 8x^3 + 6x + 5$

(*b*) $D_x\left(3x^7 + \sqrt{2}x^5 - \frac{4}{3}x^2 + 9x - \pi\right) = 21x^6 + 5\sqrt{2}x^4 - \frac{8}{3}x + 9$

Solved Problems

12.1 Prove: (*a*) Rule 1(i, ii); (*b*) Rule 2. Assume that $D_x f(x)$ and $D_x g(x)$ are defined.

(*a*)
$$\begin{aligned} D_x(f(x) \pm g(x)) &= \lim_{h\to 0} \frac{[f(x+h) \pm g(x+h)] - [f(x) \pm g(x)]}{h} \\ &= \lim_{h\to 0} \frac{[f(x+h) - f(x)] \pm [g(x+h) - g(x)]}{h} \\ &= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} \pm \frac{g(x+h)-g(x)}{h}\right] \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \pm \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \qquad \text{[by Section 8.2, Property V]} \\ &= D_x f(x) \pm D_x g(x) \end{aligned}$$

(*b*)
$$\begin{aligned} D_x(c \cdot f(x)) &= \lim_{h\to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} = \lim_{h\to 0} \frac{c[f(x+h) - f(x)]}{h} \\ &= c \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad \text{[by Section 8.2, Property III]} \\ &= c \cdot D_x f(x) \end{aligned}$$

12.2 Prove Rule 4, $D_x(x^n) = nx^{n-1}$, for any positive integer n.

We already know that Rule 4 holds when $n = 1$,

$$D_x(x^1) = D_x(x) = 1 = 1 \cdot x^0$$

(Remember that $x^0 = 1$.) We can prove the rule by mathematical induction. This involves showing that *if the rule holds for any particular positive integer k, then the rule also must hold for the next integer k + 1.* Since we know that the rule holds for $n = 1$, it would then follow that it holds for all positive integers.

Assume, then, that $D_x(x^k) = kx^{k-1}$. We have

$$\begin{aligned} D_x(x^{k+1}) &= D_x(x^k \cdot x) && [\text{since } x^{k+1} = x^k \cdot x^1 = x^k \cdot x] \\ &= x^k \cdot D_x(x) + x \cdot D_x(x^k) && [\text{by the product rule}] \\ &= x^k \cdot 1 + x(kx^{k-1}) && [\text{by the assumption that } D_x(x^k) = kx^{k-1}] \\ &= x^k + kx^k && [\text{since } x \cdot x^{k-1} = x^1 \cdot x^{k-1} = x^k] \\ &= (1+k)x^k = (k+1)x^{(k+1)-1} \end{aligned}$$

and the proof by induction is complete.

12.3 Find the derivative of the polynomial $5x^9 - 12x^6 + 4x^5 - 3x^2 + x - 2$.

By Rule 5,

$$D_x(5x^9 - 12x^6 + 4x^5 - 3x^2 + x - 2) = 45x^8 - 72x^5 + 20x^4 - 6x + 1$$

12.4 Find the slope-intercept equations of the tangent lines to the graphs of the following functions at the given points:

(*a*) $f(x) = 3x^2 - 5x + 1$, at $x = 2$ (*b*) $f(x) = x^7 - 12x^4 + 2x$, at $x = 1$

(*a*) For $f(x) = 3x^2 - 5x + 1$, Rule 5 gives $f'(x) = 6x - 5$. Then,

$$f'(2) = 6(2) - 5 = 12 - 5 = 7$$

and

$$f(2) = 3(2)^2 - 5(2) + 1 = 3(4) - 10 + 1 = 12 - 9 = 3$$

Thus, the slope of the tangent line to the graph at $(2, f(2)) = (2, 3)$ is $f'(2) = 7$, and we have as a point-slope equation of the tangent line $y - 3 = 7(x - 2)$, from which we get

$$\begin{aligned} y - 3 &= 7x - 14 \\ y &= 7x - 11 \end{aligned}$$

(*b*) For $f(x) = x^7 - 12x^4 + 2x$, Rule 5 yields $f'(x) = 7x^6 - 48x^3 + 2$. Now

$$f(1) = (1)^7 - 12(1^4) + 2(1) = 1 - 12 + 2 = -9$$

and
$$f'(1) = 7(1)^6 - 48(1)^3 + 2 = 7 - 48 + 2 = -39$$

Thus, the slope of the tangent line at $x = 1$ is -39, and a point-slope equation of the tangent line is $y - (-9) = -39(x - 1)$, whence

$$y + 9 = -39x + 39$$
$$y = -39x + 30$$

12.5 **At what point(s) of the graph of $y = x^5 + 4x - 3$ does the tangent line to the graph also pass through the point (0, 1)?**

The slope of the tangent line at a point $(x_0, y_0) = (x_0, x_0^5 + 4x_0 - 3)$ of the graph is the value of the derivative dy/dx at $x = x_0$. By Rule 5,

$$\frac{dy}{dx} = 5x^4 + 4 \qquad \text{and so} \qquad \left.\frac{dy}{dx}\right|_{x=x_0} = 5x_0^4 + 4$$

NOTATION The value of a function g at an argument $x = b$ is sometimes denoted by $g(x)|_{x=b}$. For example, $x^2|_{x=3} = (3)^2 = 9$.

The tangent line $\mathcal{T}$ to the graph at (x_0, y_0) goes through (0, 1) if and only if $\mathcal{T}$ is the line $\mathcal{L}$ that connects (x_0, y_0) and (0, 1). But that is true if and only if the slope $m_{\mathcal{T}}$ of $\mathcal{T}$ is the same as the slope $m_{\mathcal{L}}$ of $\mathcal{L}$. Now $m_{\mathcal{T}} = 5x_0^4 + 4$ and $m_{\mathcal{L}} = \dfrac{(x_0^5 + 4x_0 - 3) - 1}{x_0 - 0} = \dfrac{x_0^5 + 4x_0 - 4}{x_0}$. Thus, we must solve

$$5x_0^4 + 4 = \frac{x_0^5 + 4x_0 - 4}{x_0}$$
$$5x_0^5 + 4x_0 = x_0^5 + 4x_0 - 4$$
$$4x_0^5 = -4$$
$$x_0^5 = -1$$
$$x_0 = -1$$

Thus, the required point of the graph is

$$(-1, (-1)^5 + 4(-1) - 3) = (-1, -1 - 4 - 3) = (-1, -8).$$

Supplementary Problems

12.6 Use the basic definition of $f'(x)$ as a limit to calculate the derivatives of the following functions:

(*a*) $f(x) = 2x - 5$ (*b*) $f(x) = \frac{1}{3}x^2 - 7x + 4$ (*c*) $f(x) = 2x^3 + 3x - 1$

(*d*) $f(x) = x^4$ (*e*) $f(x) = \dfrac{1}{2 - x}$ (*f*) $f(x) = \dfrac{x}{x + 2}$ (*g*) $f(x) = \dfrac{1}{x^2 + 1}$

12.7 Use Rule 5 to find the derivatives of the following polynomials:

(*a*) $3x^3 - 4x^2 + 5x - 2$ (*b*) $-8x^5 + \sqrt{3}x^3 + 2\pi x^2 - 12$

(*c*) $3x^{13} - 5x^{10} + 10x^2$ (*d*) $2x^{51} + 3x^{12} - 14x^2 + \sqrt[3]{7}x + \sqrt{5}$

12.8 (a) Find $D_x\left(3x^7 - \frac{1}{5}x^5\right)$. (b) Find $\frac{d(3x^2 - 5x + 1)}{dx}$. (c) If $y = \frac{1}{2}x^4 + 5x$, find $\frac{dy}{dx}$.

(d) Find $\frac{d(3t^7 - 12t^2)}{dt}$. (e) If $u = \sqrt{2}x^5 - x^3$, find $D_x u$.

12.9 Find slope-intercept equations of the tangent lines to the graphs of the following functions f at the specified points:

(a) $f(x) = x^2 - 5x + 2$, at $x = -1$ (b) $f(x) = 4x^3 - 7x^2$, at $x = 3$

(c) $f(x) = -x^4 + 2x^2 + 3$, at $x = 0$

12.10 Specify all straight lines that satisfy the following conditions:

(a) Through the point (0, 2) and tangent to the curve $y = x^4 - 12x + 50$.

(b) Through the point (1, 5) and tangent to the curve $y = 3x^3 + x + 4$.

12.11 Find the slope-intercept equation of the normal line to the graph of $y = x^3 - x^2$ at the point where $x = 1$.

12.12 Find the point(s) on the graph of $y = \frac{1}{2}x^2$ at which the normal line passes through the point (4, 1).

12.13 Recalling the definition of the derivative, evaluate

(a) $\lim_{h \to 0} \frac{(3 + h)^5 - 3^5}{h}$ (b) $\lim_{h \to 0} \frac{5(\frac{1}{3} + h)^4 - 5(\frac{1}{3})^4}{h}$

12.14 A function f, defined for all real numbers, is such that: (i) $f(1) = 2$; (ii) $f(2) = 8$; (iii) $f(u + v) - f(u) = kuv - 2v^2$ for all u and v, where k is some constant. Find $f'(x)$ for arbitrary x.

12.15 Let $f(x) = 2x^2 + \sqrt{3}x$ for all x.

(a) Find the nonnegative value(s) of x for which the tangent line to the graph of f at $(x, f(x))$ is perpendicular to the tangent line to the graph at $(-x, f(-x))$.

(b) Find the point of intersection of each pair of perpendicular lines found in part (a).

12.16 If the line $4x - 9y = 0$ is tangent in the first quadrant to the graph of $y = \frac{1}{3}x^3 + c$, what is the value of c?

12.17 For what nonnegative value of b is the line $y = -\frac{1}{12}x + b$ normal to the graph of $y = x^3 + \frac{1}{3}$?

12.18 Let f be differentiable (that is, f' exists). Define a function $f^{\#}$ by the equation

$$f^{\#}(x) = \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{h}$$

(a) Find $f^{\#}(x)$ if $f(x) = x^2 - x$. (b) Find the relationship between $f^{\#}$ and the derivative f'.

$$\left[\textit{Hint: } \frac{f(x + h) - f(x - h)}{h} = \frac{f(x + h) - f(x)}{h} + \frac{f(x + k) - f(x)}{k} \text{ where } k = -h.\right]$$

12.19 Let $f(x) = x^3 + x^2 - 9x - 9$.

(a) Find the zeros of f.

ALGEBRA If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where the a_i's are integers, then any integer root k of $f(x)$ must be a divisor of the constant term a_0.[1]

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[1] If $0 = f(k) = a_n k^n + a_{n-1}k^{n-1} + \cdots + a_1 k + a_0$, then

$$a_0 = -(a_n k^n + a_{n-1}k^{n-1} + \cdots + a_1 k) = -k(a_n k^{n-1} + a_{n-1}k^{n-2} + \cdots + a_1).$$

(*b*) Find the slope-intercept equation of the tangent line to the graph of f at the point where $x = 1$.

(*c*) Find a point (x_0, y_0) on the graph of f such that the tangent line to the graph at (x_0, y_0) passes through the point $(4, -1)$.

12.20 Let $f(x) = 3x^3 - 11x^2 - 15x + 63$.

(*a*) Find the zeros of f.

(*b*) Write an equation of the line normal to the graph of f at $x = 0$.

(*c*) Find all points on the graph of f where the tangent line to the graph is horizontal.

12.21 Define $f'_+(x)$, the *right-hand derivative* of f at x, to be $\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$, and $f'_-(x)$, the *left-hand derivative* of f and x, to be $\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$. Show that the derivative $f'(x)$ exists if and only if both $f'_+(x)$ and $f'_-(x)$ exist and are equal.

12.22 Determine whether the following functions are differentiable at the given argument. [*Hint:* Use Problem 12.21.]

(*a*) The function f of Problem 10.5(*a*) at $x = 0$.

(*b*) $g(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$, at $x = 0$

(*c*) $h(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ x & \text{if } x > 1 \end{cases}$, at $x = 1$

(*d*) $G(x) = \begin{cases} x^3 & \text{if } x \leq 1 \\ 4x - 3 & \text{if } x > 1 \end{cases}$, at $x = 1$

12.23 Let $f(x) = \begin{cases} x^2 & \text{if } x > 3 \\ Ax - 1 & \text{if } x \leq 3 \end{cases}$.

(*a*) Find the value of A for which f is continuous at $x = 3$.

(*b*) For the value of A found in (*a*), is f differentiable at $x = 3$?

12.24 Use the original definition to find the derivative of the function f such that $f(x) = \frac{1}{\sqrt{x} - 1}$.

Chapter 13

More on the Derivative

13.1 DIFFERENTIABILITY AND CONTINUITY

In the formula $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ for the derivative $f'(a)$, we can let $x = a + h$ and rewrite $f'(a)$ as $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$. If f is differentiable at a, then

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a} [f(x)-f(a)+f(a)] \\ &= \lim_{x\to a} [f(x)-f(a)] + \lim_{x\to a} f(a) \\ &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\cdot(x-a)+f(a) \\ &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\cdot \lim_{x\to a}(x-a)+f(a) \\ &= f'(a)\cdot 0 + f(a) = f(a)\end{aligned}$$

Thus, f is continuous at a. This proves:

Theorem 13.1: If f is differentiable at a, then f is continuous at a.

EXAMPLE Differentiability is a *stronger condition* than continuity. In other words, the converse of Theorem 13.1 is not true. To see this, consider the absolute-value function $f(x) = |x|$ (see Fig. 13-1). f is obviously continuous at $x = 0$; but it is not differentiable at $x = 0$. In fact,

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h-0}{h} = \lim_{h\to 0^+} 1 = 1$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h-0}{h} = \lim_{h\to 0^-} -1 = -1$$

and so the two-sided limit needed to define $f'(0)$ does not exist. (The sharp corner in the graph is a tip-off. Where there is no unique tangent line, there can be no derivative.)

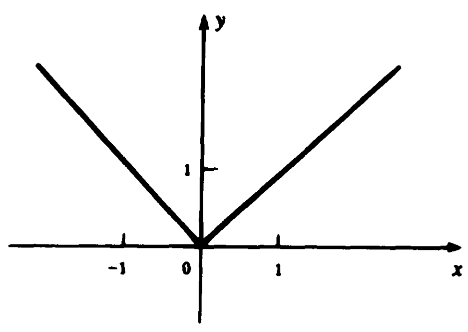


Fig. 13-1

13.2 FURTHER RULES FOR DERIVATIVES

Theorem 13.1 will enable us to justify Rule 3, the product rule, of Chapter 12 and to establish two additional rules.

RULE 6 (Quotient Rule). If f and g are differentiable at x and if $g(x) \neq 0$, then

$$D_x\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot D_x f(x) - f(x) \cdot D_x g(x)}{[g(x)]^2}$$

For a proof, see Problem 13.2.

EXAMPLES

(a) $$D_x\left(\frac{x+1}{x^2-2}\right) = \frac{(x^2-2) \cdot D_x(x+1) - (x+1) \cdot D_x(x^2-2)}{(x^2-2)^2}$$
$$= \frac{(x^2-2)(1) - (x+1)(2x)}{(x^2-2)^2} = \frac{x^2 - 2 - 2x^2 - 2x}{(x^2-2)^2}$$
$$= \frac{-x^2 - 2x - 2}{(x^2-2)^2} = -\frac{x^2 + 2x + 2}{(x^2-2)^2}$$

(b) $$D_x\left(\frac{1}{x^2}\right) = \frac{x^2 \cdot D_x(1) - 1 \cdot D_x(x^2)}{(x^2)^2} = \frac{x^2(0) - 1(2x)}{x^4} = -\frac{2x}{x^4} = -\frac{2}{x^3}$$

The quotient rule allows us to extend Rule 4 of Chapter 12:

RULE 7. $D_x(x^k) = kx^{k-1}$ for any integer k (positive, zero, or negative).

For a proof, see Problem 13.3.

EXAMPLES

(a) $$D_x\left(\frac{1}{x}\right) = D_x(x^{-1}) = (-1)x^{-2} = (-1)\frac{1}{x^2} = -\frac{1}{x^2}$$

(b) $$D_x\left(\frac{1}{x^2}\right) = D_x(x^{-2}) = -2x^{-3} = -\frac{2}{x^3}$$

Solved Problems

13.1 Prove Rule 3, the product rule: If f and g are differentiable at x, then

$$D_x(f(x) \cdot g(x)) = f(x) \cdot D_x g(x) + g(x) \cdot D_x f(x)$$

By simple algebra,

$$f(x+h)g(x+h) - f(x)g(x) = f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]$$

Hence,
$$D_x(f(x) \cdot g(x)) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h\to 0} \frac{g(x)[f(x+h) - f(x)]}{h}$$
$$= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= f(x) \cdot D_x g(x) + g(x) \cdot D_x f(x)$$

In the last step, $\lim_{h\to 0} f(x+h) = f(x)$ follows from the fact that f is continuous at x, by Theorem 13.1.

13.2 Prove Rule 6, the quotient rule: If f and g are differentiable at x and if $g(x) \neq 0$, then

$$D_x\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot D_x f(x) - f(x) \cdot D_x g(x)}{[g(x)]^2}$$

If $g(x) \neq 0$, then $1/g(x)$ is defined. Moreover, since g is continuous at x (by Theorem 13.1), $g(x + h) \neq 0$ for all sufficiently small values of h. Hence, $1/g(x + h)$ is defined for those same values of h. We may then calculate

$$\begin{aligned}
\lim_{h\to 0} \frac{\dfrac{1}{g(x+h)} - \dfrac{1}{g(x)}}{h} &= \lim_{h\to 0} \frac{g(x) - g(x+h)}{hg(x)g(x+h)} && \text{[by algebra: multiply top and bottom by } g(x)g(x+h)] \\
&= \lim_{h\to 0} \left[\frac{-1/g(x)}{g(x+h)}\right]\left[\frac{g(x+h) - g(x)}{h}\right] && \text{[by algebra]} \\
&= \lim_{h\to 0} \frac{-1/g(x)}{g(x+h)} \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} && \text{[by Property IV of limits]} \\
&= \frac{\lim_{h\to 0} [-1/g(x)]}{\lim_{h\to 0} g(x+h)} \cdot D_x g(x) && \text{[by Property VI of limits and differentiability of } g] \\
&= \frac{-1/g(x)}{g(x)} \cdot D_x g(x) && \text{[by Property II of limits and continuity of } g] \\
&= \frac{-1}{[g(x)]^2} D_x g(x)
\end{aligned}$$

Having thus proved that

$$D_x\left(\frac{1}{g(x)}\right) = \frac{-1}{[g(x)]^2} D_x g(x) \tag{1}$$

we may substitute in the product rule (proved in Problem 13.1) to obtain

$$\begin{aligned}
D_x\left(\frac{f(x)}{g(x)}\right) = D_x\left(f(x) \cdot \frac{1}{g(x)}\right) &= f(x) \frac{-1}{[g(x)]^2} D_x g(x) + \frac{1}{g(x)} D_x f(x) \\
&= \frac{-f(x)D_x g(x)}{[g(x)]^2} + \frac{g(x)D_x f(x)}{[g(x)]^2} \\
&= \frac{g(x)D_x f(x) - f(x)D_x g(x)}{[g(x)]^2}
\end{aligned}$$

which is the desired quotient rule.

13.3 Prove Rule 7: $D_x(x^k) = kx^{k-1}$ for any integer k.

When k is positive, this is just Rule 4 (Chapter 12). When $k = 0$,

$$D_x(x^k) = D_x(x^0) = D_x(1) = 0 = 0 \cdot x^{-1} = kx^{k-1}$$

Now assume k is negative; $k = -n$, where n is positive.

ALGEBRA By definition,

$$x^k = x^{-n} = \frac{1}{x^n}$$

By (*1*) of Problem 13.2,

$$D_x(x^k) = D_x\left(\frac{1}{x^n}\right) = \frac{-1}{(x^n)^2} D_x(x^n)$$

But $(x^n)^2 = x^{2n}$ and, by Rule 4, $D_x(x^n) = nx^{n-1}$. Therefore,

$$D_x(x^k) = \frac{-1}{x^{2n}} nx^{n-1} = -nx^{(n-1)-2n} = -nx^{-n-1} = kx^{k-1}$$

ALGEBRA We have used the law of exponents,

$$\frac{x^a}{x^b} = x^{a-b}$$

13.4 Find the derivative of the function f such that $f(x) = \dfrac{x^2 + x - 2}{x^3 + 4}$.

Use the quotient rule and then Rule 5 (Chapter 12),

$$D_x\left(\frac{x^2 + x - 2}{x^3 + 4}\right) = \frac{(x^3 + 4)D_x(x^2 + x - 2) - (x^2 + x - 2)D_x(x^3 + 4)}{(x^3 + 4)^2}$$

$$= \frac{(x^3 + 4)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 4)^2}$$

$$= \frac{(2x^4 + x^3 + 8x + 4) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 4)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 8x + 4}{(x^3 + 4)^2}$$

13.5 Find the slope-intercept equation of the tangent line to the graph of $y = 1/x^3$ when $x = \frac{1}{2}$.

The slope of the tangent line is the derivative

$$\frac{dy}{dx} = D_x\left(\frac{1}{x^3}\right) = D_x(x^{-3}) = -3x^{-4} = -\frac{3}{x^4}$$

When $x = \frac{1}{2}$,

$$\left.\frac{dy}{dx}\right|_{x=1/2} = -\frac{3}{(\frac{1}{2})^4} = -\frac{3}{\frac{1}{16}} = -3(16) = -48$$

So, the tangent line has slope-intercept equation $y = -48x + b$. When $x = \frac{1}{2}$, the y-coordinate of the point on the graph is

$$\frac{1}{(\frac{1}{2})^3} = \frac{1}{\frac{1}{8}} = 8$$

Substituting 8 for y and $\frac{1}{2}$ for x in $y = -48x + b$, we have

$$8 = -48(\tfrac{1}{2}) + b \quad \text{or} \quad 8 = -24 + b \quad \text{or} \quad b = 32$$

Thus, the equation is $y = -48x + 32$.

Supplementary Problems

13.6 Find the derivatives of the functions defined by the following formulas:

(*a*) $(x^{100} + 2x^{50} - 3)(7x^8 + 20x + 5)$ (*b*) $\dfrac{x^2 - 3}{x + 4}$ (*c*) $\dfrac{x^5 - x + 2}{x^3 + 7}$

(*d*) $\dfrac{3}{x^5}$ (*e*) $8x^3 - x^2 + 5 - \dfrac{2}{x} + \dfrac{4}{x^3}$ (*f*) $\dfrac{3x^7 + x^5 - 2x^4 + x - 3}{x^4}$

13.7 Find the slope-intercept equation of the tangent line to the graph of the function at the indicated point:

(*a*) $f(x) = \dfrac{1}{x^2}$, at $x = 2$ (*b*) $f(x) = \dfrac{x+2}{x^3 - 1}$, at $x = -1$

13.8 Let $f(x) = \dfrac{x+2}{x-2}$ for all $x \neq 2$. Find $f'(-2)$.

13.9 Determine the points at which the function $f(x) = |x - 3|$ is differentiable.

13.10 The parabola in Fig. 13-2 is the graph of the function $f(x) = x^2 - 4x$.

(*a*) Draw the graph of $y = |f(x)|$. (*b*) Where does the derivative of $|f(x)|$ fail to exist?

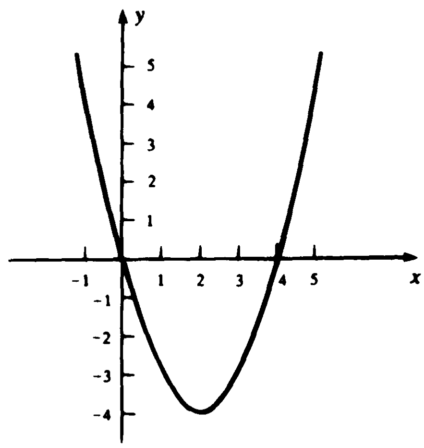


Fig. 13-2

13.11 GC Use a graphing calculator to find the discontinuities of the derivatives of the following functions:

(*a*) $f(x) = x^{2/3}$ (*b*) $f(x) = 4|x - 2| + 3$ (*c*) $f(x) = 2\sqrt{x-1}$

13.12 Evaluate $\lim_{h \to 0} \dfrac{1}{h}\left[\dfrac{1}{(1+h)^8} - 1\right]$.

Chapter 14

Maximum and Minimum Problems

14.1 RELATIVE EXTREMA

A function f is said to have a *relative maximum* at $x = c$ if

$$f(x) \leq f(c)$$

for all x near c. More precisely, f achieves a relative maximum at c if there exists $\delta > 0$ such that $|x - c| < \delta$ implies $f(x) \leq f(c)$.

EXAMPLE For the function f whose graph is shown in Fig. 14-1, relative maxima occur at $x = c_1$ and $x = c_2$. This is obvious, since the point A is higher than nearby points on the graph, and the point B is higher than nearby points on the graph.

The word "relative" is used to modify "maximum" because the value of a function at a relative maximum is not necessarily the greatest value of the function. Thus, in Fig. 14-1, the value $f(c_1)$ at c_1 is smaller than many other values of $f(x)$; in particular, $f(c_1) < f(c_2)$. In this example, the value $f(c_2)$ is the greatest value of the function.

A function f is said to have a *relative minimum* at $x = c$ if

$$f(x) \geq f(c)$$

for x near c. In Fig. 14-1, f achieves a relative minimum at $x = d$, since point D is lower than nearby points on the graph. The value at a relative minimum need not be the smallest value of the function; for example, in Fig. 14-1, the value $f(e)$ is smaller than $f(d)$.

By a *relative extremum* is meant either a relative maximum or a relative minimum. Points at which a relative extremum exists possess the following characteristic property.

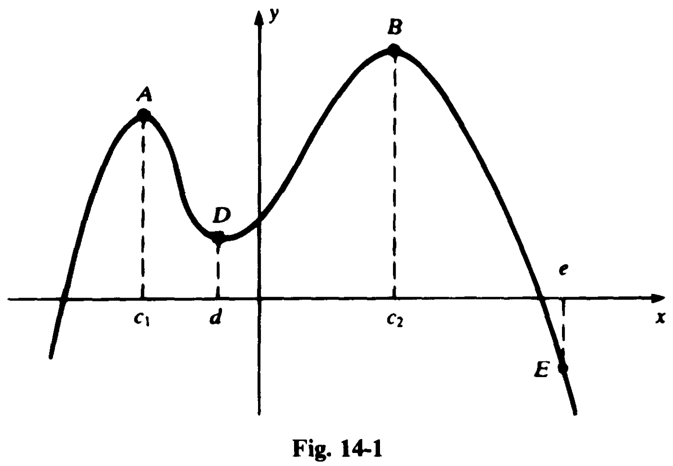


Fig. 14-1

Theorem 14.1: If f has a relative extremum at $x = c$ and if $f'(c)$ exists, then $f'(c) = 0$.

The theorem is intuitively obvious. If $f'(c)$ exists, then there is a well-defined tangent line at the point on the graph of f where $x = c$. But at a relative maximum or relative minimum, the tangent line is horizontal (see Fig. 14-2), and so its slope $f'(c)$ is zero. For a rigorous proof, see Problem 14.28.

The converse of Theorem 14.1 does not hold. If $f'(c) = 0$, then f need not have a relative extremum at $x = c$.

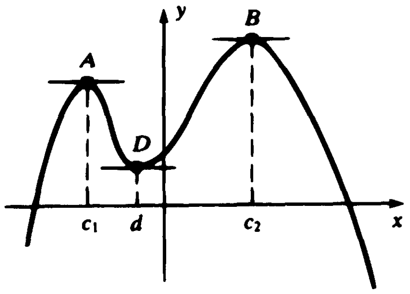


Fig. 14-2

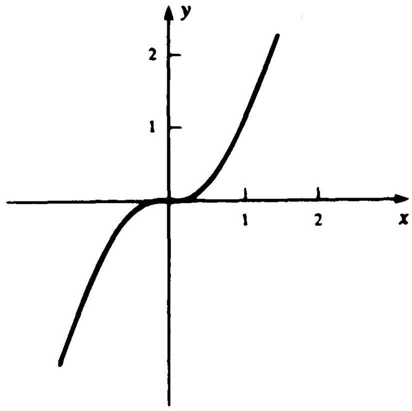


Fig. 14-3

EXAMPLE Consider the function $f(x) = x^3$. Because $f'(x) = 3x^2$, $f'(x) = 0$ if and only if $x = 0$. But from the graph of f in Fig. 14-3 it is clear that f has neither a relative maximum nor a relative minimum at $x = 0$.

In Chapter 23, a method will be given that often will enable us to determine whether a relative extremum actually exists when $f'(c) = 0$.

14.2 ABSOLUTE EXTREMA

Practical applications usually call for finding the *absolute* maximum or *absolute* minimum of a function on a given set. Let f be a function defined on a set $\mathscr{E}$ (and possibly at other points, too), and let c belong to $\mathscr{E}$. Then f is said to achieve an *absolute maximum on* $\mathscr{E}$ at c if $f(x) \le f(c)$ for all x in $\mathscr{E}$. Similarly, f is said to achieve an *absolute minimum on E* at d if $f(x) \ge f(d)$ for all x in $\mathscr{E}$.

If the set $\mathscr{E}$ is a closed interval $[a, b]$, and if the function f is continuous over $[a, b]$ (see Section 10.3), then we have a very important existence theorem (which cannot be proved in an elementary way).

Theorem 14.2 (*Extreme-Value Theorem*): Any continuous function f over a closed interval $[a, b]$ has an absolute maximum and an absolute minimum on $[a, b]$.

EXAMPLES

(*a*) Let $f(x) = x + 1$ for all x in the closed interval $[0, 2]$. The graph of f is shown in Fig. 14-4(*a*). Then f achieves an absolute maximum on $[0, 2]$ at $x = 2$; this absolute maximum value is 3. In addition, f achieves an absolute minimum at $x = 0$; this absolute minimum value is 1.

(*b*) Let $f(x) = 1/x$ for all x in the *open* interval $(0, 1)$. The graph of f is shown in Fig. 14-4(*b*). f has neither an absolute maximum nor an absolute minimum on $(0, 1)$. If we extended f to the half-open interval $(0, 1]$, then there is an absolute minimum at $x = 1$, but still no absolute maximum.

(*c*) Let $f(x) = \begin{cases} x + 1 & \text{if } -1 \le x < 0 \\ 0 & \text{if } x = 0 \\ x - 1 & \text{if } 0 < x \le 1 \end{cases}$

See Fig. 14-4(*c*) for the graph of f. f has neither an absolute maximum nor an absolute minimum on the closed interval $[-1, 1]$. Theorem 14.2 does not apply, because f is discontinuous at 0.

Critical Numbers

To actually locate the absolute extrema guaranteed by Theorem 14.2, it is useful to have the following notion.

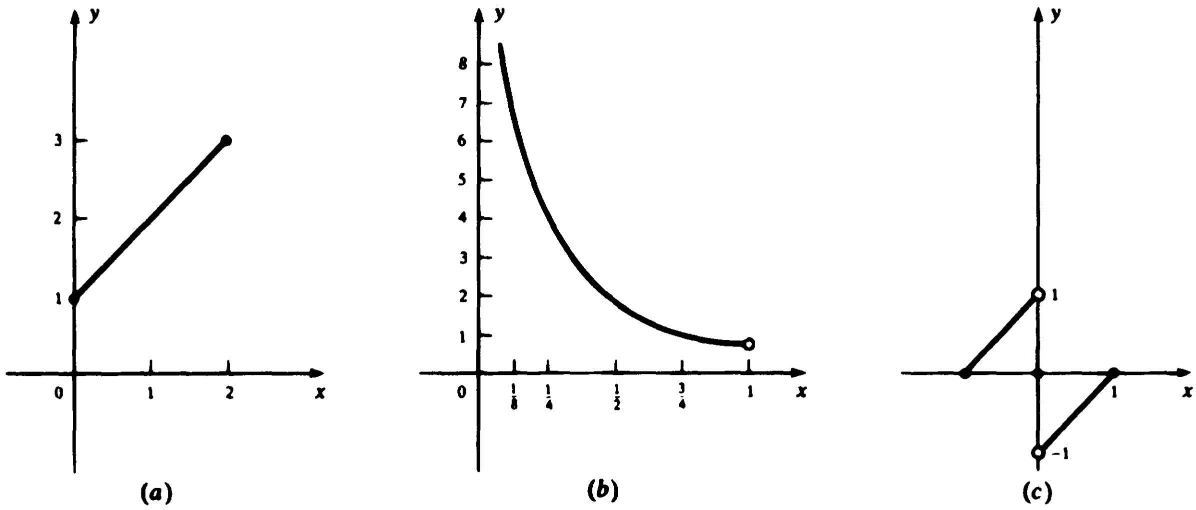


Fig. 14-4

Definition: A *critical number* of a function f is a number c in the domain of f for which either $f'(c) = 0$ or $f'(c)$ is not defined.

EXAMPLES

(*a*) Let $f(x) = 3x^2 - 2x + 4$. Then $f'(x) = 6x - 2$. Since $6x - 2$ is defined for all x, the only critical numbers are given by

$$\begin{aligned} 6x - 2 &= 0 \\ 6x &= 2 \\ x &= \tfrac{2}{6} = \tfrac{1}{3} \end{aligned}$$

Thus, the only critical number is $\frac{1}{3}$.

(*b*) Let $f(x) = x^3 - x^2 - 5x + 3$. Then $f'(x) = 3x^2 - 2x - 5$, and since $3x^2 - 2x - 5$ is defined for all x, the only critical numbers are the solutions of

$$\begin{aligned} 3x^2 - 2x - 5 &= 0 \\ (3x - 5)(x + 1) &= 0 \\ 3x - 5 = 0 \quad &\text{or} \quad x + 1 = 0 \\ 3x = 5 \quad &\text{or} \quad x = -1 \\ x = \tfrac{5}{3} \quad &\text{or} \quad x = -1 \end{aligned}$$

Hence, there are two critical numbers, -1 and $\frac{5}{3}$.

(*c*) Let $f(x) = |x|$. Thus, $f(x) = \begin{cases} x & \text{if } 0 \le x \\ -x & \text{if } x < 0 \end{cases}$.

We already know from the example in Section 13.1 that $f'(0)$ is not defined. Hence, 0 is a critical number. Since $D_x(x) = 1$ and $D_x(-x) = -1$, there are no other critical numbers.

Method for Finding Absolute Extrema

Let f be a continuous function on a closed interval $[a, b]$. Assume that there are only a finite number of critical numbers $c_1, c_2, \ldots, c_k$ of f *inside* $[a, b]$; that is, in (a, b). (This assumption holds for most functions encountered in calculus.) Tabulate the values of f at these critical numbers and at the endpoints a and b, as in Table 14-1. Then the largest tabulated value is the absolute maximum of f on $[a, b]$, and the smallest tabulated value is the absolute minimum of f on $[a, b]$. (This result is proved in Problem 14.1.)

Table 14-1

x	$f(x)$
c_1	$f(c_1)$
c_2	$f(c_2)$
$\vdots$	$\vdots$
c_k	$f(c_k)$
a	$f(a)$
b	$f(b)$

EXAMPLE Find the absolute maximum and minimum values of

$$f(x) = x^3 - 5x^2 + 3x + 1$$

on [0, 1] and find the arguments at which these values are achieved.

The function is continuous everywhere; in particular, on [0, 1]. Since $f'(x) = 3x^2 - 10x + 3$ is defined for all x, the only critical numbers are the solutions of

$$\begin{aligned} 3x^2 - 10x + 3 &= 0 \\ (3x-1)(x-3) &= 0 \\ 3x - 1 = 0 \quad &\text{or} \quad x - 3 = 0 \\ 3x = 1 \quad &\text{or} \quad x = 3 \\ x = \tfrac{1}{3} \quad &\text{or} \quad x = 3 \end{aligned}$$

Hence, the only critical number in the open interval (0, 1) is $\frac{1}{3}$. Now construct Table 14-2:

$$\begin{aligned} f\left(\frac{1}{3}\right) &= \left(\frac{1}{3}\right)^3 - 5\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right) + 1 = \frac{1}{27} - \frac{5}{9} + 1 + 1 \\ &= \frac{1}{27} - \frac{15}{27} + 2 = 2 - \frac{14}{27} = \frac{40}{27} \\ f(0) &= 0^3 - 5(0)^2 + 3(0) + 1 = 1 \\ f(1) &= 1^3 - 5(1)^2 + 3(1) + 1 = 1 - 5 + 3 + 1 = 0 \end{aligned}$$

The absolute maximum is the largest value in the second column, $\frac{40}{27}$, and it is achieved at $x = \frac{1}{3}$. The absolute minimum is the smallest value, 0, which is achieved at $x = 1$.

Table 14-2

x	$f(x)$
1/3	40/27
0	1
1	0

Solved Problems

14.1 Justify the tabular method for locating the absolute maximum and minimum of a function on a closed interval.

By the extreme-value theorem (Theorem 14.2), a function f continuous on $[a, b]$ must have an absolute maximum and an absolute minimum on $[a, b]$. Let p be an argument at which the absolute maximum is achieved.

Case 1: p is one of the endpoints, a or b. Then $f(p)$ will be one of the values in our table. In fact, it will be the largest value in the table, since $f(p)$ is the absolute maximum of f on $[a, b]$.

Case 2: p is not an endpoint and $f'(p)$ is not defined. Then p is a critical number and will be one of the numbers $c_1, c_2, \ldots, c_k$ in our list. Hence, $f(p)$ will appear as a tabulated value, and it will be the largest of the tabulated values.

Case 3: p is not an endpoint and $f'(p)$ is defined. Since p is the absolute maximum of f on $[a, b]$, $f(p) \geq f(x)$ for all x near p. Thus, f has a relative maximum at p, and Theorem 14.1 gives $f'(p) = 0$. But then p is a critical number, and the conclusion follows as in Case 2.

A completely analogous argument shows that the method yields the absolute minimum.

14.2 Find the absolute maximum and minimum of each function on the given interval:

(*a*) $f(x) = 2x^3 - 5x^2 + 4x - 1$ on $[-1, 2]$ (*b*) $f(x) = \dfrac{x^2 + 3}{x + 1}$ on $[0, 3]$

(*a*) Since $f'(x) = 6x^2 - 10x + 4$, the critical numbers are the solutions of:

$$6x^2 - 10x + 4 = 0$$
$$3x^2 - 5x + 2 = 0$$
$$(3x - 2)(x - 1) = 0$$
$$3x - 2 = 0 \quad \text{or} \quad x - 1 = 0$$
$$3x = 2 \quad \text{or} \quad x = 1$$
$$x = \tfrac{2}{3} \quad \text{or} \quad x = 1$$

Thus, the critical numbers are $\frac{2}{3}$ and 1, both of which are in $(-1, 2)$. Now construct Table 14-3:

$$f\left(\frac{2}{3}\right) = 2\left(\frac{2}{3}\right)^3 - 5\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right) - 1 = \frac{16}{27} - \frac{20}{9} + \frac{8}{3} - 1 = \frac{16}{27} - \frac{60}{27} + \frac{72}{27} - \frac{27}{27} = \frac{1}{27}$$
$$f(1) = 2(1)^3 - 5(1)^2 + 4(1) - 1 = 2 - 5 + 4 - 1 = 0$$
$$f(-1) = 2(-1)^3 - 5(-1)^2 + 4(-1) - 1 = -2 - 5 - 4 - 1 = -12$$
$$f(2) = 2(2)^3 - 5(2)^2 + 4(2) - 1 = 16 - 20 + 8 - 1 = 3$$

Thus, the absolute maximum is 3, achieved at $x = 2$, and the absolute minimum is -12, achieved at $x = -1$.

Table 14-3

x	$f(x)$
2/3	1/27
1	0
−1	−12 min
2	3 max

(*b*) $$f'(x) = \frac{(x + 1)D_x(x^2 + 3) - (x^2 + 3)D_x(x + 1)}{(x + 1)^2} = \frac{(x + 1)(2x) - (x^2 + 3)(1)}{(x + 1)^2}$$
$$= \frac{2x^2 + 2x - x^2 - 3}{(x + 1)^2} = \frac{x^2 + 2x - 3}{(x + 1)^2}$$

$f'(x)$ is not defined when $(x + 1)^2 = 0$; that is, when $x = -1$. But since -1 is not in $(0, 3)$, the only critical numbers that need to be considered are the zeros of $x^2 + 2x - 3$ in $(0, 3)$:

$$x^2 + 2x - 3 = 0$$
$$(x + 3)(x - 1) = 0$$
$$x + 3 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = -3 \quad \text{or} \quad x = 1$$

Thus, 1 is the only critical number in (0, 3).

Now construct Table 14-4:

$$f(1) = \frac{(1)^2 + 3}{1 + 1} = \frac{1 + 3}{2} = \frac{4}{2} = 2$$

$$f(0) = \frac{(0)^2 + 3}{0 + 1} = \frac{3}{1} = 3$$

$$f(3) = \frac{(3)^2 + 3}{3 + 1} = \frac{9 + 3}{4} = \frac{12}{4} = 3$$

Thus, the absolute maximum, achieved at 0 and 3, is 3; and the absolute minimum, achieved at 1, is 2.

Table 14-4

x	$f(x)$
1	2 min
0	3 max
3	3 max

14.3 Among all pairs of positive real numbers u and v whose sum is 10, which gives the greatest product uv?

Let $P = uv$. Since $u + v = 10$, $v = 10 - u$, and so

$$P = u(10 - u) = 10u - u^2$$

Here, $0 < u < 10$. But since P would take the value 0 at $u = 0$ and $u = 10$, and 0 is clearly not the absolute maximum of P, we can extend the domain of P to the *closed* interval [0, 10]. Thus, we must find the absolute maximum of $P = 10u - u^2$ on the closed interval [0, 10]. The derivative $dP/du = 10 - 2u$ vanishes only at $u = 5$, and this critical point must yield the maximum. Thus, the absolute maximum is $P(5) = 5(10 - 5) = 5(5) = 25$, which is attained for $u = 5$. When $u = 5$, $v = 10 - u = 10 - 5 = 5$.

ALGEBRA Calculus was not really needed in this problem, for $P = \frac{(u + v)^2 - (u - v)^2}{4} = \frac{10^2 - (u - v)^2}{4}$, which is largest when $u - v = 0$, that is, when $u = v$. Then $10 = u + v = 2u$ and, therefore, $u = 5$.

14.4 An open box is to be made from a rectangular piece of cardboard that is 8 feet by 3 feet by cutting out four equal squares from the corners and then folding up the flaps (see Fig. 14-5). What length of the side of a square will yield the box with the largest volume?

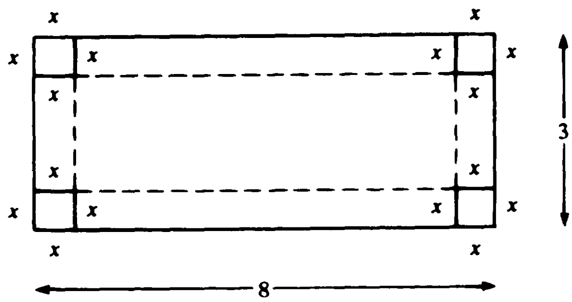


Fig. 14-5

Let x be the side of the square that is removed from each corner. The volume $V = lwh$, where l, w, and h are the length, width, and height of the box. Now $l = 8 - 2x$, $w = 3 - 2x$, and $h = x$, giving

$$V(x) = (8 - 2x)(3 - 2x)x = (4x^2 - 22x + 24)x = 4x^3 - 22x^2 + 24x$$

The width w must be positive. Hence,

$$3 - 2x > 0 \quad \text{or} \quad 3 > 2x \quad \text{or} \quad \tfrac{3}{2} > x$$

Furthermore, $x > 0$. But we also can admit the values $x = 0$ and $x = \frac{3}{2}$, which make $V = 0$ and which, therefore, cannot yield the maximum volume. Thus, we have to maximize $V(x)$ on the interval $[0, \frac{3}{2}]$. Since

$$\frac{dV}{dx} = 12x^2 - 44x + 24$$

the critical numbers are the solutions of

$$\begin{aligned} 12x^2 - 44x + 24 &= 0 \\ 3x^2 - 11x + 6 &= 0 \\ (3x - 2)(x - 3) &= 0 \\ 3x - 2 = 0 \quad &\text{or} \quad x - 3 = 0 \\ 3x = 2 \quad &\text{or} \quad x = 3 \\ x = \tfrac{2}{3} \quad &\text{or} \quad x = 3 \end{aligned}$$

The only critical number in $(0, \frac{3}{2})$ is $\frac{2}{3}$. Hence, the volume is greatest when $x = \frac{2}{3}$.

14.5 A manufacturer sells each of his TV sets for \$85. The cost C (in dollars) of manufacturing and selling x TV sets per week is

$$C = 1500 + 10x + 0.005x^2$$

If at most 10 000 sets can be produced per week, how many sets should be made and sold to maximize the weekly profit?

For x sets per week, the total income is $85x$. The profit is the income minus the cost,

$$P = 85x - (1500 + 10x + 0.005x^2) = 75x - 1500 - 0.005x^2$$

We wish to maximize P on the interval [0, 10 000], since the output is at most 10 000.

$$\frac{dP}{dx} = 75 - 0.01x$$

and the critical number is the solution of

$$\begin{aligned} 75 - 0.01x &= 0 \\ 0.01x &= 75 \\ x &= \frac{75}{0.01} = 7500 \end{aligned}$$

We now construct Table 14-5:

$$\begin{aligned} P(7500) &= 75(7500) - 1500 - 0.0005(7500)^2 \\ &= 562\,500 - 1500 - 0.0005(56\,250\,000) \\ &= 561\,000 - 281\,250 = 279\,750 \end{aligned}$$

$$P(0) = 75(0) - 1500 - 0.0005(0)^2 = -1500$$

$$\begin{aligned} P(10\,000) &= 75(10\,000) - 1500 - 0.0005(10\,000)^2 \\ &= 750\,000 - 1500 - 0.0005(100\,000\,000) \\ &= 748\,500 - 500\,000 = 248\,500 \end{aligned}$$

Thus, the maximum profit is achieved when 7500 TV sets are produced and sold per week.

Table 14-5

x	$P(x)$
7500	279 750
0	−1500
10 000	248 500

14.6 **An orchard has an average yield of 25 bushels per tree when there are at most 40 trees per acre. When there are more than 40 trees per acre, the average yield decreases by $\frac{1}{2}$ bushel per tree for every tree over 40. Find the number of trees per acre that will give the greatest yield per acre.**

Let x be the number of trees per acre, and let $f(x)$ be the total yield in bushels per acre. When $0 \le x \le 40$, $f(x) = 25x$. If $x > 40$, the number of bushels produced by each tree becomes $25 - \frac{1}{2}(x - 40)$. [Here $x - 40$ is the number of trees over 40, and $\frac{1}{2}(x - 40)$ is the corresponding decrease in bushels per tree.] Hence, for $x > 40$, $f(x)$ is given by

$$\left(25 - \frac{1}{2}(x - 40)\right)x = \left(25 - \frac{1}{2}x + 20\right)x = \left(45 - \frac{1}{2}x\right)x = \frac{x}{2}(90 - x)$$

Thus,
$$f(x) = \begin{cases} 25x & \text{if } 0 \le x \le 40 \\ \dfrac{x}{2}(90 - x) & \text{if } x > 40 \end{cases}$$

$f(x)$ is continuous everywhere, since $25x = \dfrac{x}{2}(90 - x)$ when $x = 40$. Clearly, $f(x) < 0$ when $x > 90$. Hence, we may restrict attention to the interval $[0, 90]$.

For $0 < x < 40$, $f(x) = 25x$, and $f'(x) = 25$. Thus, there are no critical numbers in the open interval $(0, 40)$. For $40 < x < 90$,

$$f(x) = \frac{x}{2}(90 - x) = 45x - \frac{x^2}{2} \qquad \text{and} \qquad f'(x) = 45 - x$$

Thus, $x = 45$ is a critical number. In addition, 40 is also a critical number since $f'(40)$ happens not to exist. We do not have to verify this fact, since there is no harm in adding 40 [or any other number in $(0, 90)$] to the list for which we compute $f(x)$.

We now construct Table 14-6:

$$f(45) = \frac{45}{2}(90 - 45) = \frac{45}{2}(45) = \frac{2025}{2} = 1012.5$$
$$f(40) = 25(40) = 1000$$
$$f(0) = 25(0) = 0$$
$$f(90) = \frac{90}{2}(90 - 90) = \frac{90}{2}(0) = 0$$

The maximum yield per acre is realized when there are 45 trees per acre.

Table 14-6

x	$f(x)$
45	1012.5
40	1000
0	0
90	0

Supplementary Problems

14.7 Find the absolute maxima and minima of the following functions on the indicated intervals:

(a) $f(x) = -4x + 5$ on $[-2, 3]$
(b) $f(x) = 2x^2 - 7x - 10$ on $[-1, 3]$
(c) $f(x) = x^3 + 2x^2 + x - 1$ on $[-1, 1]$
(d) $f(x) = 4x^3 - 8x^2 + 1$ on $[-1, 1]$
(e) $f(x) = x^4 - 2x^3 - x^2 - 4x + 3$ on $[0, 4]$
(f) $f(x) = \dfrac{2x+5}{x^2-4}$ on $[-5, -3]$
(g) $f(x) = \dfrac{x^2}{16} + \dfrac{1}{x}$ on $[1, 4]$
(h) $f(x) = \dfrac{x^3}{x+2}$ on $[-1, 1]$
(i) $f(x) = \begin{cases} x^3 - \frac{1}{3}x & \text{for } 0 \le x \le 1 \\ x^2 + x - \frac{4}{3} & \text{for } 1 < x \le 2 \end{cases}$ on $[0, 2]$

14.8 A farmer wishes to fence in a rectangular field. If north–south fencing costs \$3 per yard, and east–west fencing costs \$2 per yard, what are the dimensions of the field of maximum area that can be fenced in for \$600?

14.9 A farmer has to fence in a rectangular field alongside a straight-running stream. If the farmer has 120 yards of fencing, and the side of the field alongside the stream does not have to be fenced, what dimensions of the field will yield the largest area?

14.10 The distance by bus from New York to Boston is 225 miles. The bus driver gets paid \$12.50 per hour, while the other costs of running the bus at a steady speed of x miles per hour amount to $90 + 0.5x$ cents per mile. The minimum and maximum legal speeds on the bus route are 40 and 55 miles per hour. At what steady speed should the bus be driven to minimize the total cost?

14.11 A charter airline is planning a flight for which it is considering a price of between \$150 and \$300 per person. The airline estimates that the number of passengers taking the flight will be $200 - 0.5x$, depending on the price of x dollars that will be set. What price will maximize the income?

14.12 Suppose that a company can sell x radios per week if it charges $100 - 0.1x$ dollars per radio. Its production cost is $30x + 5000$ dollars when x radios are produced per week. How many radios should be produced to maximize the profit, and what will be the selling price per radio?

14.13 A box with square base and vertical sides is to be made from 150 square feet of cardboard. What dimensions will provide the greatest volume if: (a) the box has a top surface; (b) the box has an open top?

14.14 A farmer wishes to fence in a rectangular field, and also to divide the field in half by another fence (AB in Fig. 14-6). The outside fence costs \$2 per foot, and the fence in the middle costs \$3 per foot. If the farmer has \$840 to spend, what dimensions will maximize the total area?

14.15 On a charter flight the price per passenger is \$250 for any number of passengers up to 100. The flight will be canceled if there are fewer than 50 passengers. However, for every passenger over 100, the price per pas-

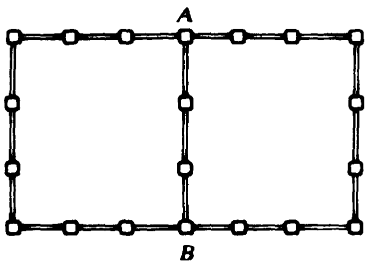


Fig. 14-6

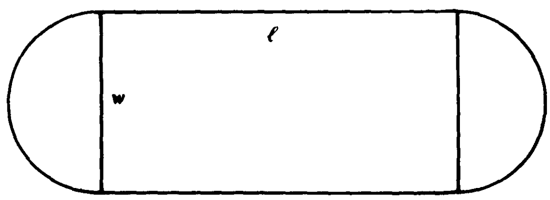


Fig. 14-7

senger will be decreased by \$1. The maximum number of passengers that can be flown is 225. What number of passengers will yield the maximum income?

14.16 Among all pairs x, y of nonnegative numbers whose sum is 100, find those pairs: (*a*) the sum of whose squares $x^2 + y^2$ is a minimum; (*b*) the sum of whose squares $x^2 + y^2$ is a maximum; (*c*) the sum of whose cubes $x^3 + y^3$ is a minimum.

14.17 A sports complex is to be built in the form of a rectangular field with two equal semicircular areas at each end (see Fig. 14-7). If the border of the entire complex is to be a running track 1256 meters long, what should be the dimensions of the complex so that the area of the rectangular field is a maximum?

14.18 A wire of length L is cut into two pieces. The first piece is bent into a circle and the second piece into a square. Where should the wire be cut so that the total area of the circle plus the square is: (*a*) a maximum; (*b*) a minimum?

GEOMETRY The area of a circle of radius r is πr^2, and the circumference is $2\pi r$.

14.19 A wire of length L is cut into two pieces. The first piece is bent into a square and the second into an equilateral triangle. Where should the wire be cut so that the total area of the square and the triangle is greatest?

GEOMETRY The area of an equilateral triangle of side s is $\sqrt{3}\,\frac{s^2}{4}$.

14.20 A company earns a profit of \$40 on every TV set it makes when it produces at most 1000 sets. If the profit per item decreases by 5 cents for every TV set over 1000, what production level maximizes the total profit?

14.21 Find the radius and the height of the right circular cylinder of greatest volume that can be inscribed in a right circular cone having a radius of 3 feet and a height of 5 feet (see Fig. 14-8).

GEOMETRY The volume of a right circular cylinder of radius r and height h is $\pi r^2 h$. By the proportionality of the sides of similar triangles (in Fig. 14-8), $\frac{5-h}{r} = \frac{5}{3}$.

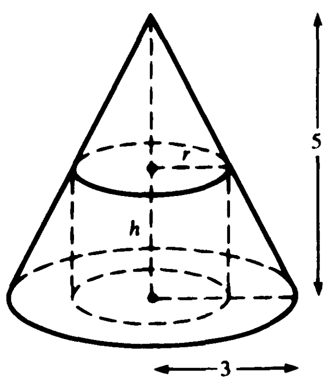


Fig. 14-8

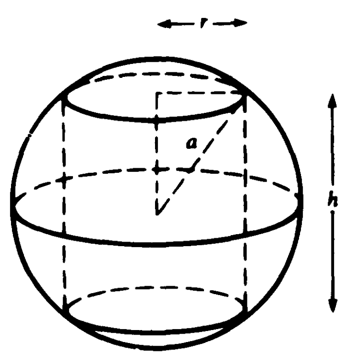


Fig. 14-9

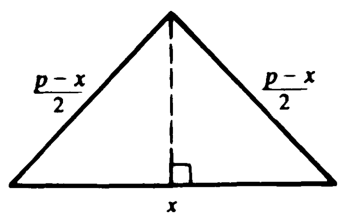


Fig. 14-10

14.22 Find the height h and the radius r of the right circular cylinder of greatest volume that can be inscribed in a sphere of radius a. [*Hint:* See Fig. 14-9. The Pythagorean theorem relates $h/2$ and r, and provides bounds on each.]

14.23 Among all isosceles triangles with a fixed perimeter p, which has the largest area? [*Hint:* See Fig. 14-10 and solve the equivalent problem of maximizing the square of the area.]

14.24 Find the point(s) on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ that is (are): (*a*) closest to the point (1, 0); (*b*) farthest from the point (1, 0). [*Hint:* It is easier to find the extrema of the square of the distance from (1, 0) to (x, y). Notice that $-5 \le x \le 5$ for points (x, y) on the ellipse.]

14.25 A rectangular swimming pool is to be built with 6-foot borders at the north and south ends, and 10-foot borders at the east and west ends. If the total area available is 6000 square feet, what are the dimensions of the largest possible water area?

14.26 A farmer has to enclose two fields. One is to be a rectangle with the length twice the width, and the other is to be a square. The rectangle is required to contain at least 882 square meters, and the square has to contain at least 400 square meters. There are 680 meters of fencing available.

(*a*) If x is the width of the rectangular field, what are the maximum and minimum possible values of x?

(*b*) What is the maximum possible total area?

14.27 It costs a company $0.1x^2 + 4x + 3$ dollars to produce x tons of gold. If more than 10 tons is produced, the need for additional labor raises the cost by $2(x - 10)$ dollars. If the price per ton is \$9, regardless of the production level, and if the maximum production capacity is 20 tons, what output maximizes the profit?

14.28 Prove Theorem 14.1. [*Hint:* Take the case of a relative minimum at $x = c$. Then $\frac{f(c+h) - f(c)}{h} \ge 0$ for h sufficiently small and positive, and $\frac{f(c+h) - f(c)}{h} \le 0$ for h negative and sufficiently small in magnitude. Since $f'(c)$ exists, $f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \ge 0$ and $f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \le 0$.]

14.29 A rectangle is inscribed in an isosceles triangle with base 9 inches and height 6 inches. Find the dimensions of the rectangle of maximum area if one side of the rectangle lies inside the base of the triangle.

14.30 A rectangular yard is to enclose an area of 200 square meters. Fencing is required on only three sides, since one side will lie along the wall of a building. The length and width of the yard are each required to measure at least 5 meters.

(*a*) What dimensions will minimize the total fencing required? What will be the minimum fencing?

(*b*) What dimensions will maximize the total fencing, and what will be the maximum fencing?

14.31 Let $f(x) = \frac{2x - 3}{x^2}$.

(*a*) Find the maximum and minimum values of f on [1, 10].

(*b*) Does the extreme-value theorem apply to f on [−10, 10]? Why?

14.32 Find the point(s) on the curve $y = \sqrt{6x^4 + 8x^3 + 11x^2 + 9}$ that is (are) closest to the origin.

14.33 Find the shortest distance between points of the curve $y = \sqrt{x^2 + 3x + 2}$ and the origin.

14.34 Find the dimensions of the rectangle of maximum area that can be inscribed in a right triangle whose sides are 3, 4, and 5, if one side of the rectangle lies on the side of the triangle of length 3 and the other side lies on the side of the triangle of length 4.

14.35 GC Find the relative extrema of $f(x) = x^4 + 2x^2 - 5x - 2$ in two ways.

(*a*) Trace the graph of f to find the relative maximum and relative minimum directly.

(*b*) Trace the graph of f' to find the critical numbers of f.

14.36 GC Find the relative extrema of $f(x) = x^3 - 3x^2 + 4x - 1$ on $(-5, 3)$.

Chapter 15

The Chain Rule

15.1 COMPOSITE FUNCTIONS

There are still many functions whose derivatives we do not know how to calculate; for example,

$$\text{(i)}\ \sqrt{x^3 - x + 2} \qquad \text{(ii)}\ \sqrt[3]{x+4} \qquad \text{(iii)}\ (x^2 + 3x - 1)^{23}$$

In case (iii), we could, of course, multiply $x^2 + 3x - 1$ by itself 22 times and then differentiate the resulting polynomial. But without a computer, this would be extremely arduous.

The above three functions have the common feature that they are combinations of simpler functions:

(i) $\sqrt{x^3 - x + 2}$ is the result of starting with the function $f(x) = x^3 - x + 2$ and then applying the function $g(x) = \sqrt{x}$ to the result. Thus,

$$\sqrt{x^3 - x + 2} = g(f(x))$$

(ii) $\sqrt[3]{x+4}$ is the result of starting with the function $F(x) = x + 4$ and then applying the function $G(x) = \sqrt[3]{x}$. Thus,

$$\sqrt[3]{x+4} = G(F(x))$$

(iii) $(x^2 + 3x - 1)^{23}$ is the result of beginning with the function $H(x) = x^2 + 3x - 1$ and then applying the function $K(x) = x^{23}$. Thus,

$$(x^2 + 3x - 1)^{23} = K(H(x))$$

Functions that are put together this way out of simpler functions are called *composite* functions.

Definition: If f and g are any functions, then the *composition* $g \circ f$ of f and g is the function such that

$$(g \circ f)(x) = g(f(x))$$

The "process" of composition is diagrammed in Fig. 15-1.

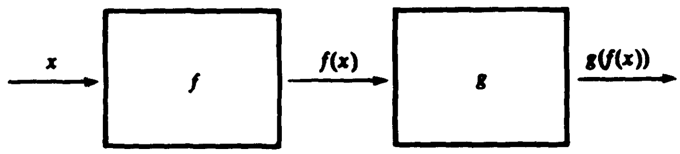


Fig. 15-1

EXAMPLES

(*a*) Let $f(x) = x - 1$ and $g(x) = x^2$. Then,

$$(g \circ f)(x) = g(f(x)) = g(x - 1) = (x - 1)^2$$

On the other hand,

$$(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 - 1$$

Thus, $f \circ g$ and $g \circ f$ are not necessarily the same function (and usually they are not the same).

(*b*) Let $f(x) = x^2 + 2x$ and $g(x) = \sqrt{x}$. Then,

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 2x) = \sqrt{x^2 + 2x}$$
$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 + 2(\sqrt{x}) = x + 2\sqrt{x}$$

Again, $g \circ f$ and $f \circ g$ are different.

A composite function $g \circ f$ is defined only for those x for which $f(x)$ is defined and $g(f(x))$ is defined. In other words, the domain of $g \circ f$ consists of those x in the domain of f for which $f(x)$ is in the domain of g.

Theorem 15.1: The composition of continuous functions is a continuous function. If f is continuous at a, and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

For a proof, see Problem 15.25.

15.2 DIFFERENTIATION OF COMPOSITE FUNCTIONS

First, let us treat an important special case. The function $[f(x)]^n$ is the composition $g \circ f$ of f and the function $g(x) = x^n$. We have:

Theorem 15.2: (*Power Chain Rule*): Let f be differentiable and let n be any integer. Then,

$$D_x((f(x))^n) = n(f(x))^{n-1}D_x(f(x)) \qquad (15.1)$$

EXAMPLES

(*a*) $D_x((x^2 - 5)^3) = 3(x^2 - 5)^2 D_x(x^2 - 5) = 3(x^2 - 5)^2(2x) = 6x(x^2 - 5)^2$

(*b*) $D_x((x^3 - 2x^2 + 3x - 1)^7) = 7(x^3 - 2x^2 + 3x - 1)^6 D_x(x^3 - 2x^2 + 3x - 1)$
$= 7(x^3 - 2x^2 + 3x - 1)^6(3x^2 - 4x + 3)$

(*c*) $$D_x\left(\frac{1}{(3x-5)^4}\right) = D_x((3x-5)^{-4}) = -4(3x-5)^{-5}D_x(3x-5)$$
$$= -\frac{4}{(3x-5)^5}(3) = -\frac{12}{(3x-5)^5}$$

Theorem 15.3 (*Chain Rule*): Assume that f is differentiable at x and that g is differentiable at $f(x)$. Then the composition $g \circ f$ is differentiable at x, and its derivative $(g \circ f)'$ is given by

$$(g \circ f)'(x) = g'(f(x))f'(x) \qquad (15.2)$$

that is,

$$D_x(g(f(x)) = g'(f(x))D_x f(x)$$

The proof of Theorem 15.3 is tricky; see Problem 15.27. The power chain rule (Theorem 15.2) follows from the chain rule (Theorem 15.3) when $g(x) = x^n$.

Applications of the general chain rule will be deferred until later chapters. Before leaving it, however, we shall point out a suggestive notation. If one writes $y = g(f(x))$ and $u = f(x)$, then $y = g(u)$, and (*15.2*) may be expressed in the form

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad (15.3)$$

just as though derivatives were fractions (which they are not) and as though the chain rule were an identity obtained by the cancellation of the du's on the right-hand side. While this "identity" makes for an easy way to remember the chain rule, it must be borne in mind that y on the left-hand side of (*15.3*) stands for a certain function of x [namely $(g \circ f)(x)$], whereas on the right-hand side it stands for a different funcion of u [namely, $g(u)$].

EXAMPLE Using (*15.3*) to rework the preceding example (*c*), we write

$$y = (3x - 5)^{-4} = u^{-4}$$

where $u = 3x - 5$. Then,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-4u^{-5})(3) = -\frac{12}{u^5} = -\frac{12}{(3x-5)^5}$$

Differentiation of Rational Powers

We want to be able to differentiate the function $f(x) = x^r$, where r is a rational number. The special case of r an integer is already covered by Rule 7 of Chapter 13.

ALGEBRA A rational number r is one that can be represented in the form $r = n/k$, where n and k are integers, with k positive. By definition,

$$a^{n/k} = (\sqrt[k]{a})^n$$

except when a is negative and k is even (in which case the kth root of a is undefined). For instance,

$$(8)^{2/3} = (\sqrt[3]{8})^2 = (2)^2 = 4$$
$$(32)^{-2/5} = (\sqrt[5]{32})^{-2} = (2)^{-2} = \frac{1}{2^2} = \frac{1}{4}$$
$$(-27)^{4/3} = (\sqrt[3]{-27})^4 = (-3)^4 = 81$$
$(-4)^{7/8}$ is not defined

Observe that

$$(\sqrt[k]{a})^n = \sqrt[k]{a^n}$$

whenever both sides are defined. In fact,

$$((\sqrt[k]{a})^n)^k = (\sqrt[k]{a})^{nk} = (\sqrt[k]{a})^{kn} = ((\sqrt[k]{a})^k)^n = a^n$$

which shows that $(\sqrt[k]{a})^n$ is the kth root of a^n. In calculations we are free to choose whichever expression for $a^{n/k}$ is the more convenient. Thus: (i) $64^{2/3}$ is easier to compute as

$$(\sqrt[3]{64})^2 = (4)^2 = 16$$

than as

$$\sqrt[3]{(64)^2} = \sqrt[3]{4096}$$

but (ii) $(\sqrt{8})^{2/3}$ is easier to compute as

$$\sqrt[3]{(\sqrt{8})^2} = \sqrt[3]{8} = 2$$

than as

$$(\sqrt[3]{\sqrt{8}})^2$$

The usual laws of exponents hold for rational exponents:

(1) $a^r \cdot a^s = a^{r+s}$

(2) $\dfrac{a^r}{a^s} = a^{r-s}$

(3) $(a^r)^s = a^{rs}$

(4) $(ab)^r = a^r b^r$

where r and s are any rational numbers.

Theorem 15.4: For any rational number r, $D_x(x^r) = rx^{r-1}$.

For a proof, see Problem 15.6.

EXAMPLES

(*a*) $D_x(\sqrt{x}) = D_x(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2}\frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$

(*b*) $D_x(x^{3/2}) = \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x}$ (*c*) $D_x(x^{3/4}) = \frac{3}{4}x^{-1/4} = \frac{3}{4}\frac{1}{x^{1/4}} = \frac{3}{4\sqrt[4]{x}}$

Theorem 15.4, together with the chain rule (Theorem 15.3), allows us to extend the power chain rule (Theorem 15.2) to rational exponents.

Corollary 15.5: If f is differentiable and r is a rational number,

$$D_x((f(x))^r) = r(f(x))^{r-1}D_x f(x)$$

EXAMPLES

(*a*) $$D_x(\sqrt{x^2-3x+1}) = D_x((x^2-3x+1)^{1/2}) = \frac{1}{2}(x^2-3x+1)^{-1/2}D_x(x^2-3x+1)$$
$$= \frac{1}{2}\frac{1}{(x^2-3x+1)^{1/2}}(2x-3) = \frac{2x-3}{2\sqrt{x^2-3x+1}}$$

(*b*) $$D_x((3x^2-1)^{5/4}) = \frac{5}{4}(3x^2-1)^{1/4}D_x(3x^2-1)$$
$$= \frac{5}{4}\sqrt[4]{3x^2-1}\,(6x) = \frac{15x}{2}\sqrt[4]{3x^2-1}$$

(*c*) $$D_x\left(\frac{1}{\sqrt[3]{7x+2}}\right) = D_x\left(\frac{1}{(7x+2)^{1/3}}\right) = D_x((7x+2)^{-1/3}) = -\frac{1}{3}(7x+2)^{-4/3}D_x(7x+2)$$
$$= -\frac{1}{3}\frac{1}{(7x+2)^{4/3}}(7) = -\frac{7}{3(\sqrt[3]{7x+2})^4}$$

Solved Problems

15.1 For each pair of functions f and g, find formulas for $g \circ f$ and $f \circ g$, and determine the domains of $g \circ f$ and $f \circ g$.

(*a*) $g(x) = \sqrt{x}$ and $f(x) = x + 1$ (*b*) $g(x) = x^2$ and $f(x) = x - 1$

(*a*) $$(g \circ f)(x) = g(f(x)) = g(x+1) = \sqrt{x+1}$$

Because $\sqrt{x+1}$ is defined if and only if $x \geq -1$, the domain of $g \circ f$ is $[-1, \infty)$.

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \sqrt{x} + 1$$

Because $\sqrt{x} + 1$ is defined if and only if $x \geq 0$, the domain of $f \circ g$ is $[0, \infty)$.

(*b*) $$(g \circ f)(x) = g(f(x))) = g(x-1) = (x-1)^2$$
$$(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 - 1$$

Both composite functions are polynomials, and so the domain of each is the set of all real numbers.

15.2 Calculate the derivatives of:

(*a*) $(x^4 - 3x^2 + 5x - 2)^3$ (*b*) $\sqrt{7x^3 - 2x^2 + 5}$ (*c*) $\frac{1}{(5x^2+4)^3}$

The power chain rule is used in each case.

(a) $D_x((x^4 - 3x^2 + 5x - 2)^3) = 3(x^4 - 3x^2 + 5x - 2)^2 D_x(x^4 - 3x^2 + 5x - 2)$
$= 3(x^4 - 3x^2 + 5x - 2)^2(4x^3 - 6x + 5)$

(b) $D_x(\sqrt{7x^3 - 2x^2 + 5}) = D_x((7x^3 - 2x^2 + 5)^{1/2}) = \frac{1}{2}(7x^3 - 2x^2 + 5)^{-1/2} D_x(7x^3 - 2x^2 + 5)$
$= \frac{1}{2}\frac{1}{(7x^3 - 2x^2 + 5)^{1/2}}(21x^2 - 4x) = \frac{x(21x - 4)}{2\sqrt{7x^3 - 2x^2 + 5}}$

(c) $D_x\left(\frac{1}{(5x^2 + 4)^3}\right) = D_x((5x^2 + 4)^{-3}) = -3(5x^2 + 4)^{-4} D_x(5x^2 + 4)$
$= \frac{-3}{(5x^2 + 4)^4}(10x) = -\frac{30x}{(5x^2 + 4)^4}$

15.3 Find the derivative of the function $f(x) = \sqrt{1 + \sqrt{(x + 1)}} = [1 + (x + 1)^{1/2}]^{1/2}$.

By the power chain rule, used twice,

$$\begin{aligned} f'(x) &= \frac{1}{2}(1 + (x + 1)^{1/2})^{-1/2} D_x(1 + (x + 1)^{1/2}) \\ &= \frac{1}{2}(1 + (x + 1)^{1/2})^{-1/2}\left(\frac{1}{2}(x + 1)^{-1/2} D_x(x + 1)\right) \\ &= \frac{1}{4}(1 + (x + 1)^{1/2})^{-1/2}(x + 1)^{-1/2}(1) \\ &= \frac{1}{4}\{(1 + (x + 1)^{1/2})(x + 1)\}^{-1/2} \\ &= \frac{1}{4}\frac{1}{\{(1 + (x + 1)^{1/2})(x + 1)\}^{1/2}} = \frac{1}{4}\frac{1}{\sqrt{((1 + \sqrt{(x + 1)})(x + 1)}} \end{aligned}$$

15.4 Find the absolute extrema of $f(x) = x\sqrt{1 - x^2}$ on [0, 1].

$$\begin{aligned} D_x(x\sqrt{1 - x^2}) &= xD_x(\sqrt{1 - x^2}) + \sqrt{1 - x^2}\,D_x(x) && \text{[by the product rule]} \\ &= xD_x((1 - x^2)^{1/2}) + \sqrt{1 - x^2} \\ &= x\left(\frac{1}{2}(1 - x^2)^{-1/2} D_x(1 - x^2)\right) + \sqrt{1 - x^2} && \text{[by the power chain rule]} \\ &= \frac{x}{2}\frac{1}{(1 - x^2)^{1/2}}(-2x) + \sqrt{1 - x^2} = \frac{-x^2}{\sqrt{1 - x^2}} + \sqrt{1 - x^2} \\ &= \frac{-x^2 + (1 - x^2)}{\sqrt{1 - x^2}} = \frac{1 - 2x^2}{\sqrt{1 - x^2}} \quad \left(\text{by } \frac{a}{c} + b = \frac{a + bc}{c}\right) \end{aligned}$$

The right-hand side is not defined when the denominator is 0; that is, when $x^2 = 1$. Hence, 1 and -1 are critical numbers. The right-hand side is 0 when the numerator is 0; that is, when

$$2x^2 = 1 \quad \text{or} \quad x^2 = \tfrac{1}{2} \quad \text{or} \quad x = \pm\sqrt{\tfrac{1}{2}}$$

Thus, $\sqrt{\frac{1}{2}}$ and $-\sqrt{\frac{1}{2}}$ are also critical numbers. The only critical number in (0, 1) is $\sqrt{\frac{1}{2}}$,

---

ALGEBRA $\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2} \approx 0.707$

---

and $f(\sqrt{\tfrac{1}{2}}) = \sqrt{\tfrac{1}{2}}\sqrt{1 - \tfrac{1}{2}} = \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{2}} = \tfrac{1}{2}$

At the endpoints, $f(0) = f(1) = 0$. Hence, $\frac{1}{2}$ is the absolute maximum (achieved at $x = \sqrt{\frac{1}{2}}$) and 0 is the absolute minimum (achieved at $x = 0$ and $x = 1$).

15.5 A spy on a submarine S, 6 kilometers off a straight shore, has to reach a point B, which is 9 kilometers down the shore from the point A opposite S (see Fig. 15-2). The spy must row a boat to some point C on the shore and then walk the rest of the way to B. If he rows at 4 kilometers per hour and walks at 5 kilometers per hour, at what point C should he land in order to reach B as soon as possible?

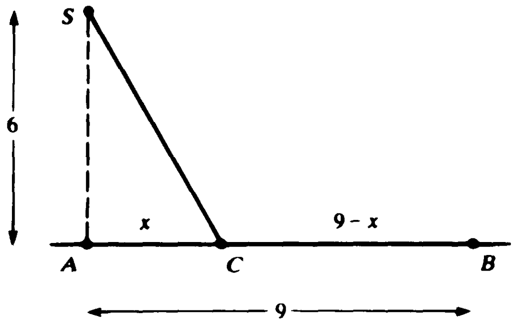


Fig. 15-2

Let $x = \overline{AC}$; then $\overline{BC} = 9 - x$. By the Pythagorean theorem,

$$\overline{SC}^2 = (6)^2 + x^2 \qquad \text{or} \qquad \overline{SC} = \sqrt{36 + x^2}$$

The time spent rowing is, therefore,

$$T_1 = \frac{\sqrt{36 + x^2}}{4} \quad \text{[hours]}$$

and the time spent walking will be $T_2 = (9 - x)/5$ [hours]. The total time $T(x)$ is given by the formula

$$T(x) = T_1 + T_2 = \frac{\sqrt{36 + x^2}}{4} + \frac{9 - x}{5}$$

We have to minimize $T(x)$ on the interval [0, 9], since x can vary from 0 (at A) to 9 (at B),

$$\begin{aligned} T'(x) &= D_x\left(\frac{(36 + x^2)^{1/2}}{4}\right) + D_x\left(\frac{9}{5} - \frac{x}{5}\right) \\ &= \frac{1}{4} \cdot \frac{1}{2}(36 + x^2)^{-1/2} \cdot D_x(36 + x^2) - \frac{1}{5} \qquad \text{[by the power chain rule]} \\ &= \frac{1}{8}\frac{1}{(36 + x^2)^{1/2}}(2x) - \frac{1}{5} = \frac{x}{4\sqrt{36 + x^2}} - \frac{1}{5} \end{aligned}$$

The only critical numbers are the solutions of

$$\begin{aligned} \frac{x}{4\sqrt{36 + x^2}} - \frac{1}{5} &= 0 \\ \frac{x}{4\sqrt{36 + x^2}} &= \frac{1}{5} \\ 5x &= 4\sqrt{36 + x^2} \qquad \text{[cross multiply]} \\ 25x^2 &= 16(36 + x^2) \qquad \text{[square both sides]} \\ 25x^2 &= 576 + 16x^2 \\ 9x^2 &= 576 \\ x^2 &= 64 \\ x &= \pm 8 \end{aligned}$$

The only critical number in (0, 9) is 8. Computing the values for the tabular method,

$$T(8) = \frac{\sqrt{36+(8)^2}}{4} + \frac{9-8}{5} = \frac{\sqrt{100}}{4} + \frac{1}{5} = \frac{10}{4} + \frac{1}{5} = \frac{5}{2} + \frac{1}{5} = \frac{27}{10}$$
$$T(0) = \frac{\sqrt{36+(0)^2}}{4} + \frac{9-0}{5} = \frac{\sqrt{36}}{4} + \frac{9}{5} = \frac{6}{4} + \frac{9}{5} = \frac{3}{2} + \frac{9}{5} = \frac{33}{10}$$
$$T(9) = \frac{\sqrt{36+(9)^2}}{4} + \frac{9-9}{5} = \frac{\sqrt{36+81}}{4} + 0 = \frac{\sqrt{117}}{4} = \frac{\sqrt{9}\sqrt{13}}{4} = \frac{3}{4}\sqrt{13}$$

we generate Table 15-1. The absolute minimum is achieved at $x = 8$; the spy should land 8 kilometers down the shore from A.

Table 15-1

x	$T(x)$
8	$\frac{27}{10}$ min
0	$\frac{33}{10}$
9	$\frac{3}{4}\sqrt{13}$

ALGEBRA $T(9) > T(8)$; for, assuming the contrary,

$$\frac{3}{4}\sqrt{13} \le \frac{27}{10}$$
$$\sqrt{13} \le \frac{36}{10} \qquad \text{[multiply by } \tfrac{4}{3}\text{]}$$
$$13 \le \frac{1296}{100} = 12.96 \qquad \text{[by squaring]}$$

which is false.

15.6 Prove Theorem 15.4: $D_x(x^r) = rx^{r-1}$ for any rational number r.

Let $r = n/k$, where n is an integer and k is a positive integer. That $x^{n/k}$ is differentiable is not easy to prove; see Problem 15.26. Assuming this, let us now derive the formula for the derivative. Let

$$f(x) = x^{n/k} = \sqrt[k]{x^n}$$

Then, since $(f(x))^k = x^n$,

$$D_x((f(x))^k) = D_x(x^n) = nx^{n-1}$$

But, by Theorem 15.2, $D_x((f(x))^k) = k(f(x))^{k-1}f'(x)$. Hence,

$$k(f(x))^{k-1}f'(x) = nx^{n-1}$$

and solving for $f'(x)$, we obtain

$$f'(x) = \frac{nx^{n-1}}{k(f(x))^{k-1}} = \frac{n}{k}\frac{x^{rk-1}}{(x^r)^{k-1}}$$
$$= r\frac{x^{rk-1}}{x^{rk-r}} = rx^{r-1}$$

Supplementary Problems

15.7 For each pair of functions $f(x)$ and $g(x)$, find formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$.

(a) $f(x) = \dfrac{2}{x+1}$, $g(x) = 3x$ (b) $f(x) = x^2 + 2x - 5$, $g(x) = x^3$

(c) $f(x) = 2x^3 - x^2 + 4$, $g(x) = 3$ (d) $f(x) = x^3$, $g(x) = x^2$

(e) $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{1}{x}$ (f) $f(x) = x$, $g(x) = x^2 - 4$

15.8 For each pair of functions f and g, find the set of solutions of the equation $(f \circ g)(x) = (g \circ f)(x)$.

(a) $f(x) = x^3$, $g(x) = x^2$ (b) $f(x) = \dfrac{2}{x+1}$, $g(x) = 3x$

(c) $f(x) = 2x$, $g(x) = \dfrac{1}{x-1}$ (d) $f(x) = x^2$, $g(x) = \dfrac{1}{x+1}$

(e) $f(x) = x^2$, $g(x) = \dfrac{1}{x^2 - 3}$

15.9 Express each of the following functions as the composition $(g \circ f)(x)$ of two simpler functions. [The functions $f(x)$ and $g(x)$ obviously will not be unique.]

(a) $(x^3 - x^2 + 2)^7$ (b) $(8 - x)^4$ (c) $\sqrt{1 + x^2}$ (d) $\dfrac{1}{x^2 - 4}$

15.10 Find the derivatives of the following functions:

(a) $(x^3 - 2x^2 + 7x - 3)^4$ (b) $(7 + 3x)^5$ (c) $(2x - 3)^{-2}$

(d) $(3x^2 + 5)^{-3}$ (e) $(4x^2 - 3)^2(x + 5)^3$ (f) $\left(\dfrac{x+2}{x-3}\right)^3$

(g) $\left(\dfrac{x^2 - 2}{2x^2 + 1}\right)^2$ (h) $\dfrac{4}{3x^2 - x + 5}$ (i) $\sqrt{1 + x^3}$

15.11 Find the derivatives of the following functions:

(a) $2x^{3/4}$ (b) $x^2(1 - 3x^3)^{1/3}$ (c) $\dfrac{x}{\sqrt{x^2 + 1}}$

(d) $(7x^3 - 4x^2 + 2)^{1/4}$ (e) $\dfrac{\sqrt{x+2}}{\sqrt{x-1}}$ (f) $8x^{3/4} + 4x^{1/4} - x^{-1/3}$

(g) $\sqrt[3]{(4x^2 + 3)^2}$ (h) $\sqrt{\dfrac{4}{x}} - \sqrt{3x}$ (i) $\sqrt{4 - \sqrt{(4 + x)}}$ (j) $\dfrac{(1 + x^3)^{2/3}}{1 - 2x}$

15.12 Find the slope-intercept equation of the tangent line to the graph of $y = \dfrac{\sqrt{x-1}}{x^2 + 1}$ at the point $(2, \frac{1}{5})$.

15.13 Find the slope-intercept equation of the normal line to the curve $y = \sqrt{x^2 + 16}$ at the point $(3, 5)$.

15.14 Let $g(x) = x^2 - 4$ and $f(x) = \dfrac{x+2}{x-2}$.

(a) Find a formula for $(g \circ f)(x)$ and then compute $(g \circ f)'(x)$.

(b) Show that the chain rule gives the same answer for $(g \circ f)'(x)$ as was found in part (a).

15.15 Find the absolute extrema of the following functions on the given intervals:

(a) $f(x) = \dfrac{x}{\sqrt{1+x^2}}$ on $[-1, 1]$ (b) $f(x) = (x-2)^2(x+3)^3$ on $[-4, 3]$

(c) $f(x) = \sqrt{5-4x}$ on $[-1, 1]$ (d) $f(x) = \dfrac{2}{3}x - x^{2/3}$ on $[0, 8]$

(e) $f(x) = x^{2/5} - \dfrac{1}{9}x^{7/5}$ on $[-1, 1]$

15.16 Two towns, P and Q, are located 2 miles and 3 miles, respectively, from a railroad line, as shown in Fig. 15-3. What point R on the line should be chosen for a new station in order to minimize the sum of the distances from P and Q to the station, if the distance between A and B is 4 miles?

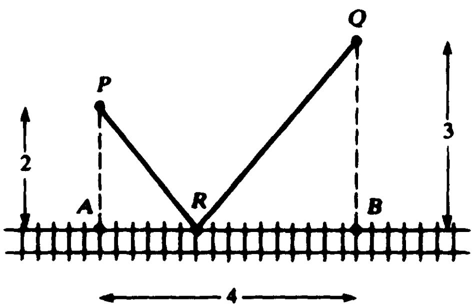


Fig. 15-3

15.17 Assume that F and G are differentiable functions such that $F'(x) = -G(x)$ and $G'(x) = -F(x)$. If $H(x) = (F(x))^2 - (G(x))^2$, find a formula for $H'(x)$.

15.18 If $y = x^3 - 2$ and $z = 3x + 5$, then y can be considered a function of z. Express dy/dz in terms of x.

15.19 Let F be a differentiable function, and let $G(x) = F'(x)$. Express $D_x(F(x^3))$ in terms of G and x.

15.20 If $g(x) = x^{1/5}(x-1)^{3/5}$, find the domain of $g'(x)$.

15.21 Let f be a differentiable odd function (Section 7.3). Find the relationship between $f'(-x)$ and $f'(x)$.

15.22 Let F and G be differentiable functions such that

$$F(3) = 5 \qquad F'(3) = 13 \qquad F'(7) = 2$$
$$G(3) = 7 \qquad G'(3) = 6 \qquad G'(7) = 0$$

If $H(x) = F(G(x))$, find $H'(3)$.

15.23 Let $F(x) = \sqrt{1+3x}$.

(a) Find the domain and the range of F.

(b) Find the slope-intercept equation of the tangent line to the graph of F at $x = 5$.

(c) Find the coordinates of the point(s) on the graph of F such that the normal line there is parallel to the line $4x + 3y = 1$.

15.24 Find the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 1 if a side of the rectangle is on the diameter.

15.25 Prove Theorem 15.1: If f is continuous at a and g is continuous at $f(a)$, prove that $g \circ f$ is continuous at a. [*Hint:* For arbitrary $\epsilon > 0$, let $\delta_1 > 0$ be such that $|g(u) - g(f(a))| < \epsilon$ whenever $|u - f(a)| < \delta_1$. Then choose $\delta > 0$ such that $|f(x) - f(a)| < \delta_1$ whenever $|x - a| < \delta$.]

15.26 Prove that $x^{n/k}$ is differentiable. [*Hint:* It is enough to show that $f(x) = x^{1/k}$ $(k > 1)$ is differentiable.] Proceed as follows:

(1) By direct multiplication, establish that

$$a^k - b^k = (a - b)(a^{k-1} + a^{k-2}b + \cdots + ab^{k-2} + b^{k-1})$$

or

$$\frac{a - b}{a^k - b^k} = \frac{1}{a^{k-1} + a^{k-2}b + \cdots + ab^{k-2} + b^{k-1}}$$

(2) Substitute $a = (x + h)^{1/k}$ and $b = x^{1/k}$ in step (1):

$$\frac{(x + h)^{1/k} - x^{1/k}}{h} = \frac{1}{(x + h)^{(k-1)/k} + (x + h)^{(k-2)/k}x^{1/k} + \cdots + (x + h)^{1/k}x^{(k-2)/k} + x^{(k-1)/k}}$$

(3) Let $h \to 0$ in step (2).

15.27 Prove the chain rule (Theorem 15.3): $(g \circ f)'(x) = g'(f(x))f'(x)$, where f is differentiable at x and g is differentiable at $f(x)$. [*Hint:* Let $H = g \circ f$. Let $y = f(x)$ and $K = f(x + h) - f(x)$. Also let $G(t) = \dfrac{g(y + t) - g(y)}{t} - g'(y)$ for $t \neq 0$. Since $\lim_{t \to 0} G(t) = 0$, let $G(0) = 0$. Then $g(y + t) - g(y) = t(G(t) + g'(y))$ holds for all t. When $t = K$,

$$g(y + K) - g(y) = K(G(K) + g'(y))$$

$$g(f(x + h)) - g(f(x)) = K(G(K) + g'(y))$$

So,

$$\frac{H(x + h) - H(x)}{h} = \frac{K}{h}(G(K) + g'(y))$$

Now $\lim_{h \to 0} \dfrac{K}{h} = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = f'(x)$. Since $\lim_{h \to 0} K = 0$, $\lim_{h \to 0} G(K) = 0$.
Hence, $H'(x) = g'(y)f'(x) = g'(f(x))f'(x)$.]

Chapter 16

Implicit Differentiation

A function is usually defined *explicitly* by means of a formula.

EXAMPLES

$$(a)\ f(x) = x^2 - x + 2 \qquad (b)\ f(x) = \sqrt{x} \qquad (c)\ f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 1 \\ 1 - x & \text{if } x < 1 \end{cases}$$

However, sometimes the value $y = f(x)$ is not given by such a direct formula.

EXAMPLES

(*a*) The equation $y^3 - x = 0$ *implicitly* determines y as a function of x. In this case, we can solve for y explicitly,

$$y^3 = x \qquad \text{or} \qquad y = \sqrt[3]{x}$$

(*b*) The equation $y^3 + 12y^2 + 48y - 8x + 64 = 0$ is satisfied when $y = 2\sqrt[3]{x} - 4$, but it is not easy to find this solution. In more complicated cases, it will be impossible to find a formula for y in terms of x.

(*c*) The equation $x^2 + y^2 = 1$ implicitly determines *two* functions of x,

$$y = \sqrt{1 - x^2} \qquad \text{and} \qquad y = -\sqrt{1 - x^2}$$

The question of how many functions an equation determines and of the properties of these functions is too complex to be considered here. We shall content ourselves with learning a method for finding the derivatives of functions determined implicitly by equations.

EXAMPLE Let us find the derivative of a function y determined by the equation $x^2 + y^2 = 4$. Since y is assumed to be some function of x, the two sides of the equation represent the same function of x, and so must have the same derivative,

$$D_x(x^2 + y^2) = D_x(4)$$
$$2x + 2yD_x y = 0 \qquad \text{[by the power chain rule]}$$
$$2yD_x y = -2x$$
$$D_x y = -\frac{2x}{2y} = -\frac{x}{y}$$

Thus, $D_x y$ has been found in terms of x and y. Sometimes this is all the information we may need. For example, if we want to know the slope of the tangent line to the graph of $x^2 + y^2 = 4$ at the point $(\sqrt{3}, 1)$, then this slope is the derivative

$$D_x y = -\frac{x}{y} = -\frac{\sqrt{3}}{1} = -\sqrt{3}$$

The process by which $D_x y$ has been found, without first solving explicitly for y, is called *implicit differentiation*.

Note that the given equation could, in this case, have been solved explicitly for y,

$$y = \pm\sqrt{4 - x^2}$$

and from this, using the power chain rule,

$$D_x y = D_x(\pm(4 - x^2)^{1/2}) = \pm\tfrac{1}{2}(4 - x^2)^{-1/2}D_x(4 - x^2)$$
$$= \pm\frac{1}{2\sqrt{4 - x^2}}(-2x) = \pm\frac{-x}{\sqrt{4 - x^2}} = \frac{x}{\mp\sqrt{4 - x^2}}$$

Solved Problems

16.1 Consider the curve $3x^2 - xy + 4y^2 = 141$.

(*a*) Find a formula in x and y for the slope of the tangent line at any point (x, y) of the curve.

(*b*) Write the slope-intercept equation of the line tangent to the curve at the point (1, 6).

(*c*) Find the coordinates of all other points on the curve where the slope of the tangent line is the same as the slope of the tangent line at (1, 6).

(*a*) We may assume that y is some function of x such that $3x^2 - xy + 4y^2 = 141$. Hence,

$$D_x(3x^2 - xy + 4y^2) = D_x(141)$$
$$6x - D_x(xy) + D_x(4y^2) = 0$$
$$6x - \left(x\frac{dy}{dx} + y \cdot 1\right) + 8y\frac{dy}{dx} = 0$$
$$-x\frac{dy}{dx} + 8y\frac{dy}{dx} = y - 6x$$
$$(-x + 8y)\frac{dy}{dx} = y - 6x$$
$$\frac{dy}{dx} = \frac{y - 6x}{8y - x}$$

which is the slope of the tangent line at (x, y).

(*b*) The slope of the tangent line at (1, 6) is obtained by substituting 1 for x and 6 for y in the result of part (*a*). Thus, the slope is

$$\frac{6 - 6(1)}{8(6) - 1} = \frac{6 - 6}{48 - 1} = \frac{0}{47} = 0$$

and the slope-intercept equation is $y = b = 6$.

(*c*) If (x, y) is a point on the curve where the tangent line has slope 0, then,

$$\frac{y - 6x}{8y - x} = 0 \quad \text{or} \quad y - 6x = 0 \quad \text{or} \quad y = 6x$$

Substitute $6x$ for y in the equation of the curve,

$$3x^2 - x(6x) + 4(6x)^2 = 141$$
$$3x^2 - 6x^2 + 144x^2 = 141$$
$$141x^2 = 141$$
$$x^2 = 1$$
$$x = \pm 1$$

Hence, $(-1, -6)$ is another point for which the slope of the tangent line is zero.

16.2 If $y = f(x)$ is a function satisfying the equation $x^3y^2 - 2x + y^3 = 36$, find a formula for the derivative dy/dx.

$$D_x(x^3y^2 - 2x + y^3) = D_x(36)$$
$$D_x(x^3y^2) - 2D_x(x) + D_x(y^3) = 0$$
$$x^3D_x(y^2) + y^2D_x(x^3) - 2(1) + 3y^2\frac{dy}{dx} = 0$$
$$x^3\left(2y\frac{dy}{dx}\right) + y^2(3x^2) - 2 + 3y^2\frac{dy}{dx} = 0$$
$$(2x^3y + 3y^2)\frac{dy}{dx} = 2 - 3x^2y^2$$
$$\frac{dy}{dx} = \frac{2 - 3x^2y^2}{2x^3y + 3y^2}$$

16.3 If $y = f(x)$ is a differentiable function satisfying the equation $x^2y^3 - 5xy^2 - 4y = 4$ and if $f(3) = 2$, find the slope of the tangent line to the graph of f at the point (3, 2).

$$D_x(x^2y^3 - 5xy^2 - 4y) = D_x(4)$$
$$x^2(3y^2y') + y^3(2x) - 5(x(2yy') + y^2) - 4y' = 0$$
$$3x^2y^2y' + 2xy^3 - 10xyy' - 5y^2 - 4y' = 0$$

Substitute 3 for x and 2 for y,

$$108y' + 48 - 60y' - 20 - 4y' = 0$$
$$44y' + 28 = 0$$
$$y' = -\frac{28}{44} = -\frac{7}{11}$$

Hence, the slope of the tangent line at (3, 2) is $-\frac{7}{11}$.

Supplementary Problems

16.4 (*a*) Find a formula for the slope of the tangent line to the curve $x^2 - xy + y^2 = 12$ at any point (x, y). Also, find the coordinates of all points on the curve where the tangent line is: (*b*) horizontal; (*c*) vertical.

16.5 Consider the hyperbola $5x^2 - 2y^2 = 130$.

(*a*) Find a formula for the slope of the tangent line to this hyperbola at (x, y).

(*b*) For what value(s) of k will the line $x - 3y + k = 0$ be normal to the hyperbola at a point of intersection?

16.6 Find y' by implicit differentiation.

(*a*) $x^2 + y^2 = 25$ (*b*) $x^3 = \frac{2x + y}{2x - y}$ (*c*) $\frac{1}{x} + \frac{1}{y} = 1$

(*d*) $\sqrt{x} + \sqrt{y} = 1$ (*e*) $x^3 - y^3 = 2xy$ (*f*) $(7x - 1)^3 = 2y^4$

(*g*) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ (*h*) $y + xy^3 = 2x$ (*i*) $x^2 = \frac{x + y}{x - y}$

16.7 Use implicit differentiation to find the slope-intercept equation of the tangent line at the indicated point.

(*a*) $y^3 - xy = 2$ at (3, 2) (*b*) $\frac{x^2}{16} + y^2 = 1$ at $\left(2, \frac{\sqrt{3}}{2}\right)$

(*c*) $(y - x)^2 + y^3 = xy + 7$ at (1, 2) (*d*) $x^3 - y^3 = 7xy$ at (4, 2)

(*e*) $4xy^2 + 98 = 2x^4 - y^4$ at (3, 2) (*f*) $4x^3 - xy - 2y^3 = 1$ at (1, 1)

(*g*) $\frac{x^3 - y}{1 - y^3} = x$ at (1, −1) (*h*) $2y = xy^3 + 2x^3 - 3$ at (1, −1)

16.8 Use implicit differentiation to find the slope-intercept equation of the normal line at the indicated point.

(*a*) $y^3x + 2y = x^2$ at (2, 1) (*b*) $2x^3y + 2y^4 - x^4 = 2$ at (2, 1)

(*c*) $y\sqrt{x} - x\sqrt{y} = 12$ at (9, 16) (*d*) $x^2 + y^2 = 25$ at (3, 4)

16.9 Use implicit differentiation to find the slope of the tangent line to the graph of $y = \sqrt{1 - \sqrt{1 - x}}$ at $x = \frac{7}{16}$. [*Hint*: Eliminate the radicals by squaring twice.]

Chapter 17

The Mean-Value Theorem and the Sign of the Derivative

17.1 ROLLE'S THEOREM AND THE MEAN-VALUE THEOREM

Let us consider a function f that is continuous over a closed interval $[a, b]$ and differentiable at every point of the open interval (a, b). We also suppose that $f(a) = f(b) = 0$. Graphs of some examples of such a function are shown in Fig. 17-1. It seems clear that there must always be some point between $x = a$ and $x = b$ at which the tangent line is horizontal and, therefore, at which the derivative of f is 0.

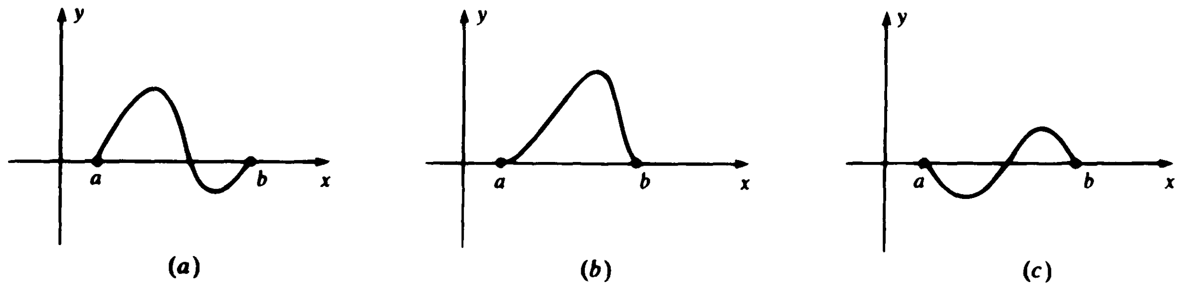


Fig. 17-1

Theorem 17.1 (*Rolle's Theorem*): If f is continuous over a closed interval $[a, b]$, differentiable on the open interval (a, b), and if $f(a) = f(b) = 0$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

See Problem 17.6 for the proof.

Rolle's theorem enables us to prove the following basic theorem (which is also referred to as the *law of the mean for derivatives*).

Theorem 17.2 (*Mean-Value Theorem*): Let f be continuous over the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is a number c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

For a proof, see Problem 17.7.

EXAMPLE In graphic terms, the mean-value theorem states that at some point along an arc of a curve, the tangent line is parallel to the line connecting the initial and the terminal points of the arc. This can be seen in Fig. 17-2, where there are three numbers (c_1, c_2, and c_3) between a and b for which the slope of the tangent line to the graph $f'(c)$ is equal to the slope of the line AB, $\dfrac{f(b) - f(a)}{b - a}$.

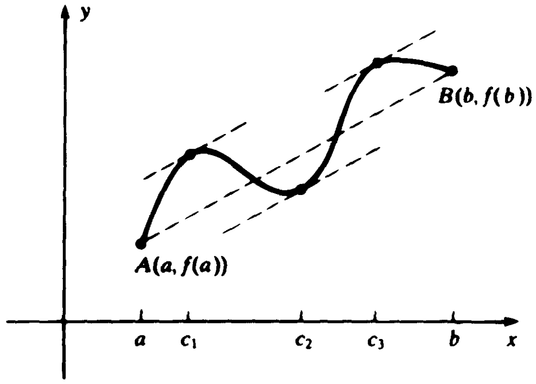


Fig. 17-2

17.2 THE SIGN OF THE DERIVATIVE

A function f is said to be *increasing* on a set $\mathscr{A}$ if, for any u and v in $\mathscr{A}$, $u < v$ implies $f(u) < f(v)$. Similarly, f is *decreasing* on a set $\mathscr{A}$ if, for any u and v in $\mathscr{A}$, $u < v$ implies $f(u) > f(v)$.

Of course, on a given set, a function is not necessarily either increasing or decreasing (see Fig. 17-3).

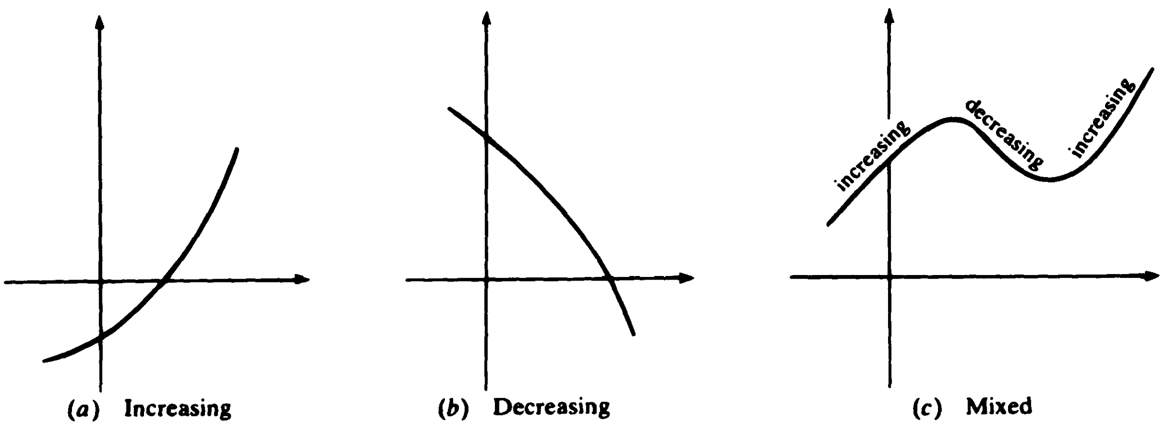


Fig. 17-3

Theorem 17.3: If $f'(x) > 0$ for all x in the open interval (a, b), then f is increasing on (a, b). If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).

For the proof, see Problem 17.8. The converse of Theorem 17.3 does not hold. In fact, the function $f(x) = x^3$ is differentiable and increasing on $(-1, 1)$—and everywhere else—but $f'(x) = 3x^2$ is zero for $x = 0$ [see Fig. 7-3(*b*)].

The following important property of continuous functions will often be useful.

Theorem 17.4 *(Intermediate-Value Theorem)*: Let f be a continuous function over a closed interval $[a, b]$, with $f(a) \neq f(b)$. Then any number between $f(a)$ and $f(b)$ is assumed as the value of f for some argument between a and b.

While Theorem 17.4 is not elementary, its content is intuitively obvious. The function could not "skip" an intermediate value unless there were a break in the graph; that is, unless the function were discontinuous. As illustrated in Fig. 17-4, a function f satisfying Theorem 17.4 may also take on values that are not between $f(a)$ and $f(b)$. In Problem 17.9, we prove:

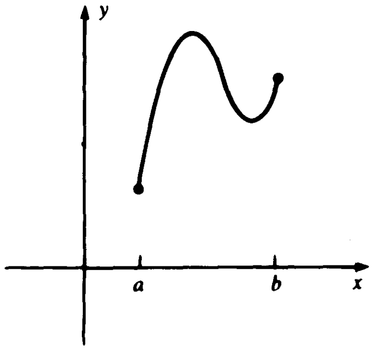


Fig. 17-4

Corollary 17.5: If f is a continuous function with domain $[a, b]$, then the range of f is either a closed interval or a point.

Solved Problems

17.1 Verify Rolle's theorem for $f(x) = x^3 - 3x^2 - x + 3$ on the interval $[1, 3]$.

f is differentiable everywhere and, therefore, also continuous. Furthermore,

$$f(1) = (1)^3 - 3(1)^2 - 1 + 3 = 1 - 3 - 1 + 3 = 0$$
$$f(3) = (3)^3 - 3(3)^2 - 3 + 3 = 27 - 27 - 3 + 3 = 0$$

so that all the hypotheses of Rolle's theorem are valid. There must then be some c in $(1, 3)$ for which $f'(c) = 0$.

Now, by the quadratic formula, the roots of $f'(x) = 3x^2 - 6x - 1 = 0$ are

$$x = \frac{6 \pm \sqrt{6^2 - 4(3)(-1)}}{2(3)} = \frac{6 \pm \sqrt{36 + 12}}{6} = \frac{6 \pm \sqrt{48}}{6} = \frac{6 \pm \sqrt{16}\sqrt{3}}{6}$$
$$= \frac{6 \pm 4\sqrt{3}}{6} = 1 \pm \frac{2}{3}\sqrt{3}$$

Consider the root $c = 1 + \frac{2}{3}\sqrt{3}$. Since $\sqrt{3} < 3$,

$$1 < 1 + \frac{2}{3}\sqrt{3} < 1 + \frac{2}{3}(3) = 1 + 2 = 3$$

Thus, c is in $(1, 3)$ and $f'(c) = 0$.

17.2 Verify the mean-value theorem for $f(x) = x^3 - 6x^2 - 4x + 30$ on the interval $[4, 6]$.

f is differentiable and, therefore, continuous for all x.

$$f(6) = (6)^3 - 6(6)^2 - 4(6) + 30 = 216 - 216 - 24 + 30 = 6$$
$$f(4) = (4)^3 - 6(4)^2 - 4(4) + 30 = 64 - 96 - 16 + 30 = -18$$

whence,
$$\frac{f(6) - f(4)}{6 - 4} = \frac{6 - (-18)}{6 - 4} = \frac{24}{2} = 12$$

We must therefore find some c in (4, 6) such that $f'(c) = 12$.

Now, $f'(x) = 3x^2 - 12x - 4$, so that c will be a solution of

$$3x^2 - 12x - 4 = 12 \qquad \text{or} \qquad 3x^2 - 12x - 16 = 0$$

By the quadratic formula,

$$x = \frac{12 \pm \sqrt{144 - 4(3)(-16)}}{2(3)} = \frac{12 \pm \sqrt{144 + 192}}{6} = \frac{12 \pm \sqrt{336}}{6}$$
$$= \frac{12 \pm \sqrt{16}\sqrt{21}}{6} = \frac{12 \pm 4\sqrt{21}}{6} = 2 \pm \tfrac{2}{3}\sqrt{21}$$

Choose $c = 2 + \frac{2}{3}\sqrt{21}$. Since $4 < \sqrt{21} < 5$,

$$4 < 2 + \frac{8}{3} = 2 + \frac{2}{3}(4) < 2 + \frac{2}{3}\sqrt{21} < 2 + \frac{2}{3}(5) < 2 + 4 = 6$$

Thus, c is in (4, 6) and $f'(c) = 12$.

17.3 **Determine when the function $f(x) = x^3 - 6x^2 + 9x + 2$ is increasing and when it is decreasing, and sketch its graph.**

We have $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$. The crucial points are 1 and 3 [see Fig. 17-5(*a*)].

(i) When $x < 1$, both $(x - 1)$ and $(x - 3)$ are negative and so $f'(x) > 0$ in $(-\infty, 1)$.

(ii) When x moves from $(-\infty, 1)$ into (1, 3), the factor $(x - 1)$ changes from negative to positive, but $(x - 3)$ remains negative. Hence, $f'(x) < 0$ in (1, 3).

(iii) When x moves from (1, 3) into $(3, \infty)$, $(x - 3)$ changes from negative to positive, but $(x - 1)$ remains positive. Hence, $f'(x) > 0$ in $(3, \infty)$.

Thus, by Theorem 17.3, f is increasing for $x < 1$, decreasing for $1 < x < 3$, and increasing for $x > 3$. Note that $f(1) = 6$, $f(3) = 2$, $\lim_{x \to +\infty} f(x) = +\infty$, and $\lim_{x \to -\infty} f(x) = -\infty$. A rough sketch of the graph is shown in Fig. 17-5(*b*).

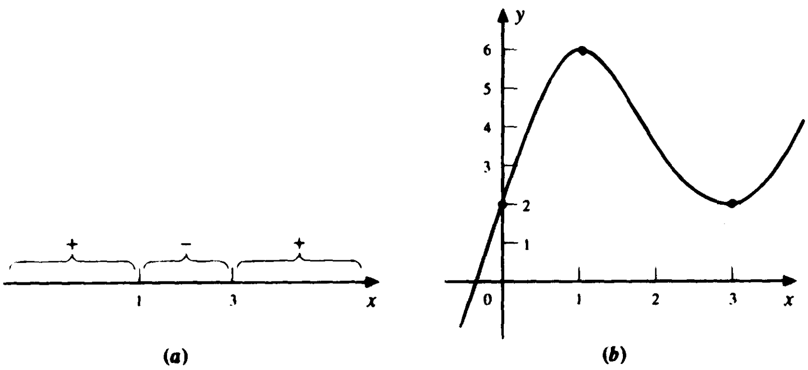


Fig. 17-5

17.4 **Verify Rolle's theorem for $f(x) = 2x^6 - 8x^5 + 6x^4 - x^3 + 6x^2 - 11x + 6$ on [1, 3].**

f is differentiable everywhere, and

$$f(1) = 2 - 8 + 6 - 1 + 6 - 11 + 6 = 0$$
$$f(3) = 1458 - 1944 + 486 - 27 + 54 - 33 + 6 = 0$$

It is difficult to compute a value of x in (1, 3) for which

$$f'(x) = 12x^5 - 40x^4 + 24x^3 - 3x^2 + 12x - 11 = 0$$

However, $f'(x)$ is itself a continuous function such that

$$f'(1) = 12 - 40 + 24 - 3 + 12 - 11 = -6 < 0$$
$$f'(3) = 2916 - 3240 + 648 - 27 + 36 - 11 = 322 > 0$$

Hence, the intermediate-value theorem assures us that there must be some number c between 1 and 3 for which $f'(c) = 0$.

17.5 Show that $f(x) = 2x^3 + x - 4 = 0$ has exactly one real solution.

Since $f(0) = -4$ and $f(2) = 16 + 2 - 4 = 14$, the intermediate-value theorem guarantees that f has a zero between 0 and 2; call it x_0.

Because $f'(x) = 6x^2 + 1 > 0$, $f(x)$ is increasing everywhere (Theorem 17.3). Therefore, when $x > x_0$, $f(x) > 0$; and when $x < x_0$, $f(x) < 0$. In other words, there is no zero other than x_0.

17.6 Prove Rolle's theorem (Theorem 17.1).

Case 1: $f(x) = 0$ for all x in $[a, b]$. Then $f'(x) = 0$ for all x in (a, b), since the derivative of a constant function is 0.

Case 2: $f(x) > 0$ for some x in (a, b). Then, by the extreme-value theorem (Theorem 14.2), an absolute maximum of f on $[a, b]$ exists, and must be positive [since $f(x) > 0$ for some x in (a, b)]. Because $f(a) = f(b) = 0$, the maximum is achieved at some point c in the open interval (a, b). Thus, the absolute maximum is also a relative maximum and, by Theorem 14.1, $f'(c) = 0$.

Case 3: $f(x) < 0$ for some x in (a, b). Let $g(x) = -f(x)$. Then, by Case 2, $g'(c) = 0$ for some c in (a, b). Consequently, $f'(c) = -g'(c) = 0$.

17.7 Prove the mean-value theorem (Theorem 17.2).

Let

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

Then g is continuous over $[a, b]$ and differentiable on (a, b). Moreover,

$$g(a) = f(a) - \frac{f(b) - f(a)}{b - a}(a - a) - f(a) = f(a) - 0 - f(a) = 0$$
$$g(b) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) - f(a) = f(b) - (f(b) - f(a)) - f(a)$$
$$= f(b) - f(b) + f(a) - f(a) = 0$$

By Rolle's theorem, applied to g, there exists c in (a, b) for which $g'(c) = 0$. But,

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

whence,
$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad f'(c) = \frac{f(b) - f(a)}{b - a}$$

17.8 Prove Theorem 17.3.

Assume that $f'(x) > 0$ for all x in (a, b) and that $a < u < v < b$. We must show that $f(u) < f(v)$. By the mean-value theorem, applied to f on the closed interval $[u, v]$, there is some number c in (u, v) such that

$$f'(c) = \frac{f(v) - f(u)}{v - u} \quad \text{or} \quad f(v) - f(u) = f'(c)(v - u)$$

But $f'(c) > 0$ and $v - u > 0$; hence, $f(v) - f(u) > 0$, $f(u) < f(v)$.

The case $f'(x) < 0$ is handled similarly.

17.9 Prove Corollary 17.5.

By the extreme-value theorem, f has an absolute maximum value $f(d)$ at some argument d in $[a, b]$, and an absolute minimum value $f(c)$ at some argument c in $[a, b]$. If $f(c) = f(d) = k$, then f is constant on $[a, b]$, and its range is the single point k. If $f(c) \neq f(d)$, then the intermediate-value theorem, applied to the closed subinterval bounded by d and c, ensures that f assumes every value between $f(c)$ and $f(d)$. The range of f is then the closed interval $[f(c), f(d)]$ (which includes the values assumed on that part of $[a, b]$ that lies outside the subinterval).

Supplementary Problems

17.10 Determine whether the hypotheses of Rolle's theorem hold for each function f, and if they do, verify the conclusion of the theorem.

(*a*) $f(x) = x^2 - 2x - 3$ on $[-1, 3]$ (*b*) $f(x) = x^3 - x$ on $[0, 1]$

(*c*) $f(x) = 9x^3 - 4x$ on $[-\frac{2}{3}, \frac{2}{3}]$ (*d*) $f(x) = x^3 - 3x^2 + x + 1$ on $[1, 1 + \sqrt{2}]$

(*e*) $f(x) = \dfrac{x^2 - x - 6}{x - 1}$ on $[-2, 3]$ (*f*) $f(x) = \begin{cases} \dfrac{x^3 - 2x^2 - 5x + 6}{x - 1} & \text{if } x \neq 1 \\ -6 & \text{if } x = 1 \end{cases}$ on $[-2, 3]$

(*g*) $f(x) = x^{2/3} - 2x^{1/3}$ on $[0, 8]$ (*h*) $f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \end{cases}$ on $[0, 2]$

17.11 Verify that the hypotheses of the mean-value theorem hold for each function f on the given interval, and find a value c satisfying the conclusion of the theorem.

(*a*) $f(x) = 2x + 3$ on $[1, 4]$ (*b*) $f(x) = 3x^2 - 5x + 1$ on $[2, 5]$

(*c*) $f(x) = x^{3/4}$ on $[0, 16]$ (*d*) $f(x) = \dfrac{x + 3}{x - 4}$ on $[1, 3]$

(*e*) $f(x) = \sqrt{25 - x^2}$ on $[-3, 4]$ (*f*) $f(x) = \dfrac{1}{x - 4}$ on $[0, 2]$

17.12 Determine where the function f is increasing and where it is decreasing. Then sketch the graph of f.

(*a*) $f(x) = 3x + 1$ (*b*) $f(x) = -2x + 2$ (*c*) $f(x) = x^2 - 4x + 7$

(*d*) $f(x) = 1 - 4x - x^2$ (*e*) $f(x) = \sqrt{1 - x^2}$ (*f*) $f(x) = \frac{1}{3}\sqrt{9 - x^2}$

(*g*) $f(x) = x^3 - 9x^2 + 15x - 3$ (*h*) $f(x) = x + \dfrac{1}{x}$ (*i*) $f(x) = x^3 - 12x + 20$

17.13 Let f be a differentiable function such that $f'(x) \neq 0$ for all x in the open interval (a, b). Prove that there is at most one zero of $f(x)$ in (a, b). [*Hint:* Assume, for the sake of contradiction, that c and d are two zeros of f, with $a < c < d < b$, and apply Rolle's theorem on the interval $[c, d]$.]

17.14 Consider the polynomial $f(x) = 5x^3 - 2x^2 + 3x - 4$.

(*a*) Show that f has a zero between 0 and 1.

(*b*) Show that f has only one real zero. [*Hint:* Use Problem 17.13.]

17.15 Assume f continuous over $[0, 1]$ and assume that $f(0) = f(1)$. Which one(s) of the following assertions must be true?

(*a*) If f has an absolute maximum at c in $(0, 1)$, then $f'(c) = 0$.

(*b*) f' exists on $(0, 1)$.

(c) $f'(c) = 0$ for some c in $(0, 1)$.

(d) $\lim_{x \to c} f(x) = f(c)$ for all c in $(0, 1)$.

(e) f has an absolute maximum at some point c in $(0, 1)$.

17.16 Let f and g be differentiable functions.

(a) If $f(a) = g(a)$ and $f(b) = g(b)$, where $a < b$, show that $f'(c) = g'(c)$ for some c in (a, b).

(b) If $f(a) \geq g(a)$ and $f'(x) > g'(x)$ for all x, show that $f(x) > g(x)$ for all $x > a$.

(c) If $f'(x) > g'(x)$ for all x, show that the graphs of f and g intersect at most once. [*Hint:* In each part, apply the appropriate theorem to the function $h(x) = f(x) - g(x)$.]

17.17 Let f be a differentiable function on an open interval (a, b).

(a) If f is increasing on (a, b), prove that $f'(x) \geq 0$ for every x in (a, b).
$\left[\textit{Hint}: f'(x) = \lim_{h \to 0^+} \dfrac{f(x+h) - f(x)}{h} \text{ and Problem 9.10}(a) \text{ applies.} \right]$

(b) If f is decreasing on (a, b), prove that $f'(x) \leq 0$ for every x in (a, b).

17.18 The mean-value theorem predicts the existence of what point on the graph of $y = \sqrt[3]{x}$ between (27, 3) and (125, 5)?

17.19 (*Generalized Rolle's Theorem*) Assume f is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, prove that there is a point c in (a, b) such that $f'(c) = 0$. [*Hint:* Apply Rolle's theorem to $g(x) = f(x) - f(a)$.]

17.20 Let $f(x) = x^3 - 4x^2 + 4x$ and $g(x) = 1$ for all x.

(a) Find the intersection of the graphs of f and g.

(b) Find the zeros of f.

(c) If the domain of f is restricted to the closed interval [0, 3], what would be the range of f?

17.21 Prove that $8x^3 - 6x^2 - 2x + 1$ has a zero between 0 and 1. [*Hint:* Apply Rolle's theorem to the function $2x^4 - 2x^3 - x^2 + x$.]

17.22 Show that $x^3 + 2x - 5 = 0$ has exactly one real root.

17.23 Prove that the equation $x^4 + x = 1$ has at least one solution in the interval [0, 1].

17.24 Find a point on the graph of $y = x^2 + x + 3$, between $x = 1$ and $x = 2$, where the tangent line is parallel to the line connecting (1, 5) and (2, 9).

17.25 (a) Show that $f(x) = x^5 + x - 1$ has exactly one real zero.

(b) GC Locate the real zero of $x^5 + x - 1$ correct to the first decimal place.

17.26 (a) GC Use a graphing calculator to estimate the intervals in which the function $f(x) = x^4 - 3x^2 + x - 4$ is increasing and the intervals in which it is decreasing.

(b) As in part (a), but for the function $f(x) = x^3 - 2x^2 + x - 2$.

Chapter 18

Rectilinear Motion and Instantaneous Velocity

Rectilinear motion is motion along a straight line. Consider, for instance, an automobile moving along a straight road. We can imagine a coordinate system imposed on the line containing the road (see Fig. 18-1). (On many highways there actually is such a coordinate system, with markers along the side of the road indicating the distance from one end of the highway.) If s designates the coordinate of the automobile and t denotes the time, then the motion of the automobile is specified by expressing s, its position, as a function of t: $s = f(t)$.

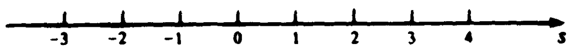


Fig. 18-1

The speedometer indicates how fast the automobile is moving. Since the speedometer reading often varies continuously, it is obvious that the speedometer indicates how fast the car is moving *at the moment* when it is read. Let us analyze this notion in order to find the mathematical concept that lies behind it.

If the automobile moves according to the equation $s = f(t)$, its position at time t is $f(t)$, and at time $t + h$, very close to time t, its position is $f(t + h)$. The distance[1] between its position at time t and its position at time $t + h$ is $f(t + h) - f(t)$ (which can be negative). The time elapsed between t and $t + h$ is h. Hence, the *average velocity*[2] during this time interval is

$$\frac{f(t + h) - f(t)}{h}$$

(Average velocity = displacement ÷ time.) Now as the elapsed time h gets closer to 0, the average velocity approaches what we intuitively think of as the *instantaneous velocity* v at time t. Thus,

$$v = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}$$

In other words, the instantaneous velocity v is the derivative $f'(t)$.

EXAMPLES

(*a*) The height s of a water column is observed to follow the law $s = f(t) = 3t + 2$. Thus, the instantaneous velocity v of the top surface is $f'(t) = 3$.

(*b*) The position s of an automobile along a highway is given by $s = f(t) = t^2 - 2t$. Hence, its instantaneous velocity is $v = f'(t) = 2t - 2$. At time $t = 3$, its velocity v is $2(3) - 2 = 4$.

The sign of the instantaneous velocity v indicates the direction in which the object is moving. If $v = ds/dt > 0$ over a time interval, Theorem 17.3 tells us that s is increasing in that interval. Thus, if the s-axis is horizontal and directed to the right, as in Fig. 18-2(*a*), then the object is moving to the right; but if the s-axis is vertical and directed upward, as in Fig. 18-2(*b*), then the object is moving upward. On

[1] More precisely, the *displacement*, since it can be positive, negative, or zero.

[2] We use the term *velocity* rather than *speed* because the quantity referred to can be negative. Speed is defined as the magnitude of the velocity and is never negative.

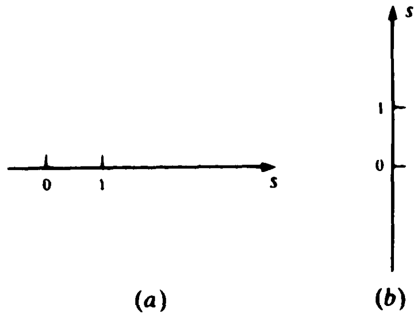


Fig. 18-2

the other hand, if $v = ds/dt < 0$ over a time interval, then s must be decreasing in that interval. In Fig. 18-2(a), the object would be moving to the left (in the direction of decreasing s); in Fig. 18-2(b), the object would be moving downward.

A consequence of these facts is that *at an instant* t *when a continuously moving object reverses direction, its instantaneous velocity* v *must be 0.* For if v were, say, positive at t, it would be positive in a small interval of time surrounding t; the object would therefore be moving in the same direction just before and just after t. Or, to say the same thing in a slightly different way, a reversal in direction means a relative extremum of s, which in turn implies (Theorem 14.1) $ds/dt = 0$.

EXAMPLE An object moves along a straight line as indicated in Fig. 18-3(a). In functional form,

$$s = f(t) = (t + 2)^2 \quad [s \text{ in meters, } t \text{ in seconds}]$$

as graphed in Fig. 18-3(b). The object's instantaneous velocity is

$$v = f'(t) = 2(t + 2) \quad [\text{meters per second}]$$

For $t + 2 < 0$, or $t < -2$, v is negative and the object is moving to the left; for $t + 2 > 0$, or $t > -2$, v is positive and the object is moving to the right. The object reverses direction at $t = -2$, and at that instant $v = 0$. [Note that $f(t)$ has a relative minimum at $t = -2$.]

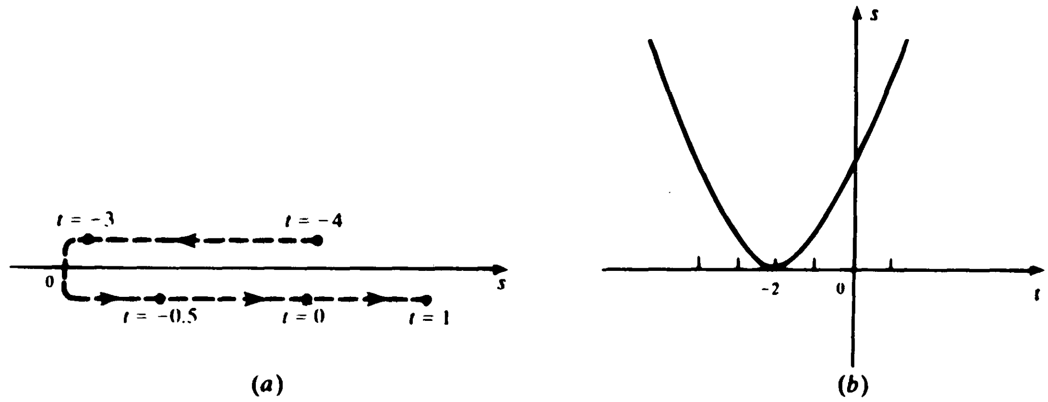


Fig. 18-3

Free Fall

Consider an object that has been thrown straight up or down, or has been dropped from rest, and which is acted upon solely by the gravitational pull of the earth. The ensuing rectilinear motion is called *free fall.*

Let us put a coordinate system on the vertical line along which the object moves, such that the s-axis is directed upward, away from the earth, with $s = 0$ located at the surface of the earth (Fig. 18-4).

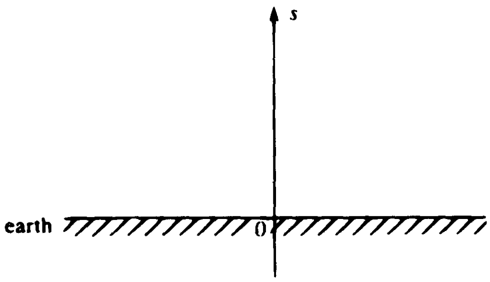


Fig. 18-4

Then the equation of free fall is

$$s = s_0 + v_0 t - 16t^2 \tag{18.1}$$

where s is measured in feet and t in seconds.[3] Here s_0 and v_0 are, respectively, the position (height) and the velocity of the object at time $t = 0$. The instantaneous velocity v is obtained by differentiating (*18.1*),

$$v = \frac{ds}{dt} = v_0 - 32t \tag{18.2}$$

EXAMPLES

(*a*) At $t = 0$, a rock is dropped from rest from the top of a building 256 feet high. When, and with what velocity, does it strike the ground?

With $s_0 = 256$ and $v_0 = 0$, (*18.1*) becomes $s = 256 - 16t^2$ and the time of striking the ground is given by the solution of

$$\begin{aligned} 0 &= 256 - 16t^2 \\ 16t^2 &= 256 \\ t^2 &= 16 \\ t &= \pm 4 \text{ seconds} \end{aligned}$$

Since we are assuming that the motion takes place when $t \geq 0$, the only solution is $t = 4$ seconds.

The velocity equation (*18.2*) is $v = -32t$, and so, for $t = 4$,

$$v = -32(4) = -128 \text{ feet per second}$$

the minus sign indicating that the rock is moving downward when it hits the ground.

ALGEBRA

$$\begin{aligned} x \text{ feet per second} &= 60x \text{ feet per minute} \\ &= 60(60x) \text{ feet per hour} \\ &= \frac{3600x}{5280} \text{ miles per hour} \\ &= \frac{15}{22}\, x \text{ miles per hour} \end{aligned} \tag{18.3}$$

For example, 128 feet per second $= \frac{15}{22}(128) = 87\frac{3}{11}$ miles per hour.

(*b*) A rocket is shot vertically from the ground with an initial velocity of 96 feet per second. When does the rocket reach its maximum height, and what is its maximum height?

With $s_0 = 0$ and $v_0 = 96$, (*18.1*) and (*18.2*) become

$$s = 96t - 16t^2 \qquad \text{and} \qquad v = \frac{ds}{dt} = 96 - 32t$$

[3] If the position is measured in meters, the equation reads $s = s_0 + v_0 t - 4.9t^2$.

At a maximum value, or turning point, $v = 0$. Hence,

$$0 = 96 - 32t$$
$$32t = 96$$
$$t = 3$$

Thus, it takes 3 seconds for the rocket to reach its maximum height, which is

$$s = 96(3) - 16(3)^2 = 288 - 16(9) = 288 - 144 = 144 \text{ feet}$$

(c) When does the rocket of part (b) hit the ground?

It suffices to set $s = 0$ in the free-fall equation *(18.1)*,

$$0 = 96t - 16t^2$$
$$0 = 6t - t^2 \qquad \text{[divide by 16]}$$
$$0 = t(6 - t)$$

from which $t = 0$ or $t = 6$. Hence, the rocket hits the ground again after 6 seconds.

Notice that the rocket rose for 3 seconds to its maximum height, and then took 3 more seconds to fall back to the ground. In general, the upward flight from point P to point Q will take exactly the same time as the downward flight from Q to P. In addition, the rocket will return to a given height with the same speed (magnitude of the velocity) that it had upon leaving that height.

Solved Problems

18.1 A stone is thrown straight down from the top of an 80-foot tower. If the initial speed is 64 feet per second, how long does it take to hit the ground, and with what speed does it hit the ground?

Here $s_0 = 80$ and $v_0 = -64$. (The speed is the magnitude of the velocity. The minus sign for v_0 indicates that the object is moving downward.) Hence,

$$s = 80 - 64t - 16t^2 \qquad \text{and} \qquad v = \frac{ds}{dt} = -64 - 32t$$

The stone hits the ground when $s = 0$,

$$0 = 80 - 64t - 16t^2$$
$$0 = t^2 + 4t - 5 \qquad \text{[divide by } -16]$$
$$0 = (t + 5)(t - 1)$$
$$t + 5 = 0 \quad \text{or} \quad t - 1 = 0$$
$$t = -5 \quad \text{or} \quad t = 1$$

Since the time of fall must be positive, $t = 1$ second. The velocity v when the stone hits the ground is

$$v(1) = -64 - 32(1) = -64 - 32 = -96 \text{ feet per second}$$

By *(18.3)*, 96 feet per second $= \frac{15}{22}(96) = 65\frac{5}{11}$ miles per hour.

18.2 A rocket, shot straight up from the ground, reaches a height of 256 feet after 2 seconds. What was its initial velocity, what will be its maximum height, and when does it reach its maximum height?

Since $s_0 = 0$,

$$s = v_0 t - 16t^2 \qquad \text{and} \qquad v = \frac{ds}{dt} = v_0 - 32t$$

When $t = 2$, $s = 256$,

$$256 = v_0(2) - 16(2)^2$$
$$256 = 2v_0 - 64$$
$$320 = 2v_0$$
$$160 = v_0$$

The initial velocity was 160 feet per second, so that

$$s = 160t - 16t^2 \quad \text{and} \quad v = 160 - 32t$$

To find the time when the maximum height is reached, set $v = 0$,

$$0 = 160 - 32t$$
$$32t = 160$$
$$t = 5 \text{ seconds}$$

To find the maximum height, substitute $t = 5$ in the formula for s,

$$s = 160(5) - 16(5)^2 = 800 - 16(25) = 800 - 400 = 400 \text{ feet}$$

18.3 A car is moving along a straight road according to the equation

$$s = f(t) = 2t^3 - 3t^2 - 12t$$

Describe its motion by indicating when and where the car is moving to the right, and when and where it is moving to the left. When is the car at rest?

We have $v = f'(t) = 6t^2 - 6t - 12 = 6(t^2 - t - 2) = 6(t - 2)(t + 1)$. The key points are $t = 2$ and $t = -1$ (see Fig. 18-5).

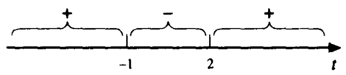


Fig. 18-5

(i) When $t > 2$, both $t - 2$ and $t + 1$ are positive. So, $v > 0$ and the car is moving to the right.

(ii) As t moves from $(2, \infty)$ through $t = 2$ into $(-1, 2)$, the sign of $t - 2$ changes, but the sign of $t + 1$ remains the same. Hence, v changes from positive to negative. Thus, for $-1 < t < 2$, the car is moving to the left.

(iii) As t moves through $t = -1$ from $(-1, 2)$ into $(-\infty, -1)$, the sign of $t + 1$ changes but the sign of $t - 2$ remains the same. Hence, v changes from negative to positive. So, the car is moving to the right when $t < -1$.

When $t = -1$,

$$s = 2(-1)^3 - 3(-1)^2 - 12(-1) = -2 - 3 + 12 = 7$$

When $t = 2$

$$s = 2(2)^3 - 3(2)^2 - 12(2) = 16 - 12 - 24 = -20$$

Thus, the car moves to the right until, at $t = -1$, it reaches $s = 7$, where it reverses direction and moves left until, at $t = 2$, it reaches $s = -20$, where it reverses direction again and keeps moving to the right thereafter (see Fig. 18-6).

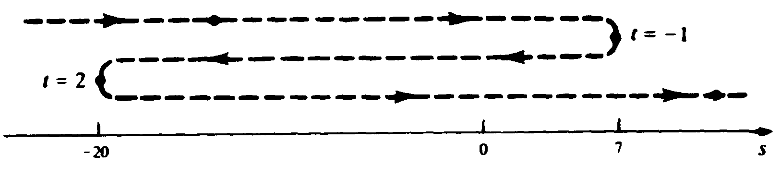


Fig. 18-6

The car is never at rest. It makes sense to talk about the car being at rest only when the position of the car is constant over an interval of time (not at just a single point). In such a case, the velocity would be zero on an entire interval.

Supplementary Problems

18.4 (*a*) If an object is released from rest at any given height, show that, after t seconds, it has dropped $16t^2$ feet (assuming that it has not yet struck the ground).

(*b*) How many seconds does it take the object in part (*a*) to fall: (i) 1 foot; (ii) 16 feet; (iii) 64 feet; (iv) 100 feet?

18.5 A rock is dropped down a well that is 256 feet deep. When will it hit the bottom of the well?

18.6 Assuming that one story of a building is 10 feet, with what speed, in miles per hour, does an object dropped from the top of a 40-story building hit the ground?

18.7 A rocket is shot straight up into the air with an initial velocity of 128 feet per second.

(*a*) How far has it traveled in 1 second? in 2 seconds? (*b*) When does it reach its maximum height? (*c*) What is its maximum height? (*d*) When does it hit the ground again? (*e*) What is its speed when it hits the ground?

18.8 A rock is thrown straight down from a height of 480 feet with an initial velocity of 16 feet per second.

(*a*) How long does it take to hit the ground? (*b*) With what speed does it hit the ground? (*c*) How long does it take before the rock is moving at a speed of 112 feet per second? (*d*) When has the rock traveled a distance of 60 feet?

18.9 An automobile moves along a straight highway, with its position given by $s = 12t^3 - 18t^2 + 9t - \frac{3}{2}$ [s in miles, t in hours].

(*a*) Describe the motion of the car: when it is moving to the right, when to the left, where and when it changes direction.

(*b*) What distance has it traveled in 1 hour from $t = 0$ to $t = 1$?

18.10 The position of a moving object on a line is given by the formula $s = (t - 1)^3(t - 5)$.

(*a*) When is the object moving to the right? (*b*) When is it moving to the left? (*c*) When is it changing direction? (*d*) When is it at rest? (*e*) What is the farthest to the left of the origin that it moves?

18.11 A particle moves on a straight line so that its position s (miles) at time t (hours) is given by $s = (4t - 1)(t - 1)^2$.

(*a*) When is the particle moving to the right? (*b*) When is the particle moving to the left? (*c*) When does it change direction? (*d*) When the particle is moving to the left, what is the maximum *speed* that it achieves? (The speed is the absolute value of the velocity.)

18.12 A particle moves along the x-axis according to the equation $x = 10t - 2t^2$. What is the *total* distance covered by the particle between $t = 0$ and $t = 3$?

18.13 A rocket was shot straight up from the ground. What must have been its initial velocity if it returned to earth in 20 seconds?

18.14 Two particles move along the x-axis. Their positions $f(t)$ and $g(t)$ are given by $f(t) = 6t - t^2$ and $g(t) = t^2 - 4t$.

(*a*) When do they have the same position? (*b*) When do they have the same velocity? (*c*) When they have the same position, are they moving in the same direction?

18.15 A rock is dropped and strikes the ground with a velocity of -49 meters per second. (*a*) How long did it fall? (*b*) Find the height from which it was dropped.

18.16 A ball is thrown vertically upward from the top of a 96-foot tower. Two seconds later, the velocity of the ball is 16 feet per second. Find: (*a*) the maximum height that the ball reaches; (*b*) the speed of the ball when it hits the ground.

Chapter 19

Instantaneous Rate of Change

One quantity, y, may be related to another quantity, x, by a function f: $y = f(x)$. A change in the value of x usually induces a corresponding change in the value of y.

EXAMPLE Let x be the length of the side of a cube, and let y be the volume of the cube. Then $y = x^3$. In the case where the side has length $x = 2$ units, consider a small change Δx in the length.

NOTATION Δx (read "delta-ex") is the traditional symbol in calculus for a small change in x. Δx is considered a single symbol, *not* a product of Δ and x. In earlier chapters, the role of Δx often was taken by the symbol h.

The new volume will be $(2 + \Delta x)^3$, and so the *change* in the value of the volume y is $(2 + \Delta x)^3 - 2^3$. This change in y is denoted traditionally by Δy,

$$\Delta y = (2 + \Delta x)^3 - 2^3$$

Now the natural way to compare the change Δy in y to the change Δx in x is to calculate the ratio $\Delta y/\Delta x$. This ratio depends of course on Δx, but if we let Δx approach 0, then the limit of $\Delta y/\Delta x$ will define the *instantaneous rate of change* of y compared to x, when $x = 2$. We have (ALGEBRA, Problem 11.2)

$$\Delta y = (2 + \Delta x)^3 - 2^3 = [(2)^3 + 3(2)^2(\Delta x)^1 + 3(2)^1(\Delta x)^2 + (\Delta x)^3] - 2^3$$
$$= 12\Delta x + 6(\Delta x)^2 + (\Delta x)^3 = (\Delta x)(12 + 6\Delta x + (\Delta x)^2)$$

Hence,
$$\frac{\Delta y}{\Delta x} = 12 + 6\Delta x + (\Delta x)^2$$

and
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (12 + 6\Delta x + (\Delta x)^2) = 12$$

Therefore, when the side is 2, the rate of change of the volume with respect to the side is 12. This means that, for sides close to 2, the change Δy in the volume is approximately 12 times the change Δx in the side (since $\Delta y/\Delta x$ is close to 12). Let us look at a few numerical cases.

If $\Delta x = 0.1$, then the new side $x + \Delta x$ is 2.1, and the new volume is $(2.1)^3 = 9.261$. So, $\Delta y = 9.261 - 8 = 1.261$, and

$$\frac{\Delta y}{\Delta x} = \frac{1.261}{0.1} = 12.61$$

If $\Delta x = 0.01$, then the new side $x + \Delta x$ is 2.01, and the new volume is $(2.01)^3 = 8.120\,601$. So, $\Delta y = 8.120\,601 - 8 = 0.120\,601$, and

$$\frac{\Delta y}{\Delta x} = \frac{0.120\,601}{0.01} = 12.0601$$

If $\Delta x = 0.001$, a similar computation yields

$$\frac{\Delta y}{\Delta x} = 12.006\,001$$

Let us extend the result of the above example from $y = x^3$ to an arbitrary differentiable function $y = f(x)$. Consider a small change Δx in the value of the argument x. The new value of the argument is then $x + \Delta x$, and the new value of y will be $f(x + \Delta x)$. Hence, the change Δy in the value of the function is

$$\Delta y = f(x + \Delta x) - f(x)$$

The ratio of the change in the function value to the change in the argument is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The *instantaneous rate of change of y with respect to x* is defined to be

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

The *instantaneous rate of change is evaluated by the derivative.* It follows that, for Δx close to 0, $\Delta y/\Delta x$ will be close to $f'(x)$, so that

$$\Delta y \approx f'(x)\,\Delta x \qquad (19.1)$$

Solved Problems

19.1 The weekly profit P, in dollars, of a corporation is determined by the number x of radios produced per week, according to the formula

$$P = 75x - 0.03x^2 - 15\,000$$

(*a*) Find the rate at which the profit is changing when the production level x is 1000 radios per week. (*b*) Find the change in weekly profit when the production level x is increased to 1001 radios per week.

(*a*) The rate of change of the profit P with respect to the production level x is $dP/dx = 75 - 0.06x$. When $x = 1000$,

$$\frac{dP}{dx} = 75 - 0.06(1000) = 75 - 60 = 15 \text{ dollars per radio}$$

(*b*) In economics, the rate of change of profit with respect to the production level is called the *marginal profit*. According to (*19.1*), the marginal profit is an approximate measure of how much the profit will change when the production level is increased by one unit. In the present case, we have

$$\begin{aligned} P(1000) &= 75(1000) - 0.03(1000)^2 - 15\,000 \\ &= 75\,000 - 30\,000 - 15\,000 = 30\,000 \\ P(1001) &= 75(1001) - 0.03(1001)^2 - 15\,000 \\ &= 75\,075 - 30\,060.03 - 15\,000 = 30\,014.97 \\ \Delta P &= P(1001) - P(1000) = 14.97 \text{ dollars per week} \end{aligned}$$

which is very closely approximated by the marginal profit, 15 dollars, as computed in part (*a*).

19.2 The volume V of a sphere of radius r is given by the formula $V = 4\pi r^3/3$. (*a*) How fast is the volume changing relative to the radius when the radius is 10 millimeters? (*b*) What is the change in volume when the radius changes from 10 to 10.1 millimeters?

(*a*) $$\frac{dV}{dr} = D_r\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi(3r^2) = 4\pi r^2$$

When $r = 10$,

$$\frac{dV}{dr} = 4\pi(10)^2 = 400\pi \approx 400(3.14) = 1256$$

(b)

$$V(10) = \frac{4}{3}\pi(10)^3 = \frac{4000\pi}{3}$$

$$V(10.1) = \frac{4}{3}\pi(10.1)^3 = \frac{4}{3}\pi(1030.301) = \frac{4121.204\pi}{3}$$

$$\Delta V = V(10.1) - V(10) = \frac{4121.204\pi}{3} - \frac{4000\pi}{3}$$

$$= \frac{\pi}{3}(4121.204 - 4000) = \frac{\pi}{3}(121.204) \approx \frac{3.14}{3}(121.204)$$

$$= 126.86 \text{ cubic millimeters}$$

The change predicted from (*19.1*) and part (*a*) is

$$\Delta V \approx \frac{dV}{dr}\Delta r = 1256(0.1) = 125.6 \text{ cubic millimeters}$$

19.3 An oil tank is being filled. The oil volume V, in gallons, after t minutes is given by

$$V = 1.5t^2 + 2t$$

How fast is the volume increasing when there is 10 gallons of oil in the tank? [*Hint:* To answer the question, "how fast?," you must always find the derivative with respect to *time.*]

When there are 10 gallons in the tank,

$$1.5t^2 + 2t = 10 \qquad \text{or} \qquad 1.5t^2 + 2t - 10 = 0$$

Solving by the quadratic formula,

$$t = \frac{-2 \pm \sqrt{4 - 4(1.5)(-10)}}{2(1.5)} = \frac{-2 \pm \sqrt{4 + 60}}{3} = \frac{-2 \pm \sqrt{64}}{3} = \frac{-2 \pm 8}{3} = 2 \qquad \text{or} \qquad -\frac{10}{3}$$

Since t must be positive, $t = 2$ minutes. The rate at which the oil volume is growing is

$$\frac{dV}{dt} = D_t(1.5t^2 + 2t) = 3t + 2$$

Hence, at the instant $t = 2$ minutes when $V = 10$ gallons,

$$\frac{dV}{dt} = 3(2) + 2 = 6 + 2 = 8 \text{ gallons per minute}$$

Supplementary Problems

19.4 The cost C, in dollars per day, of producing x TV sets per day is given by the formula

$$C = 7000 + 50x - 0.05x^2$$

Find the rate of change of C with respect to x (called the *marginal cost*) when 200 sets are being produced each day.

19.5 The profit P, in dollars per day, resulting from making x units per day of an antibiotic, is

$$P = 5x + 0.02x^2 - 120$$

Find the marginal profit when the production level x is 50 units per day.

19.6 Find the rate at which the surface area of a cube of side x is changing with respect to x, when $x = 2$ feet.

19.7 The number of kilometers a rocket ship is from earth is given by the formula

$$E = 30t + 0.005t^2$$

where t is measured in seconds. How fast is the distance changing when the rocket ship is 35 000 kilometers from earth?

19.8 As a gasoline tank is being emptied, the number G of gallons left after t seconds is given by $G = 3(15 - t)^2$.

(*a*) How fast is gasoline being emptied after 12 seconds? (*b*) What was the average rate at which the gasoline was being drained from the tank over the first 12 seconds? [*Hint:* The average rate is the total amount emptied divided by the time during which it was emptied.]

19.9 If $y = 3x^2 - 2$, find: (*a*) the average rate at which y changes with respect to x over the interval [1, 2]; (*b*) the instantaneous rate of change of y with respect to x when $x = 1$.

19.10 If $y = f(x)$ is a function such that $f'(x) \neq 0$ for any x, find those values of y for which the rate of increase of y^4 with respect to x is 32 times that of y with respect to x.

Chapter 20

Related Rates

Most quantities encountered in science or in everyday life vary with time. If two such quantities are related by an equation, and if we know the rate at which one of them changes, then, by differentiating the equation with respect to time, we can find the rate at which the other quantity changes.

EXAMPLES

(*a*) A 6-foot man is running away from the base of a streetlight that is 15 feet high (see Fig. 20-1). If he moves at the rate of 18 feet per second, how fast is the length of his shadow changing?

Let x be the distance of the man from the base A of the streetlight, and let y be the length of the man's shadow.

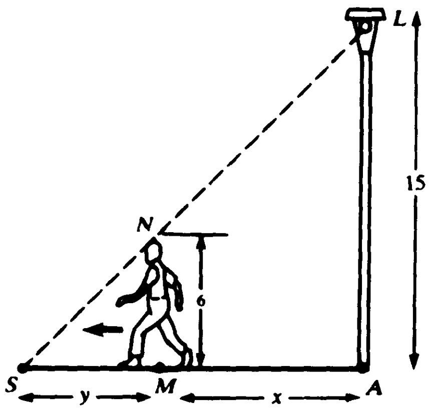
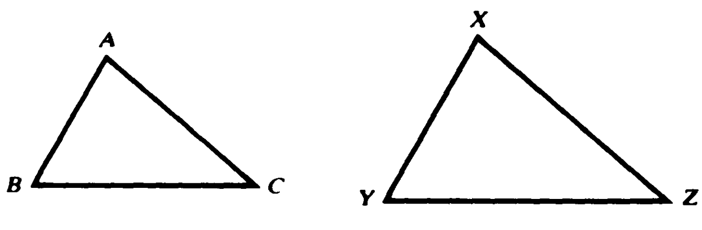


Fig. 20-1

GEOMETRY Two triangles,



are *similar* if their angles are equal in pairs: $\measuredangle A = \measuredangle X$, $\measuredangle B = \measuredangle Y$, $\measuredangle C = \measuredangle Z$. (For this condition to hold, it suffices that *two* angles of the one triangle be equal to two angles of the other.) Similar triangles have corresponding sides in fixed ratio:

$$\frac{\overline{AB}}{\overline{XY}} = \frac{\overline{AC}}{\overline{XZ}} = \frac{\overline{BC}}{\overline{YZ}}$$

In Fig. 20-1, $\triangle SMN$ and $\triangle SAL$ are similar, whence

$$\frac{\overline{SM}}{\overline{SA}} = \frac{\overline{NM}}{\overline{LA}} \quad \text{or} \quad \frac{y}{y+x} = \frac{6}{15} \tag{1}$$

which is the desired relation between x and y. In this case, it is convenient to solve (1) for y in terms of x,

$$\frac{y}{y+x} = \frac{2}{5}$$
$$5y = 2y + 2x$$
$$3y = 2x$$
$$y = \frac{2}{3}x \tag{2}$$

Differentiation of (2) with respect to t gives

$$\frac{dy}{dt} = \frac{2}{3}\frac{dx}{dt} \tag{3}$$

Now because the man is running away from A at the rate of 18 feet per second, x is increasing at that rate. Hence,

$$\frac{dx}{dt} = 18 \text{ feet per second} \quad \text{and} \quad \frac{dy}{dt} = \frac{2}{3}(18) = 12 \text{ feet per second}$$

that is, the shadow is lengthening at the rate of 12 feet per second.

(b) A cube of ice is melting. The side s of the cube is decreasing at the constant rate of 2 inches per minute. How fast is the volume V decreasing?

Since $V = s^3$,

$$\frac{dV}{dt} = \frac{d(s^3)}{dt} = 3s^2\frac{ds}{dt} \quad \text{[by the power chain rule]}$$

The fact that s is *decreasing* at the rate of 2 inches per minute translates into the mathematical statement

$$\frac{ds}{dt} = -2$$

Hence,
$$\frac{dV}{dt} = 3s^2(-2) = -6s^2$$

Thus, although s is decreasing at a constant rate, V is decreasing at a rate proportional to the square of s. For instance, when $s = 3$ inches, V is decreasing at a rate of 54 cubic inches per minute.

(c) Two small airplanes start from a common point A at the same time. One flies east at the rate of 300 kilometers per hour and the other flies south at the rate of 400 kilometers per hour. After 2 hours, how fast is the distance between them changing?

Refer to Fig. 20-2. We are given that $dx/dt = 300$ and $dy/dt = 400$ and wish to find the value of du/dt at $t = 2$ hours. The necessary relation between u, x, and y is furnished by the Pythagorean theorem,

$$u^2 = x^2 + y^2$$

Therefore,
$$\frac{d(u^2)}{dt} = \frac{d(x^2+y^2)}{dt}$$
$$2u\frac{du}{dt} = \frac{d(x^2)}{dt} + \frac{d(y^2)}{dt} \quad \text{[by the power chain rule]}$$
$$2u\frac{du}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad \text{[by the power chain rule]}$$
$$u\frac{du}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = 300x + 400y \tag{4}$$

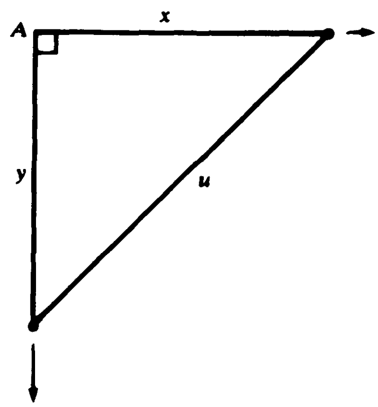


Fig. 20-2

Now we must find x, y, and u after 2 hours. Since x is increasing at the constant rate of 300 kilometers per hour and t is measured from the beginning of the flight, $x = 300t$ (distance = speed × time, when speed is constant). Similarly, $y = 400t$. Hence, at $t = 2$,

$$x = 300(2) = 600 \qquad y = 400(2) = 800$$

and

$$u^2 = (600)^2 + (800)^2 = 360\,000 + 640\,000 = 1\,000\,000$$

$$u = 1000$$

Substituting in (*4*),

$$1000 \frac{du}{dt} = 300(600) + 400(800) = 180\,000 + 320\,000 = 500\,000$$

$$\frac{du}{dt} = \frac{500\,000}{1000} = 500 \text{ kilometers per hour}$$

Solved Problems

20.1 Air is leaking out of a spherical balloon at the rate of 3 cubic inches per minute. When the radius is 5 inches, how fast is the radius decreasing?

Since air is leaking out at the rate of 3 cubic inches per minute, the volume V of the balloon is decreasing at the rate of $dV/dt = -3$. But the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Hence,

$$-3 = \frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \frac{d(r^3)}{dt} = \frac{4}{3}\pi\left(3r^2 \frac{dr}{dt}\right) = 4\pi r^2 \frac{dr}{dt}$$

So,

$$\frac{dr}{dt} = -\frac{3}{4\pi r^2}$$

Substituting $r = 5$,

$$\frac{dr}{dt} = -\frac{3}{100\pi} \approx -\frac{3}{314} \approx -0.009\,55$$

Thus, when the radius is 5 inches, the radius is decreasing at about 0.01 inch per minute.

20.2 A 13-foot ladder leans against a vertical wall (see Fig. 20-3). If the bottom of the ladder is slipping away from the base of the wall at the rate of 2 feet per second, how fast is the top of the ladder moving down the wall when the bottom of the ladder is 5 feet from the base?

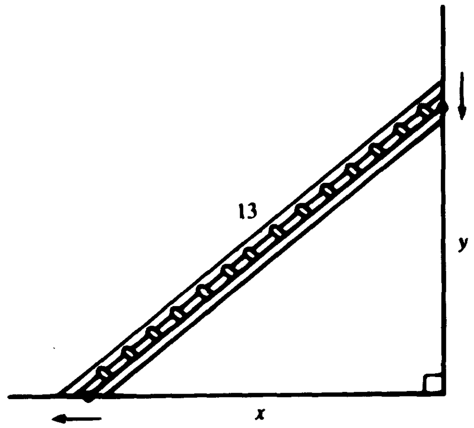


Fig. 20-3

Let x be the distance of the bottom of the ladder from the base of the wall, and let y be the distance of the top of the ladder from the base of the wall. Since the bottom of the ladder is moving away from the base of the wall at 2 feet per second, $dx/dt = 2$. We wish to compute dy/dt when $x = 5$ feet. Now, by the Pythagorean theorem,

$$(13)^2 = x^2 + y^2 \tag{1}$$

Differentiation of this, as in example (*c*), give

$$0 = x\frac{dx}{dt} + y\frac{dy}{dt} = 2x + y\frac{dy}{dt} \tag{2}$$

But when $x = 5$, (*1*) gives

$$y = \sqrt{(13)^2 - (5)^2} = \sqrt{169 - 25} = \sqrt{144} = 12$$

so that (*2*) becomes

$$0 = 2(5) + 12\frac{dy}{dt}$$

$$\frac{dy}{dt} = -\frac{2(5)}{12} = -\frac{5}{6}$$

Hence, the top of the ladder is moving *down* the wall ($dy/dt < 0$) at $\frac{5}{6}$ feet per second when the bottom of the ladder is 5 feet from the wall.

20.3 **A cone-shaped paper cup (see Fig. 20-4) is being filled with water at the rate of 3 cubic centimeters per second. The height of the cup is 10 centimeters and the radius of the base is 5 centimeters. How fast is the water level rising when the level is 4 centimeters?**

At time t (seconds), when the water depth is h, the volume of water in the cup is given by the cone formula $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the top surface. But by similar triangles in Fig. 20-4,

$$\frac{r}{5} = \frac{h}{10} \quad \text{or} \quad r = \frac{5h}{10} = \frac{h}{2}$$

(Only h is of interest, so we are eliminating r.) Thus,

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{3}\pi\left(\frac{h^2}{4}\right)h = \frac{\pi}{12}h^3$$

and, by the power chain rule,

$$\frac{dV}{dt} = \frac{\pi}{12}\frac{d(h^3)}{dt} = \frac{\pi}{12}\left(3h^2\frac{dh}{dt}\right) = \left(\frac{\pi h^2}{4}\right)\frac{dh}{dt}$$

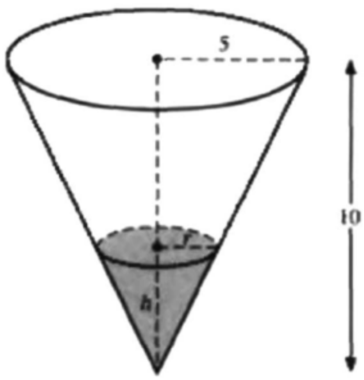


Fig. 20-4

Substituting $dV/dt = 3$ and $h = 4$, we obtain

$$3 = \left(\frac{\pi 16}{4}\right)\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{3}{4\pi} \approx \frac{3}{4(3.14)} \approx 0.24 \text{ centimeter per second}$$

Hence, at the moment when the water level is 4 centimeters, the level is rising at about 0.24 centimeter per second.

20.4 A ship B is moving westward toward a fixed point A at a speed of 12 knots (nautical miles per hour). At the moment when ship B is 72 nautical miles from A, ship C passes through A, heading due south at 10 knots. How fast is the distance between the ships changing 2 hours after ship C has passed through A?

Figure 20-5 shows the situation at time $t > 0$. At $t = 0$, ship C was at A. Since

$$u^2 = x^2 + y^2 \tag{1}$$

we obtain, as in example (c),

$$u\frac{du}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = -12x + 10y \tag{2}$$

since x is decreasing at 12 knots and y is increasing at 10 knots. At $t = 2$, we have (since distance = speed × time)

$$y = 10 \times 2 = 20$$
$$72 - x = 12 \times 2 = 24$$
$$x = 72 - 24 = 48$$

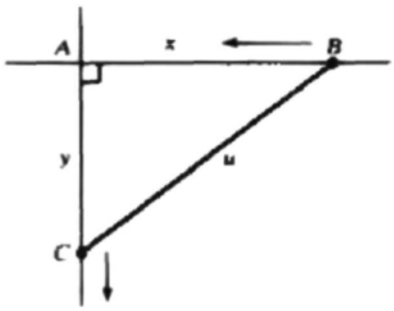


Fig. 20-5

and, by (1),

$$u = \sqrt{(48)^2 + (20)^2} = \sqrt{2304 + 400} = \sqrt{2704} = 52$$

Substitution in (2) yields

$$52\frac{du}{dt} = -12(48) + 10(20) = -576 + 200 = -376$$

$$\frac{du}{dt} = -\frac{376}{52} \approx -7.23$$

which shows that, after 2 hours, the distance between ships B and C is *decreasing* at the rate of 7.23 knots.

Supplementary Problems

20.5 The top of a 25-foot ladder, leaning against a vertical wall, is slipping down the wall at the rate of 1 foot per minute. How fast is the bottom of the ladder slipping along the ground when the bottom of the ladder is 7 feet away from the base of the wall?

20.6 A cylindrical tank of radius 10 feet is being filled with wheat at the rate of 314 cubic feet per minute. How fast is the depth of the wheat increasing? [*Hint:* The volume of a cylinder is $\pi r^2 h$, where r is its radius and h its height.]

20.7 A 5-foot girl is walking toward a 20-foot lamppost at the rate of 6 feet per second. How fast is the tip of her shadow (cast by the lamp) moving?

20.8 A rocket is shot vertically upward with an initial velocity of 400 feet per second. Its height s after t seconds is $s = 400t - 16t^2$. How fast is the distance between the rocket and an observer on the ground 1800 feet away from the launching site changing when the rocket is still rising and is 2400 feet above the ground (see Fig. 20-6)?

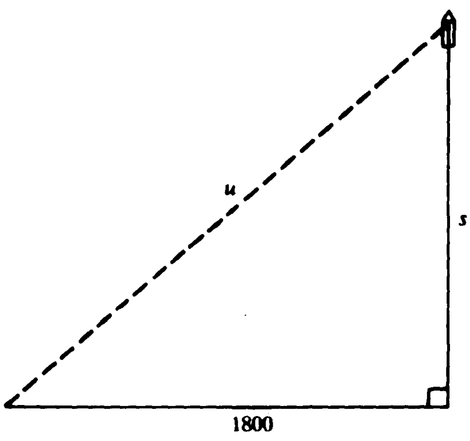


Fig. 20-6

20.9 A small funnel in the shape of a cone is being emptied of fluid at the rate of 12 cubic millimeters per second. The height of the funnel is 20 millimeters and the radius of the base is 4 millimeters. How fast is the fluid level dropping when the level stands 5 millimeters above the vertex of the cone? [Remember that the volume of a cone is $\frac{1}{3}\pi r^2 h$.]

20.10 A balloon is being inflated by pumping in air at the rate of 2 cubic inches per second. How fast is the diameter of the balloon increasing when the radius is one-half inch?

20.11 Oil from an uncapped oil well in the ocean is radiating outward in the form of a circular film on the surface of the water. If the radius of the circle is increasing at the rate of 2 meters per minute, how fast is the area of the oil film growing when the radius reaches 100 meters?

20.12 The length of a rectangle having a constant area of 800 square millimeters is increasing at the rate of 4 millimeters per second. (*a*) What is the width of the rectangle at the moment when the width is decreasing at the rate of 0.5 millimeter per second? (*b*) How fast is the diagonal of the rectangle changing when the width is 20 millimeters?

20.13 A particle moves on the hyperbola $x^2 - 18y^2 = 9$ in such a way that its y-coordinate increases at a constant rate of nine units per second. How fast is its x-coordinate changing when $x = 9$?

20.14 An object moves along the graph of $y = f(x)$. At a certain point, the slope of the curve (that is, the slope of the tangent line to the curve) is $\frac{1}{2}$ and the abscissa (x-coordinate) of the object is decreasing at the rate of three units per second. At that point, how fast is the ordinate (y-coordinate) of the object changing? [*Hint:* $y = f(x)$, and x is a function of t. So y is a composite function of t, which may be differentiated by the chain rule.]

20.15 If the radius of a sphere is increasing at the constant rate of 3 millimeters per second, how fast is the volume changing when the surface area ($4\pi r^2$) is 10 square millimeters?

20.16 What is the radius of an expanding circle at a moment when the rate of change of its area is numerically twice as large as the rate of change of its radius?

20.17 A particle moves along the curve $y = 2x^3 - 3x^2 + 4$. At a certain moment, when $x = 2$, the particle's x-coordinate is increasing at the rate of 0.5 unit per second. How fast is its y-coordinate changing at that moment?

20.18 A plane flying parallel to the ground at a height of 4 kilometers passes over a radar station R (see Fig. 20-7). A short time later, the radar equipment reveals that the plane is 5 kilometers away and that the distance between the plane and the station is increasing at a rate of 300 kilometers per hour. At that moment, how fast is the plane moving horizontally?

20.19 A boat passes a fixed buoy at 9 A.M., heading due west at 3 miles per hour. Another boat passes the same buoy at 10 A.M., heading due north at 5 miles per hour. How fast is the distance between the boats changing at 11:30 A.M.?

20.20 Water is pouring into an inverted cone at the rate of 3.14 cubic meters per minute. The height of the cone is 10 meters and the radius of its base is 5 meters. How fast is the water level rising when the water stands 7.5 meters in the cone?

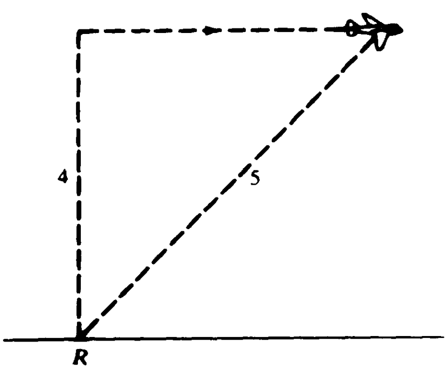


Fig. 20-7

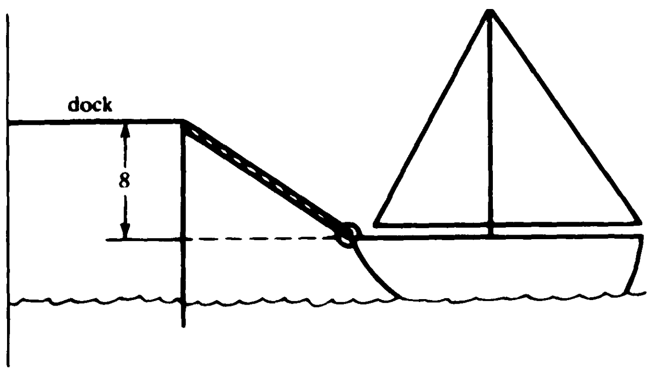


Fig. 20-8

20.21 A particle moves along the curve $y = \frac{1}{2}x^2 + 2x$. At what point(s) on the curve are the abscissa and the ordinate of the particle changing at the same rate?

20.22 A boat is being pulled into a dock by a rope that passes through a ring on the bow of the boat (see Fig. 20-8). The dock is 8 feet higher than the bow ring. How fast is the boat approaching the dock when the length of rope between the dock and the boat is 10 feet, if the rope is being pulled in at the rate of 3 feet per second?

20.23 A girl is flying a kite, which is at a height of 120 feet. The wind is carrying the kite horizontally away from the girl at a speed of 10 feet per second. How fast must the string be let out when the kite is 150 feet away from the girl?

20.24 The bottom of a 17-foot ladder is on the ground and the top rests against a vertical wall. If the bottom is pushed toward the wall at the rate of 3 feet per second, how fast is the top moving up the wall when the bottom of the ladder is 15 feet from the base of the wall?

20.25 At a given moment, one person is 5 miles north of an intersection and is walking straight toward the intersection at a constant rate of 3 miles per hour. At the same moment, a second person is 1 mile east of the intersection and is walking away from the intersection at the constant rate of 2 miles per hour. How fast is the distance between the two people changing 1 hour later? Interpret your answer.

20.26 An object is moving along the graph of $y = 3x - x^3$, and its x-coordinate is changing at the rate of two units per second. How fast is the slope of the tangent line to the graph changing when $x = -1$?

Chapter 21

Approximation by Differentials; Newton's Method

In Chapter 19, we obtained an approximate relation between the change

$$\Delta y = f(x + \Delta x) - f(x)$$

in a differentiable function f and the change Δx in the argument of f. For convenience, we repeat (*19.1*) here and name it the *approximation principle*,

$$\Delta y \approx f'(x)\,\Delta x \tag{21.1}$$

21.1 ESTIMATING THE VALUE OF A FUNCTION

Many practical problems involve finding a value $f(c)$ of some function f. A direct calculation of $f(c)$ may be difficult or, quite often, impossible. However, let us assume that an argument x close to c, the closer the better, can be found such that $f(x)$ and $f'(x)$ can be computed exactly. If we set $\Delta x = c - x$, then $c = x + \Delta x$ and the approximation principle (*21.1*) yields

$$\begin{aligned} f(x + \Delta x) - f(x) &\approx f'(x)\,\Delta x \\ f(c) - f(x) &\approx f'(x)\,\Delta x \\ f(c) &\approx f(x) + f'(x)\,\Delta x \end{aligned} \tag{21.2}$$

EXAMPLE Let us estimate $\sqrt{9.2}$. Here f is the square root function and $c = 9.2$. If we choose $x = 9$, then $\Delta x = 9.2 - 9 = 0.2$. Both $f(x)$ and $f'(x)$ can be computed easily,

$$f(x) = \sqrt{9} = 3$$

$$f'(x) = D_x(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}$$

and (*21.2*) yields

$$\sqrt{9.2} = f(9.2) \approx 3 + \frac{1}{6}(0.2) = 3 + 0.0333\ldots = 3.0333\ldots$$

The actual value of $\sqrt{9.2}$, correct to four decimal places, is 3.0331.

21.2 THE DIFFERENTIAL

The product on the right side of (*21.1*) is called the *differential* of f and is denoted by df.

Definition: Let f be a differentiable function. Then, for a given argument x and increment Δx, the *differential* df of f is defined by

$$df = f'(x)\,\Delta x$$

Notice that df depends on two quantities, x and Δx. Although Δx is usually taken to be small, this is not explicitly required in the definition. However, if Δx is small, then the content of the approximation principle is that

$$f(x + \Delta x) - f(x) \approx df \tag{21.3}$$

EXAMPLE A graphic picture of this last form of the approximation principle is given in Fig. 21-1. Line $\mathscr{L}$ is tangent to the graph of f at P; its slope is therefore $f'(x)$. But then

$$f'(x) = \frac{\overline{RT}}{\overline{PR}} = \frac{\overline{RT}}{\Delta x} \quad \text{or} \quad \overline{RT} = f'(x)\,\Delta x \equiv df$$

Now it is clear that, for Δx very small, $\overline{RT} \approx \overline{RQ}$; that is,

$$df \approx f(x + \Delta x) - f(x)$$

If the value of a function f is given by a formula, say, $f(x) = x^2 + 2x^{-3}$, let us agree that the differential df may also be written

$$d(x^2 + 2x^{-3}) = df = f'(x)\,\Delta x = (2x - 6x^{-4})\,\Delta x$$

In particular, if $f(x) = x$, we shall write

$$dx = df = f'(x)\,\Delta x = 1 \cdot \Delta x = \Delta x$$

Since $dx = \Delta x$, the definition of the differential df can be rewritten as

$$df = f'(x)\,dx$$

Assuming that $dx = \Delta x \neq 0$, we may divide both sides by dx, obtaining the result

$$f'(x) = \frac{df}{dx}$$

If we let $y = f(x)$, this may explain the traditional notation dy/dx for the derivative.

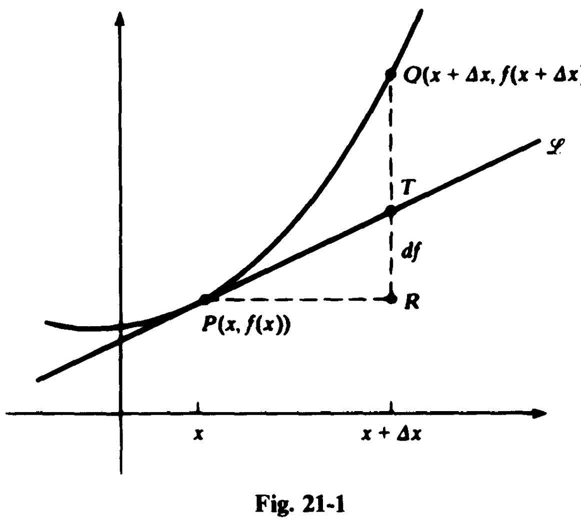


Fig. 21-1

21.3 NEWTON'S METHOD

Let us assume that we are trying to find a solution of the equation

$$f(x) = 0 \qquad (21.4)$$

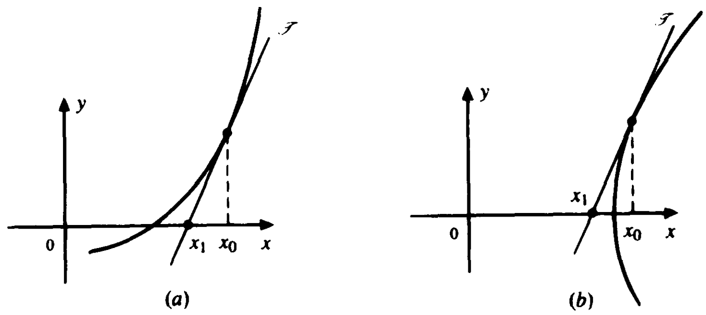


Fig. 21-2

and let us also assume that we know that x_0 is close to a solution. If we draw the tangent line $\mathcal{T}$ to the graph of f at the point with abscissa x_0, then $\mathcal{T}$ will usually intersect the x-axis at a point whose abscissa x_1 is closer to the solution of *(21.4)* than x_0 (see Fig. 21-2).

A point-slope equation of the tangent line $\mathcal{T}$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

If $\mathcal{T}$ intersects the x-axis at the point $(x_1, 0)$, then

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

If $f'(x_0) \neq 0$,

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If we repeat the same procedure, but now starting with x_1 instead of x_0, we obtain a value x_2 that should be still closer to the solution of *(21.4)*,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

If we keep applying this procedure, the resulting sequence of numbers $x_0, x_1, x_2, \ldots, x_n, \ldots$ is determined by the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{21.5}$$

This process for finding better and better approximations to a solution of the equation $f(x) = 0$ is known as *Newton's method.* It is not always guaranteed that the numbers generated by Newton's method will approach a solution of $f(x) = 0$. Some difficulties that may arise will be discussed in the problems below.

EXAMPLE If we wish to approximate $\sqrt{2}$, we should try to find an approximate solution of $x^2 - 2 = 0$. Here $f(x) = x^2 - 2$. Then $f'(x) = 2x$, and *(21.5)* becomes

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{2x_n^2 - (x_n^2 - 2)}{2x_n} = \frac{x_n^2 + 2}{2x_n} \tag{21.6}$$

If we take the first approximation x_0 to be 1 (since we know that $\sqrt{2}$ is between 1 and 2), then we obtain by successively substituting $n = 0, 1, 2, \ldots$ in *(21.6)*,[1]

[1] The computations required by Newton's method are usually too tedious to be done by hand. A calculator, preferably a programmable calculator, should be used.

$$x_1 = \frac{1+2}{2} = 1.5$$

$$x_2 = \frac{(1.5)^2 + 2}{2(1.5)} = \frac{2.25 + 2}{3} = \frac{4.25}{3} \approx 1.416\,666\,667$$

$$x_3 \approx \frac{(1.416\,666\,667)^2 + 2}{2(1.416\,666\,667)} \approx 1.414\,215\,686$$

$$x_4 \approx \frac{(1.414\,215\,686)^2 + 2}{2(1.414\,215\,686)} \approx 1.414\,213\,562$$

$$x_5 \approx \frac{(1.414\,213\,562)^2 + 2}{2(1.414\,213\,562)} \approx 1.414\,213\,562$$

Since our approximations for x_4 and x_5 are equal, all future values will be the same, and we have obtained the best approximation to nine decimal places within the limits of our calculator. Thus, $\sqrt{2} \approx 1.414\,213\,562$.

Solved Problems

21.1 **Using the approximation principle, estimate the value of $\sqrt{62}$.**

Letting $f(x) = \sqrt{x}$ and $c = 62$, choose $x = 64$ (the perfect square closest to 62). Then,

$$\Delta x = c - x = 62 - 64 = -2$$
$$f(x) = \sqrt{64} = 8$$
$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{64}} = \frac{1}{2(8)} = \frac{1}{16}$$

and *(21.2)* yields

$$\sqrt{62} \approx 8 + \frac{1}{16}(-2) = 8 - \frac{1}{8} = 7\frac{7}{8} = 7.875$$

Actually, $\sqrt{62} = 7.8740\ldots$.

21.2 **Use the approximation principle to estimate the value of $\sqrt[5]{33}$.**

Let $f(x) = \sqrt[5]{x}$, $c = 33$, and $x = 32$. Then $\Delta x = c - x = 1$, $f(x) = \sqrt[5]{32} = 2$, and

$$f'(x) = D_x(x^{1/5}) = \frac{1}{5}x^{-4/5} = \frac{1}{5x^{4/5}} = \frac{1}{5(\sqrt[5]{x})^4} = \frac{1}{5(\sqrt[5]{32})^4} = \frac{1}{5(2)^4} = \frac{1}{5(16)} = \frac{1}{80}.$$

Hence, by *(21.2)*,

$$\sqrt[5]{33} = f(c) \approx 2 + \frac{1}{80}(1) = 2.0125$$

Since the actual value is $2.0123\ldots$, the approximation is correct to three decimal places.

21.3 **Measurement of the side of a square room yields the result of 18.5 feet. Hence, the area is $A = (18.5)^2 = 342.25$ square feet. If the measuring device has a possible error of at most 0.05 foot, estimate the maximum possible error in the area.**

The formula for the area is $A = x^2$, where x is the side of the room. Hence, $dA/dx = 2x$. Let $x = 18.5$, and let $18.5 + \Delta x$ be the true length of the side of the room. By assumption, $|\Delta x| \leq 0.05$. The approximation principle yields

$$A(x + \Delta x) - A(x) \approx \frac{dA}{dx} \Delta x$$
$$A(18.5 + \Delta x) - 342.25 \approx 2(18.5)\, \Delta x$$
$$|A(18.5 + \Delta x) - 342.25| \approx |37\, \Delta x| \leq 37(0.05) = 1.85$$

Hence, the error in the area should be at most 1.85 square feet, putting the actual area in the range of (342.25 ± 1.85) square feet. See Problem 21.13.

21.4 Use Newton's method to find the positive solutions of

$$x^4 + x - 3 = 0$$

Let $f(x) = x^4 + x - 3$. Then $f'(x) = 4x^3 + 1$. Since $f(1) = -1$ and $f(2) = 15$, the intermediate-value theorem tells us that there is a solution between 1 and 2. [The interval (1, 2) is suggested by drawing the graph of f with a graphing calculator.] Since $f'(x) > 0$ for $x \geq 0$, f is increasing for $x > 0$, and, therefore, there is exactly one positive real solution. Start with $x_0 = 1$. Equation *(21.5)* becomes

$$x_{n+1} = x_n - \frac{x_n^4 + x_n - 3}{4x_n^3 + 1} = \frac{4x_n^4 + x_n - (x_n^4 + x_n - 3)}{4x_n^3 + 1} = \frac{3x_n^4 + 3}{4x_n^3 + 1}$$

Successive calculations yield $x_1 = 1.2$, $x_2 = 1.165\,419\,616$, $x_3 = 1.164\,037\,269$, $x_4 = 1.164\,035\,141$, and $x_5 = 1.164\,035\,141$. Thus, the approximate solution is $x = 1.164\,035\,141$.

21.5 Show that if Newton's method is applied to the equation $x^{1/3} = 0$ with $x_0 = 1$, the result is a divergent sequence of values (which certainly does not converge to the root $x = 0$).

Let $f(x) = x^{1/3}$. So, $f'(x) = \dfrac{1}{3x^{2/3}}$ and *(21.5)* becomes

$$x_{n+1} = x_n - \frac{x_n^{1/3}}{1/(3x_n^{2/3})} = x_n - 3x_n = -2x_n$$

Hence, $x_1 = -2$, $x_2 = 4$, $x_3 = -8$, and, in general, $x_n = (-2)^n$.

Note: If we are seeking a solution r of an equation $f(x) = 0$, then it can be shown that a sufficient condition that Newton's method yields a sequence of values tht converges to r is that

$$\left| \frac{f(x) \cdot f''(x)}{[f'(x)]^2} \right| < 1$$

for all x in an interval around r that includes x_0. However, this is not a necessary condition.

Supplementary Problems

21.6 Use the approximation principle to estimate the following quantities:

(*a*) $\sqrt{51}$ (*b*) $\sqrt{78}$ (*c*) $\sqrt[3]{123}$ (*d*) $(8.35)^{2/3}$ (*e*) $(33)^{-1/5}$

(*f*) $\sqrt[4]{\frac{17}{81}}$ (*g*) $\sqrt[3]{0.065}$ (*h*) $\sqrt{80.5}$ (*i*) $\sqrt[3]{215}$

21.7 The measurement of the side of a cubical container yields 8.14 centimeters, with a possible error of at most 0.005 centimeter. Give an estimate of the maximum possible error in the value of $V = (8.14)^3 = 539.353\,14$ cubic centimeters for the volume of the container.

21.8 It is desired to give a spherical tank 20 feet (240 inches) in diameter a coat of paint 0.1 inch thick. Use the approximation principle to estimate how many gallons of paint will be required. ($V = \frac{4}{3}\pi r^3$, and 1 gallon is about 231 cubic inches.)

21.9 A solid steel cylinder has a radius of 2.5 centimeters and a height of 10 centimeters. A tight-fitting sleeve is to be made that will extend the radius to 2.6 centimeters. Find the amount of steel needed for the sleeve: (*a*) by the approximation principle; (*b*) by an exact calculation.

21.10 If the side of a cube is measured with a percentage error of at most 3%, estimate the maximum percentage error in the volume of the cube. (If ΔQ is the error in measurement of a quantity Q, then $|\Delta Q/Q| \times 100\%$ is the *percentage error*.)

21.11 Assume, contrary to fact, that the Earth is a perfect sphere, with a radius of 4000 miles. The volume of ice at the North and South Poles is estimated to be about 8 000 000 cubic miles. If this ice were melted and if the resulting water were distributed uniformly over the globe, approximately what would be the depth of the added water at any point of the Earth?

21.12 (*a*) Let $y = x^{3/2}$. When $x = 4$ and $dx = 2$, find the value of dy.

(*b*) Let $y = 2x\sqrt{1 + x^2}$. When $x = 0$ and $dx = 3$, find the value of dy.

21.13 For Problem 21.3, calculate exactly the largest possible error in the area.

21.14 Establish the very useful approximation formula $(1 + u)^r \approx 1 + ru$, where r is any rational exponent and $|u|$ is small compared to 1. [*Hint:* Apply the approximation principle to $f(x) = x^r$, letting $x = 1$ and $\Delta x = u$.]

21.15 GC Use Newton's method to approximate the following quantities:

(*a*) $\sqrt[4]{2}$ (*b*) $\sqrt[3]{4}$ (*c*) $\sqrt[5]{23}$ (*d*) $\sqrt[3]{6}$

21.16 (*a*) Show that Newton's method for finding $\sqrt{c}$ yields the equation

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$$

for the sequence of approximating values.

(*b*) GC Use part (*a*) to approximate $\sqrt{3}$ by Newton's method.

21.17 GC Use Newton's method to approximate solutions of the following equations:

(*a*) $x^3 - x - 1 = 0$ (*b*) $x^3 + x - 1 = 0$ (*c*) $x^4 - 2x^2 + 0.5 = 0$ (*d*) $x^3 + 2x - 4 = 0$

(*e*) $x^3 - 3x^2 + 3 = 0$ (*f*) $x^3 - x + 3 = 0$ (*g*) $x^3 - 2x - 1 = 0$

21.18 Show that $x^3 + x^2 - 10 = 0$ has a unique root in (1, 2) and approximate this root by Newton's method, with $x_0 = 2$.

21.19 Show that $x^5 + 5x - 7 = 0$ has a unique solution in (1, 2) and approximate this root by Newton's method.

21.20 Explain why Newton's method does not work in the following cases:

(*a*) Solve $x^3 - 6x^2 + 12x - 7$ with $x_0 = 2$.

(*b*) Solve $x^3 - 3x^2 + x - 1 = 0$ with $x_0 = 1$.

(*c*) Solve $f(x) = 0$, where $f(x) = \begin{cases} \sqrt{x-1} & \text{for } x \geq 1 \\ -\sqrt{1-x} & \text{for } x < 1 \end{cases}$ and $x_0 > 1$ (say, $x = 1 + b$, $b > 0$).

Chapter 22

Higher-Order Derivatives

The derivative f' of a function f is itself a function, which may be differentiable. If f' is differentiable, its derivative will be denoted by f''. The derivative of f'', if it exists, is denoted by f''', and so on.

Definition:

$$f''(x) = D_x(f'(x))$$
$$f'''(x) = D_x(f''(x))$$
$$f^{(4)}(x) = D_x(f'''(x))$$
$$\vdots$$

We call f' the *first derivative* of f, f'' the *second derivative* of f, and f''' the *third derivative* of f. If the order n exceeds 3, we write $f^{(n)}$ for the nth derivative of f.

EXAMPLES

(*a*) If $f(x) = 3x^4 - 7x^3 + 5x^2 + 2x - 1$, then,

$$f'(x) = 12x^3 - 21x^2 + 10x + 2$$
$$f''(x) = 36x^2 - 42x + 10$$
$$f'''(x) = 72x - 42$$
$$f^{(4)}(x) = 72$$
$$f^{(n)} = 0 \qquad \text{for all } n \geq 5$$

(*b*) If $f(x) = \frac{1}{2}x^3 - 5x^2 + x + 4$, then,

$$f'(x) = \frac{3}{2}x^2 - 10x + 1$$
$$f''(x) = 3x - 10$$
$$f'''(x) = 3$$
$$f^{(n)}(x) = 0 \qquad \text{for all } n \geq 4$$

It is clear that if f is a polynomial of degree k, then the nth derivative $f^{(n)}$ will be 0 for all $n > k$.

(*c*) If $f(x) = \frac{1}{x} = x^{-1}$, then,

$$f'(x) = -x^{-2} = -\frac{1}{x^2}$$
$$f''(x) = 2x^{-3} = \frac{2}{x^3}$$
$$f'''(x) = -6x^{-4} = -\frac{6}{x^4}$$
$$f^{(4)}(x) = 24x^{-5} = \frac{24}{x^5}$$
$$\vdots$$

In this case, the nth derivative will never be the constant function 0.

Alternative Notation

First derivative: $$f'(x) = D_x f(x) = \frac{df}{dx} = \frac{dy}{dx} = D_x y = y'$$

Second derivative: $$f''(x) = D_x^2 f(x) = \frac{d^2f}{dx^2} = \frac{d^2y}{dx^2} = D_x^2 y = y''$$

Third derivative: $f'''(x) = D_x^3 f(x) = \frac{d^3f}{dx^3} = \frac{d^3y}{dx^3} = D_x^3 y = y'''$

nth derivative: $f^{(n)}(x) = D_x^n f(x) = \frac{d^nf}{dx^n} = \frac{d^ny}{dx^n} = D_x^n y = y^{(n)}$

Higher-Order Implicit Differentiation

EXAMPLE Let $y = f(x)$ be a differentiable function satisfying the equation

$$x^2 + y^2 = 9 \tag{0}$$

(We know that $y = \sqrt{9 - x^2}$ or $y = -\sqrt{9 - x^2}$; their graphs are shown in Fig. 22-1.)

Let us find a formula for the second derivative y'', where y stands for either of the two functions.

$$D_x(x^2 + y^2) = D_x(9)$$
$$2x + 2yy' = 0 \qquad [D_x y^2 = 2yy' \text{ by the power chain rule}]$$
$$x + yy' = 0 \tag{1}$$

Next, differentiate both sides of (1) with respect to x,

$$D_x(x + yy') = D_x(0)$$
$$1 + yD_x(y') + y'D_x y = 0$$
$$1 + yy'' + y' \cdot y' = 0$$
$$1 + yy'' + (y')^2 = 0 \tag{2}$$

Solve (1) for y' in terms of x and y,

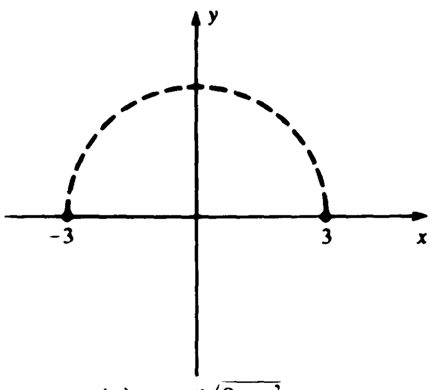
$$y' = -\frac{x}{y}$$

Substitute $-(x/y)$ for y' in (2),

$$1 + yy'' + \frac{x^2}{y^2} = 0$$
$$y^2 + y^3y'' + x^2 = 0 \qquad [\text{multiplying by } y^2]$$

Solve for y'',

$$y'' = -\frac{x^2 + y^2}{y^3}$$



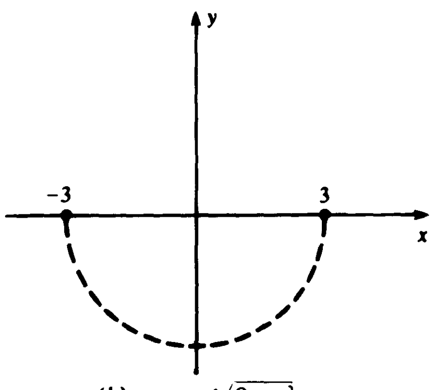


Fig. 22-1

From (0), we may substitute 9 for $x^2 + y^2$,

$$y'' = -\frac{9}{y^3}$$

Acceleration

Let an object move along a coordinate axis according to the equation $s = f(t)$, where s is the coordinate of the object and t is the time. From Chapter 18, the object's velocity is given by

$$v = \frac{ds}{dt} = f'(t)$$

The rate at which the velocity changes is called the *acceleration a*.

Definition: $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t)$

EXAMPLES

(*a*) For an object in free fall, $s = s_0 + v_0 t - 16t^2$, where s, measured in feet, is positive in the upward direction and t is measured in seconds. Recall that s_0 and v_0 denote the initial position and velocity; that is, the values of s and v when $t = 0$. Hence,

$$v = \frac{ds}{dt} = v_0 - 32t$$

$$a = \frac{dv}{dt} = -32$$

Thus, the velocity decreases by 32 feet per second every second. This is sometimes expressed by saying that the (downward) acceleration due to gravity is 32 feet per second per second, which is abbreviated as 32 ft/sec^2.

(*b*) An object moves along a straight line according to the equation $s = 2t^3 - 3t^2 + t - 1$. Then,

$$v = \frac{ds}{dt} = 6t^2 - 6t + 1$$

$$a = \frac{dv}{dt} = 12t - 6$$

In this case, the acceleration is not constant. Notice that $a > 0$ when $12t - 6 > 0$, or $t > \frac{1}{2}$. This implies (by Theorem 17.3) that the velocity is increasing for $t > \frac{1}{2}$.

Solved Problems

22.1 Describe all the derivatives (first, second, etc.) of the following functions:

(*a*) $f(x) = \frac{1}{4}x^4 - 5x - \pi$ (*b*) $f(x) = \frac{x}{x+1}$

(*a*) $f'(x) = x^3 - 5 \quad f''(x) = 3x^2 \quad f'''(x) = 6x \quad f^{(4)}(x) = 6 \quad f^{(n)}(x) = 0$ for $n \geq 5$

(*b*)
$$f'(x) = \frac{(x+1)D_x x - xD_x(x+1)}{(x+1)^2} \qquad \text{[by the quotient rule]}$$
$$= \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} = (x+1)^{-2}$$
$$f''(x) = -2(x+1)^{-3}D_x(x+1) \qquad \text{[by the power chain rule]}$$
$$= -2(x+1)^{-3}(1) = \frac{-2}{(x+1)^3}$$
$$f'''(x) = -2(-3)(x+1)^{-4} = \frac{+6}{(x+1)^4}$$
$$f^{(4)}(x) = -2(-3)(-4)(x+1)^{-5} = \frac{-24}{(x+1)^5}$$
$$\vdots$$
$$f^{(n)}(x) = -2(-3)(-4)(-5)\cdots(-n)(x+1)^{-(n+1)} = \frac{(-1)^{n-1}n!}{(x+1)^{n+1}}$$

ALGEBRA $(-1)^{n-1}$ will be 1 when n is odd and -1 when n is even. $n!$ stands for the product $1 \times 2 \times 3 \times \cdots \times n$ of the first n positive integers.

22.2 Find y'' if

$$y^3 - xy = 1 \qquad (0)$$

Differentiation of (*0*), using the power chain rule for $D_x y^3$ and the product rule for $D_x(xy)$, gives

$$3y^2y' - (xy' + y) = 0$$
$$3y^2y' - xy' - y = 0$$
$$(3y^2 - x)y' - y = 0 \qquad (1)$$

Next, differentiate (*1*),

$$(3y^2 - x)D_x y' + y'D_x(3y^2 - x) - y' = 0 \qquad \text{[by the product rule]}$$
$$(3y^2 - x)y'' + y'(6yy' - 1) - y' = 0 \qquad \text{[by the power chain rule]}$$
$$(3y^2 - x)y'' + y'((6yy' - 1) - 1) = 0 \qquad \text{[factor } y'\text{]}$$
$$(3y^2 - x)y'' + y'(6yy' - 2) = 0 \qquad (2)$$

Now solve (*1*) for y',

$$y' = \frac{y}{3y^2 - x}$$

Finally, substitute into (*2*) and solve for y'',

$$(3y^2 - x)y'' + \frac{y}{3y^2 - x}\cdot\left(\frac{6y^2}{3y^2 - x} - 2\right) = 0$$
$$(3y^2 - x)^3y'' + (3y^2 - x)\frac{y}{3y^2 - x}\cdot(3y^2 - x)\left(\frac{6y^2}{3y^2 - x} - 2\right) = 0 \qquad \text{[multiply by } (3y^2 - x)^2\text{]}$$
$$(3y^2 - x)^3y'' + y\cdot(6y^2 - 2(3y^2 - x)) = 0 \qquad \left[a\left(\frac{b}{a} - c\right) = b - ca\right]$$
$$(3y^2 - x)^3y'' + y(6y^2 - 6y^2 + 2x) = 0$$
$$(3y^2 - x)^3y'' + 2xy = 0$$
$$y'' = \frac{-2xy}{(3y^2 - x)^3}$$

22.3 If y is a function of x such that

$$x^3 - 2xy + y^3 = 8 \qquad (0)$$

and such that $y = 2$ when $x = 2$ [note that these values satisfy (0)], find the values of y' and y'' when $x = 2$.

Proceed as in Problem 22.2.

$$x^3 - 2xy + y^3 = 8$$
$$D_x(x^3 - 2xy + y^3) = D_x(8)$$
$$3x^2 - 2(xy' + y) + 3y^2y' = 0$$
$$3x^2 - 2xy' - 2y + 3y^2y' = 0 \qquad (1)$$

$$D_x(3x^2 - 2xy' - 2y + 3y^2y') = D_x(0)$$
$$6x - 2(xy'' + y') - 2y' + 3(y^2y'' + y'(2yy')) = 0$$
$$6x - 2xy'' - 2y' - 2y' + 3y^2y'' + 6y(y')^2 = 0 \qquad (2)$$

Substitute 2 for x and 2 for y in (1),

$$12 - 4y' - 4 + 12y' = 0 \quad \text{or} \quad 8y' + 8 = 0 \quad \text{or} \quad y' = -1$$

Substitute 2 for x, 2 for y, and -1 for y' in (2),

$$12 - 4y'' + 2 + 2 + 12y'' + 12 = 0 \quad \text{or} \quad 8y'' + 28 = 0 \quad \text{or} \quad y'' = -\frac{28}{8} = -\frac{7}{2}$$

22.4 Let $s = t^3 - 9t^2 + 24t$ describe the position s at time t of an object moving on a straight line. (*a*) Find the velocity and acceleration. (*b*) Determine when the velocity is positive and when it is negative. (*c*) Determine when the acceleration is positive and when it is negative. (*d*) Describe the motion of the object.

(*a*) $v = \frac{ds}{dt} = 3t^2 - 18t + 24 = 3(t^2 - 6t + 8) = 3(t - 2)(t - 4)$

$a = \frac{dv}{dt} = 6t - 18 = 6(t - 3)$

(*b*) v is positive when $t - 2 > 0$ and $t - 4 > 0$ or when $t - 2 < 0$ and $t - 4 < 0$; that is, when

$$t > 2 \text{ and } t > 4 \quad \text{or} \quad t < 2 \text{ and } t < 4$$

which is equivalent to $t > 4$ or $t < 2$. $v = 0$ if and only if $t = 2$ or $t = 4$. Hence, $v < 0$ when $2 < t < 4$.

(*c*) $a > 0$ when $t > 3$, and $a < 0$ when $t < 3$.

(*d*) Assuming that s increases to the right, positive velocity indicates movement to the right, and negative velocity movement to the left. The object moves right until, at $t = 2$, it is at $s = 20$, where it reverses direction. It then moves left until, at $t = 4$, it is at $s = 16$, where it reverses direction again. Thereafter, it continues to move right (see Fig. 22-2).

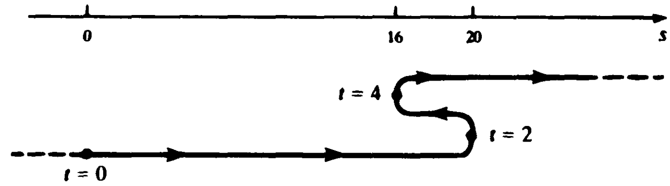


Fig. 22-2

Supplementary Problems

22.5 Find the second derivative $D_x^2 y$ of the following functions y:

(a) $y = x - \frac{1}{x}$ (b) $y = \pi x^3 - 7x$ (c) $y = \sqrt{x+5}$ (d) $y = \sqrt[3]{x-1}$

(e) $y = \sqrt[4]{x^2+1}$ (f) $y = (x+1)^2(x-3)^3$ (g) $y = \frac{x}{(1-x)^2}$

22.6 Use implicit differentiation to find the second derivative y'' in the following cases:

(a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$ (c) $x^3 - y^3 = 1$ (d) $xy + y^2 = 1$

22.7 If $\sqrt{x} + \sqrt{y} = 1$, calculate y'': (a) by explicitly solving for y and then differentiating twice; (b) by implicit differentiation. (c) Which of methods (a) and (b) is the simpler one?

22.8 Find all derivatives (first, second, etc.) of y:

(a) $y = 4x^4 - 2x^2 + 1$ (b) $y = 2x^2 + x - 1 + \frac{1}{x}$ (c) $y = \sqrt{x}$

(d) $y = \frac{x+1}{x-1}$ (e) $y = \frac{1}{3+x}$ (f) $y = \frac{1}{x^2}$

22.9 Find the velocity the first time that the acceleration is 0 if the equation of motion is:

(a) $s = t^2 - 5t + 7$ (b) $s = t^3 - 3t + 2$ (c) $s = t^4 - 4t^3 + 6t^2 - 4t + 3$

22.10 At the point (1, 2) of the curve $x^2 - xy + y^2 = 3$, find the rate of change with respect to x of the slope of the tangent line to the curve.

22.11 If $x^2 + 2xy + 3y^2 = 2$, find the values of dy/dx and d^2y/dx^2 when $y = 1$.

22.12 Let $f(x) = \begin{cases} 1 + 3K(x-2) + (x-2)^2 & \text{if } x \le 2 \\ Lx + K & \text{if } x > 2 \end{cases}$ where L and K are constants.

(a) If $f(x)$ is differentiable at $x = 2$, find L and K. (b) With L and K as found in part (a), is $f''(x)$ continuous for all x?

22.13 Let $h(x) = f(x)g(x)$ and assume that f and g have derivatives of all orders. (a) Find formulas for $h''(x)$, $h'''(x)$, and $h^{(4)}(x)$. (b) Find a general formula for $h^{(n)}(x)$.

22.14 Let $H(x) = f(x)/g(x)$, where f and g have first and second derivatives. Find a formula for $H''(x)$.

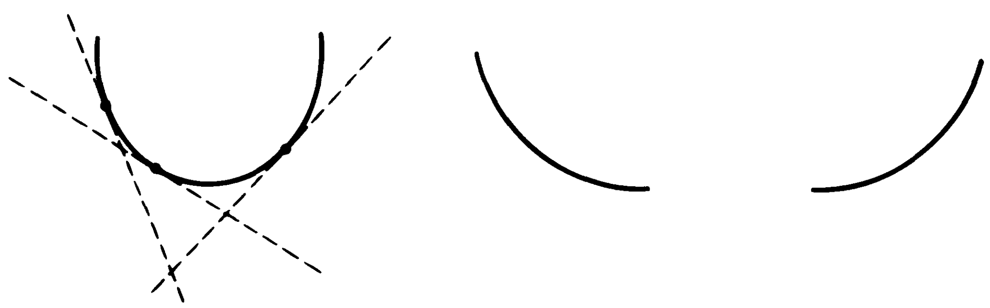
22.15 The height s of an object in free fall on the moon is approximately given by $s = s_0 + v_0 t - \frac{27}{10}t^2$, where s is measured in feet and t in seconds. (a) What is the acceleration due to "gravity" on the moon? (b) If an object is thrown upward from the surface of the moon with an initial velocity of 54 feet per second, what is the maximum height it will reach, and when will it reach that height?

Chapter 23

Applications of the Second Derivative and Graph Sketching

23.1 CONCAVITY

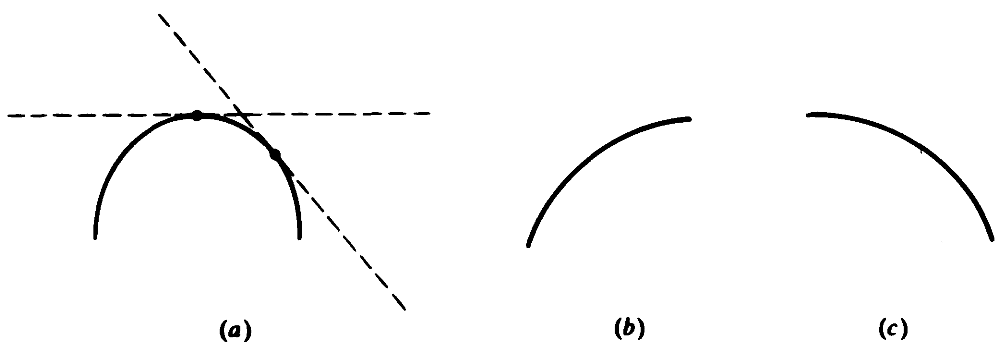
If a curve has the shape of a cup or part of a cup (like the curves in Fig. 23-1), then we say that it is *concave upward*. (To remember the sense of "*c*oncave *up*ward," notice that the letters *c* and *up* form the word *cup*.) A mathematical description of this notion can be given. A curve is concave upward if the curve lies above the tangent line at any given point of the curve. Thus, in Fig. 23-1(*a*) the curve lies above all three tangent lines.



(*a*) (*b*) (*c*)

Fig. 23-1 Concavity upward

A curve is said to be *concave downward* if it has the shape of a cap or part of a cap (see Fig. 23-2). In mathematical terms, a curve is concave downward if it lies below the tangent line at an arbitrary point of the curve [see Fig. 23-2(*a*)].



(*a*) (*b*) (*c*)

Fig. 23-2 Concavity downward

A curve may, of course, be composed of parts of different concavity. The curve in Fig. 23-3 is concave downward from A to B, concave upward from B to C, concave downward from C to D, and concave upward from D to E. A point on the curve at which the concavity changes is called an *inflection point*. B, C, and D are inflection points in Fig. 23-3.

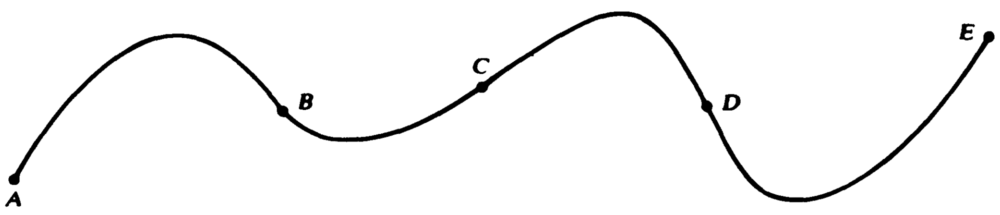


Fig. 23-3

From Fig. 23-1 we see that if we move from left to right along a curve that is concave upward, the slope of the tangent line increases. The slope either becomes less negative or more positive. Conversely, if the tangent line has this property, the curve must be concave upward. Now for a curve $y = f(x)$, the tangent line will certainly have this property if $f''(x) > 0$ since, in that case, Theorem 17.3 implies that the slope $f'(x)$ of the tangent line will be an increasing function. By a similar argument, we see that if $f''(x) < 0$, the slope of the tangent line is decreasing, and from Fig. 23-2 we see that the curve $y = f(x)$ is concave downward. This yields:

Theorem 23.1: If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave upward between $x = a$ and $x = b$. If $f''(x) < 0$ for all x in (a, b), then the graph of f is concave downward between $x = a$ and $x = b$.

For a rigorous proof of Theorem 23.1, see Problem 23.17.

Corollary 23.2: If the graph of f has an inflection point at $x = c$, and f'' exists and is continuous at $x = c$, then $f''(c) = 0$.

In fact, if $f''(c) \neq 0$, the $f''(c) > 0$ or $f''(c) < 0$. If $f''(c) > 0$, then $f''(x) > 0$ for all x in some open interval containing c, and the graph would be concave upward in that interval, contradicting the assumption that there is an inflection point at $x = c$. We get a similar contradiction if $f''(c) < 0$, for in that case, the graph would be concave downward in an open interval containing c.

EXAMPLES

(*a*) Consider the graph of $y = x^3$ [see Fig. 23-4(*a*)]. Here $y' = 3x^2$ and $y'' = 6x$. Since $y'' > 0$ when $x > 0$, and $y'' < 0$ when $x < 0$, the curve is concave upward when $x > 0$, and concave downward when $x < 0$. There is an inflection point at the origin, where the concavity changes. This is the only possible inflection point, for if $y'' = 6x = 0$, then x must be 0.

(*b*) If $f''(c) = 0$, the graph of f need not have an inflection point at $x = c$. For instance, the graph of $f(x) = x^4$ [see Fig. 23-4(*b*)] has a relative minimum, not an inflection point, at $x = 0$, where $f''(x) = 12x^2 = 0$.

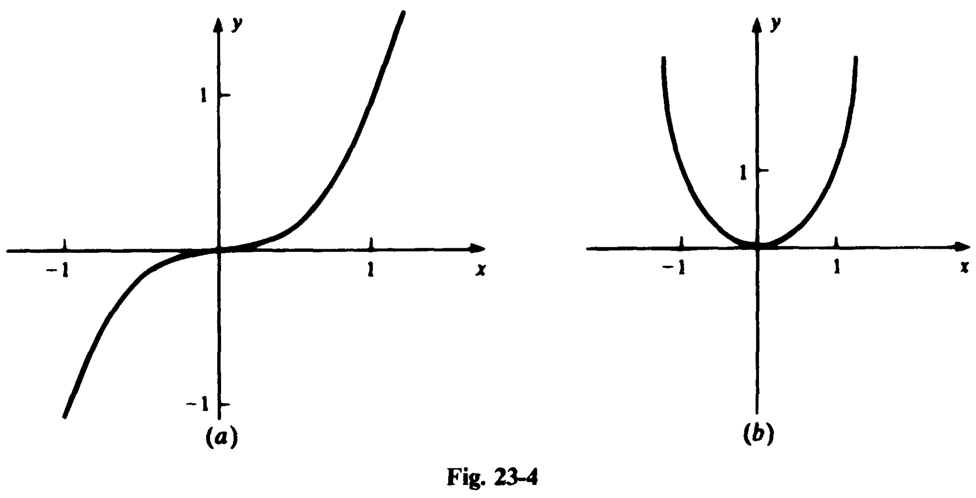


Fig. 23-4

23.2 TEST FOR RELATIVE EXTREMA

We already know, from Chapter 14, that the condition $f'(c) = 0$ is necessary, but not sufficient, for a differentiable function f to have a relative maximum or minimum at $x = c$. We need some additional information that will tell us whether a function actually has a relative extremum at a point where its derivative is zero.

Theorem 23.3 (*Second-Derivative Test for Relative Extrema*): If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at c. If $f'(c) = 0$ and $f''(c) > 0$, then f has a relative minimum at c.

Proof: If $f'(c) = 0$, the tangent line to the graph of f is horizontal at $x = c$. If, in addition, $f''(c) < 0$, then, by Theorem 23.1,[1] the graph of f is concave downward near $x = c$. Hence, near $x = c$, the graph of f must lie below the horizontal line through $(c, f(c))$; f thus has a relative maximum at $x = c$ [see Fig. 23-5(a)]. A similar argument leads to a relative minimum when $f''(c) > 0$ [see Fig. 23-5(b)].

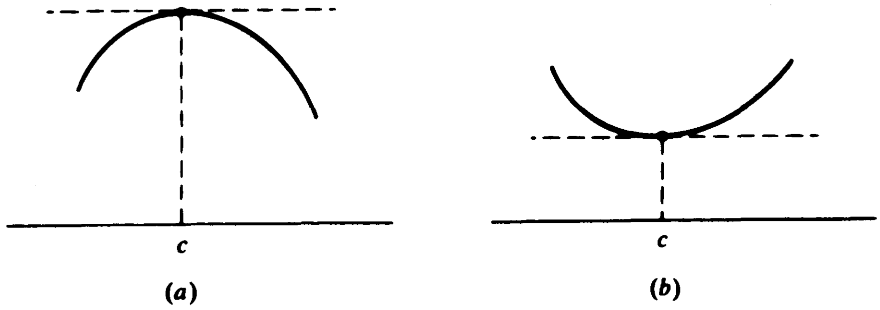


Fig. 23-5

EXAMPLE Consider the function $f(x) = 2x^3 + x^2 - 4x + 2$. Then,

$$f'(x) = 6x^2 + 2x - 4 = 2(3x^2 + x - 2) = 2(3x - 2)(x + 1)$$

Hence, if $f'(x) = 0$, then $3x - 2 = 0$ or $x + 1 = 0$; that is, $x = \frac{2}{3}$ or $x = -1$. Now $f''(x) = 12x + 2$. Hence,

$$f''(-1) = 12(-1) + 2 = -12 + 2 = -10 < 0$$
$$f''(\tfrac{2}{3}) = 12(\tfrac{2}{3}) + 2 = 8 + 2 = 10 > 0$$

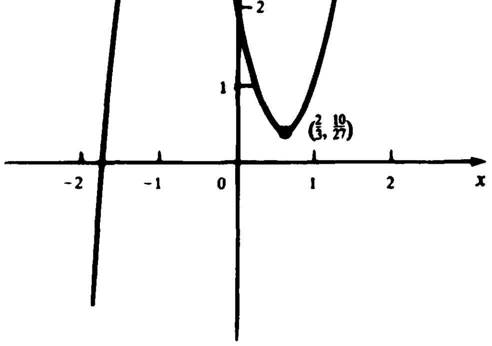


Fig. 23-6

[1] In order to use Theorem 23.1, we must assume that f'' is continuous at c and exists in an open interval around c. However, a more complicated argument can avoid that assumption.

Since $f''(-1) < 0$, f has a relative maximum at $x = -1$, with

$$f(-1) = 2(-1)^3 + (-1)^2 - 4(-1) + 2 = -2 + 1 + 4 + 2 = 5$$

Since $f''(\tfrac{2}{3}) > 0$, f has a relative minimum at $x = \tfrac{2}{3}$, with

$$f\left(\frac{2}{3}\right) = 2\left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 2 = \frac{16}{27} + \frac{4}{9} - \frac{8}{3} + 2 = \frac{16}{27} + \frac{12}{27} - \frac{72}{27} + \frac{54}{27} = \frac{10}{27}$$

The graph of f is shown in Fig. 23-6. Now because

$$f''(x) = 12x + 2 = 12\left(x + \frac{1}{6}\right) = 12\left[x - \left(-\frac{1}{6}\right)\right]$$

$f''(x) > 0$ when $x > -\tfrac{1}{6}$, and $f''(x) < 0$ when $x < -\tfrac{1}{6}$. Hence, the curve is concave upward for $x > -\tfrac{1}{6}$ and concave downward for $x < -\tfrac{1}{6}$. So there must be an inflection point I, where $x = -\tfrac{1}{6}$.

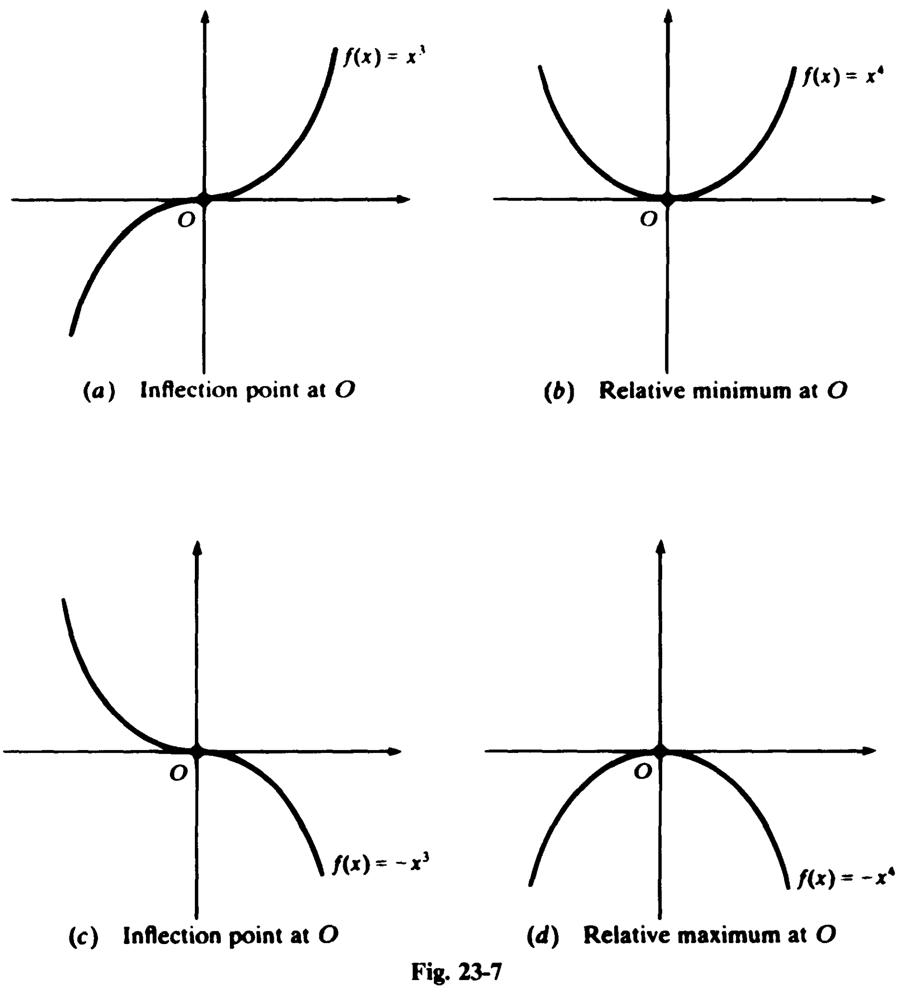
From Problem 9.1 we know that

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} 2x^3 = +\infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} 2x^3 = -\infty$$

Thus, the curve moves upward without bound toward the right, and downward without bound toward the left.

The second-derivative test tells us *nothing* when $f'(c) = 0$ and $f''(c) = 0$. This is shown by the examples in Fig. 23-7, where, in each case, $f'(0) = f''(0) = 0$.

To distinguish among the four cases shown in Fig. 23-7, consider the sign of the derivative f' just to the left and just to the right of the critical point. Recalling that the sign of the derivative is the sign of the slope of the tangent line, we have the four combinations shown in Fig. 23-8. These lead directly to Theorem 23-4.



(*a*) Inflection point at O (*b*) Relative minimum at O

(*c*) Inflection point at O (*d*) Relative maximum at O

Fig. 23-7

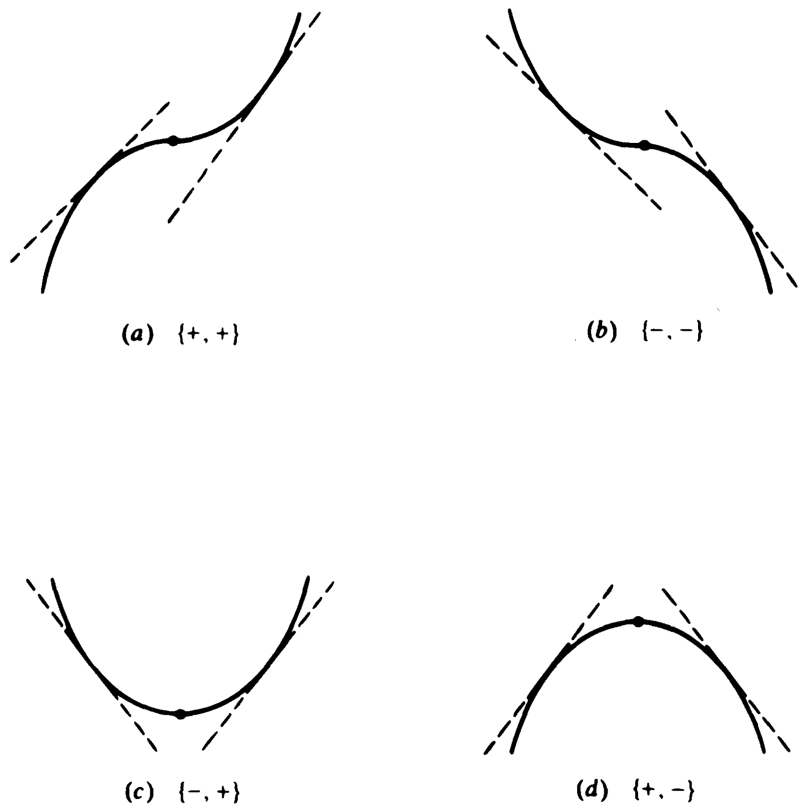


Fig. 23-8

Theorem 23.4 (*First-Derivative Test for Relative Extrema*): Assume $f'(c) = 0$.

$\{-, +\}$ If f' is negative to the left of c and positive to the right of c, then f has a *relative minimum* at c.

$\{+, -\}$ If f' is positive to the left of c and negative to the right of c, then f has a *relative maximum* at c.

$\{+, +\}$ $\{-, -\}$ If f' has the same signs to the left and to the right of c, then f has an *inflection point* at c.

23.3 GRAPH SKETCHING

We are now equipped to sketch the graphs of a great variety of functions. The most important features of such graphs are:

(i) Relative extrema (if any)

(ii) Inflection points (if any)

(iii) Concavity

(iv) Vertical and horizontal asymptotes (if any)

(v) Behavior as x approaches $+\infty$ and $-\infty$

The procedure was illustrated for the function $f(x) = 2x^3 + x^2 - 4x + 2$ in Section 23.2. An additional example follows.

EXAMPLE Sketch the graph of the rational function

$$f(x) = \frac{x}{x^2+1}$$

First of all, the function is odd [that is, $f(-x) = -f(x)$], so that it need be graphed only for positive x. The graph is then completed by reflection in the origin (see Section 7.3).

Compute the first two derivatives of f,

$$f'(x) = \frac{(x^2+1)D_x(x) - xD_x(x^2+1)}{(x^2+1)^2} = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$$\begin{aligned} f''(x) = D_x f'(x) &= \frac{(x^2+1)^2 D_x(1-x^2) - (1-x^2)D_x((x^2+1)^2)}{(x^2+1)^4} \\ &= \frac{(x^2+1)^2(-2x) - (1-x^2)[2(x^2+1)(2x)]}{(x^2+1)^4} \\ &= \frac{-2x(x^2+1)[x^2+1+2(1-x^2)]}{(x^2+1)^4} = \frac{-2x(3-x^2)}{(x^2+1)^3} = \frac{2x(x-\sqrt{3})(x+\sqrt{3})}{(x^2+1)^3} \end{aligned}$$

Since $1-x^2 = (1-x)(1+x)$, $f'(x)$ has a single positive root, $x = 1$, at which $f''(1) = [-2(2)]/(2)^3 = -\frac{1}{2}$. Hence, by the second-derivative test, f has a relative maximum at $x = 1$. The maximum value is $f(1) = \frac{1}{2}$.

If we examine the formula for $f''(x)$,

$$f''(x) = \frac{2x(x-\sqrt{3})(x+\sqrt{3})}{(x^2+1)^3}$$

we see that $f''(x) > 0$ when $x > \sqrt{3}$ and that $f''(x) < 0$ when $0 < x < \sqrt{3}$. By Theorem 23.1, the graph of f is concave upward for $x > \sqrt{3}$ and concave downward for $0 < x < \sqrt{3}$. Thus, there is an inflection point I at $x = \sqrt{3}$, where the concavity changes.

Now calculate

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x}{x^2+1} = \lim_{x\to+\infty} \frac{1/x}{1+(1/x^2)} = \frac{0}{1+0} = 0$$

which shows that the positive x-axis is a horizontal asymptote to the right.

The graph, with its extension to negative x (dashed), is sketched in Fig. 23-9. Note how concavity of one kind reflects into concavity of the other kind. Thus, there is an inflection point at $x = -\sqrt{3}$ and another inflection point at $x = 0$. The value $f(1) = \frac{1}{2}$ is the absolute maximum of f, and $f(-1) = -\frac{1}{2}$ is the absolute minimum.

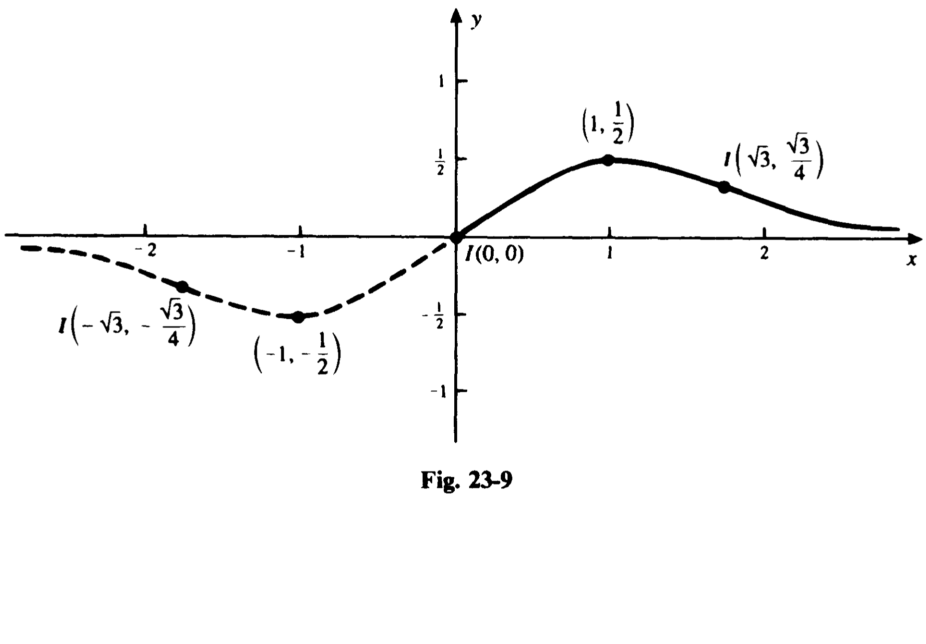


Fig. 23-9

Solved Problems

23.1 Sketch the graph of $f(x) = x - 1/x$.

The function is odd. Hence, we can first sketch the graph for $x > 0$ and, later, reflect in the origin to obtain the graph for $x < 0$.

The first and second derivatives are

$$f'(x) = D_x(x - x^{-1}) = 1 - (-1)x^{-2} = 1 + \frac{1}{x^2}$$

$$f''(x) = D_x(1 + x^{-2}) = -2x^{-3} = -\frac{2}{x^3}$$

Since $f'(x) = 1 + (1/x^2) > 0$, f is an increasing function. Moreover, for $x > 0$, the graph of f is concave downward, since $f''(x) = -(2/x^3) < 0$ when $x > 0$. The line $y = x$ turns out to be an asymptote, because

$$\lim_{x \to +\infty} [x - f(x)] = \lim_{x \to +\infty} \frac{1}{x} = 0$$

Since

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(x - \frac{1}{x}\right) = \lim_{x \to 0^+} \left(-\frac{1}{x}\right) = -\infty$$

the graph of f has the negative y-axis as a vertical asymptote.

Notice that $x = 0$, at which f is undefined, is the only critical number. The graph is sketched, for all x, in Fig. 23-10. Although the concavity changes at $x = 0$, there is no inflection point there because $f(0)$ is not defined.

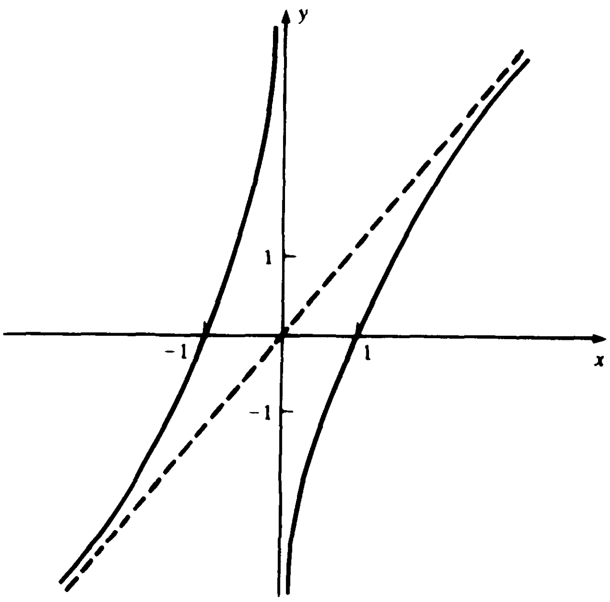


Fig. 23-10

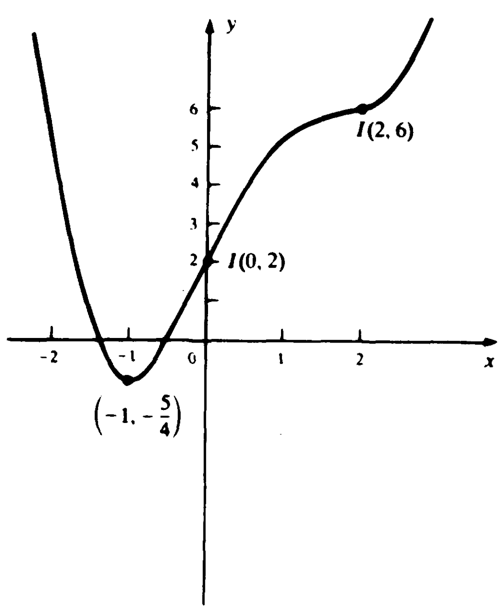


Fig. 23-11

23.2 Sketch the graph of $f(x) = \frac{1}{4}x^4 - x^3 + 4x + 2$.

The first derivative is $f'(x) = x^3 - 3x^2 + 4$. We can determine that -1 is a root of $x^3 - 3x^2 + 4$.

ALGEBRA When looking for roots of a polynomial, first test the integral factors of the constant. In this case, the factors of 4 are ± 1, ± 2, ± 4.

So (Theorem 7.2), $f'(x)$ is divisible by $x + 1$. The division yields

$$x^3 - 3x^2 + 4 = (x + 1)(x^2 - 4x + 4) = (x + 1)(x - 2)^2$$

Hence, the critical numbers are $x = -1$ and $x = 2$. Now

$$f''(x) = 3x^2 - 6x = 3x(x - 2)$$

Thus, $f''(-1) = 3(-1)(-1 - 2) = 9$. Hence, by the second-derivative test, f has a relative minimum at $x = -1$.

Since $f''(2) = 3(2)(2 - 2) = 0$, we do the first-derivative test at $x = 2$.

$$f'(x) = (x + 1)(x - 2)^2$$

On both sides of $x = 2$, $f'(x) > 0$, since $x + 1 > 0$ and $(x - 2)^2 > 0$. This is the case $\{+, +\}$. There is an inflection point at (2, 6). Furthermore, $f''(x)$ changes sign at $x = 0$, so that there is also an inflection point at (0, 2). Because

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{1}{4}x^4 = +\infty$$

the graph moves upward without bound on the left and the right. The graph is shown in Fig. 23-11.

23.3 Sketch the graph of $f(x) = x^4 - 8x^2$.

As the function is even, we restrict attention to $x \geq 0$.

$$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x - 2)(x + 2)$$

$$f''(x) = 12x^2 - 16 = 4(3x^2 - 4) = 12\left(x^2 - \frac{4}{3}\right) = 12\left(x + \frac{2}{\sqrt{3}}\right)\left(x - \frac{2}{\sqrt{3}}\right)$$

The nonnegative critical numbers are $x = 0$ and $x = 2$. Testing,

x	$f(x)$	$f''(x)$	
0	0	-16	rel. max.
2	-16	32	rel. min.
$+\frac{2}{\sqrt{3}}$	$-\frac{80}{9}$	0	infl. pt.

Checking the sign of $f''(x)$, we see that the graph will be concave downward for $0 < x < 2/\sqrt{3}$ and concave upward for $x > 2/\sqrt{3}$. Because $\lim_{x \to +\infty} f(x) = +\infty$, the graph moves upward without bound on the right.

The graph is sketched in Fig. 23-12. Observe that, on the set of all real numbers, f has an absolute minimum value of -16, assumed at $x = \pm 2$, but no absolute maximum value.

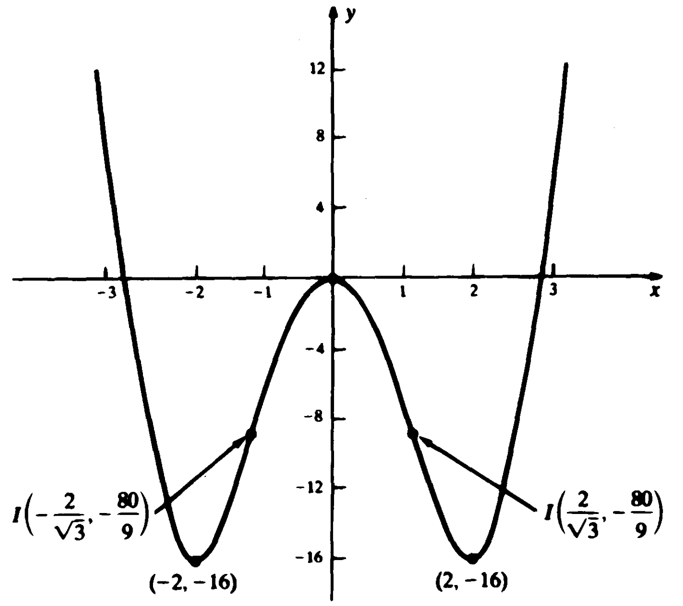


Fig. 23-12

Supplementary Problems

23.4 Determine the intervals where the graphs of the following functions are concave upward and the intervals where they are concave downward. Find all inflection points. GC Check your solutions with a graphing calculator.

(a) $f(x) = x^2 - x + 12$ (b) $f(x) = x^4 + 18x^3 + 120x^2 + x + 1$

(c) $f(x) = x^3 + 15x^2 + 6x + 1$ (d) $f(x) = \dfrac{x}{2x-1}$ (e) $f(x) = 5x^4 - x^5$

23.5 Find the critical numbers of the following functions and determine whether they yield relative maxima, relative minima, inflection points, or none of these. GC Check your solutions with a graphing calculator.

(a) $f(x) = 8 - 3x + x^2$ (b) $f(x) = x^4 - 18x^2 + 9$

(c) $f(x) = x^3 - 5x^2 - 8x + 3$ (d) $f(x) = \dfrac{x^2}{x-1}$ (e) $f(x) = \dfrac{x^2}{x^2+1}$

23.6 Sketch the graphs of the following functions, showing extrema (relative or absolute), inflection points, asymptotes, and behavior at infinity. GC Check your solutions with a graphing calculator.

(a) $f(x) = (x^2 - 1)^3$ (b) $f(x) = x^3 - 2x^2 - 4x + 3$ (c) $f(x) = x(x-2)^2$

(d) $f(x) = x^4 + 4x^3$ (e) $f(x) = 3x^5 - 20x^3$ (f) $f(x) = \sqrt[3]{x-1}$

(g) $f(x) = x^2 + \dfrac{2}{x}$ (h) $f(x) = \dfrac{x^2 - 3}{x^3}$ (i) $f(x) = \dfrac{(x-1)^3}{x^2}$

23.7 If, for all $x, f'(x) > 0$ and $f''(x) < 0$, which of the curves in Fig. 23-13 could be part of the graph of f?

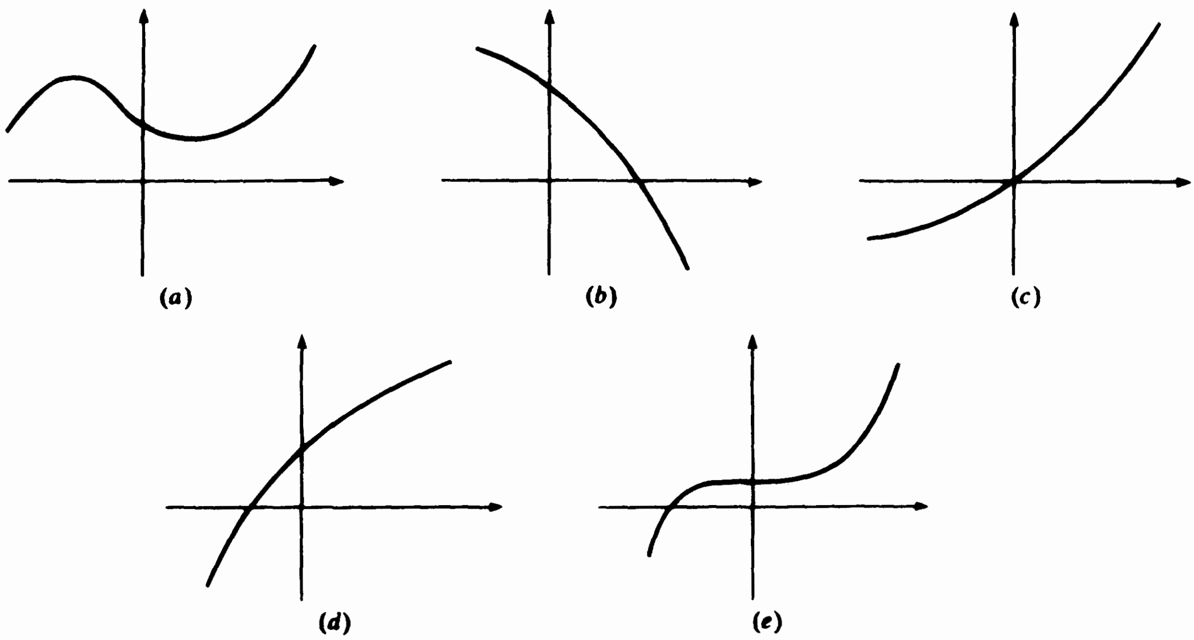


Fig. 23-13

23.8 At which of the five indicated points on the graph in Fig. 23-14 do y' and y'' have the same sign?

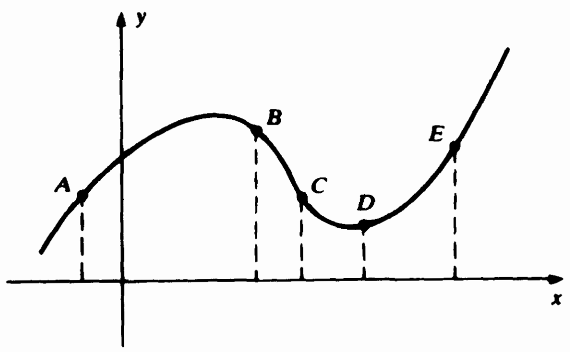


Fig. 23-14

23.9 Let $f(x) = ax^2 + bx + c$, with $a \neq 0$. (*a*) How many relative extrema does f have? (*b*) How many points of inflection does the graph of f have? (*c*) What kind of curve is the graph of f?

23.10 Let f be continuous for all x, with a relative maximum at $(-1, 4)$ and a relative minimum at $(3, -2)$. Which of the following *must* be true? (*a*) The graph of f has a point of inflection for some x in $(-1, 3)$. (*b*) The graph of f has a vertical asymptote. (*c*) The graph of f has a horizontal asymptote. (*d*) $f'(3) = 0$. (*e*) The graph of f has a horizontal tangent line at $x = -1$. (*f*) The graph of f intersects both the x-axis and the y-axis. (*g*) f has an absolute maximum on the set of all real numbers.

23.11 If $f(x) = x^3 + 3x^2 + k$ has three distinct real roots, what are the bounds on k? [*Hint:* Sketch the graph of f, using f' and f''. At how many points does the graph cross the x-axis?]

23.12 Sketch the graph of a continuous function f such that:

(*a*) $f(1) = -2, f'(1) = 0, f''(x) > 0$ for all x

(*b*) $f(2) = 3, f'(2) = 0, f''(x) < 0$ for all x

(*c*) $f(1) = 1, f''(x) < 0$ for $x > 1, f''(x) > 0$ for $x < 1$, $\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to-\infty} f(x) = -\infty$

(*d*) $f(0) = 0, f''(x) < 0$ for $x > 0, f''(x) > 0$ for $x < 0$, $\lim_{x\to+\infty} f(x) = 1$, $\lim_{x\to-\infty} f(x) = -1$

(*e*) $f(0) = 1, f''(x) < 0$ for $x \neq 0$, $\lim_{x\to0^+} f'(x) = +\infty$, $\lim_{x\to0^-} f'(x) = -\infty$

(*f*) $f(0) = 0, f''(x) > 0$ for $x < 0, f''(x) < 0$ for $x > 0$, $\lim_{x\to0^-} f'(x) = +\infty$, $\lim_{x\to0^+} f'(x) = +\infty$

(*g*) $f(0) = 1, f''(x) < 0$ if $x \neq 0$, $\lim_{x\to0^+} f'(x) = 0$, $\lim_{x\to0^-} f'(x) = -\infty$

23.13 Let $f(x) = x|x - 1|$ for x in $[-1, 2]$. (*a*) At what values of x is f continuous? (*b*) At what values of x is f differentiable? Calculate $f'(x)$. [*Hint:* Distinguish the cases $x > 1$ and $x < 1$.] (*c*) Where is f an increasing function? (*d*) Calculate $f''(x)$. (*e*) Where is the graph of f concave upward, and where concave downward? (*f*) Sketch the graph of f.

23.14 Given functions f and g such that, for all x, (i) $(g(x))^2 - (f(x))^2 = 1$; (ii) $f'(x) = (g(x))^2$; (iii) $f''(x)$ and $g''(x)$ exist; (iv) $g(x) < 0$; (v) $f(0) = 0$. Show that: (*a*) $g'(x) = f(x)g(x)$; (*b*) g has a relative maximum at $x = 0$; (*c*) f has a point of inflection at $x = 0$.

23.15 For what value of k will $x - kx^{-1}$ have a relative maximum at $x = -2$?

23.16 Let $f(x) = x^4 + Ax^3 + Bx^2 + Cx + D$. Assume that the graph of $y = f(x)$ is symmetric with respect to the y-axis, has a relative maximum at (0, 1), and has an absolute minimum at $(k, -3)$. Find A, B, C, and D, as well as the possible value(s) of k.

23.17 Prove Theorem 23.1. [*Hint:* Assume that $f''(x) > 0$ on (a, b), and let c be in (a, b). The equation of the tangent line at $x = c$ is $y = f'(c)(x - c) + f(c)$. It must be shown that $f(x) > f'(c)(x - c) + f(c)$. But the mean-value theorem gives

$$f(x) = f'(x^*)(x - c) + f(c)$$

where x^* is between x and c, and since $f''(x) > 0$ on (a, b), f' is increasing.]

23.18 Give a rigorous proof of the second-derivative test (Theorem 23.3). [*Hint:* Assume $f'(c) = 0$ and $f''(c) < 0$. Since $f''(c) < 0$, $\lim_{h\to0} \dfrac{f'(c + h) - f'(c)}{h} < 0$. So, there exists $\delta > 0$ such that, for $|h| < \delta$, $\dfrac{f'(c + h) - f'(c)}{h} < 0$, and since $f'(c) = 0$, $f'(c + h) < 0$ for $h > 0$ and $f'(c + h) > 0$ for $h < 0$. By the mean-value theorem, if $|h| < \delta$, $\dfrac{f(c + h) - f(c)}{h} = f'(c + h_1)$ for some $c + h_1$ between c and $c + h$. So, $|h_1| < |h|$, and whether $h > 0$ or $h < 0$, we can deduce that $f(c + h) - f(c) < 0$; that is, $f(c + h) < f(c)$. Thus, f has a relative maximum at c. The case when $f''(c) > 0$ is reduced to the first case by considering $-f$.]

23.19 Consider $f(x) = \dfrac{3(x^2 - 1)}{x^2 + 3}$.

(*a*) Find all open intervals where f is increasing. (*b*) Find all critical points and determine whether they correspond to relative maxima, relative minima, or neither. (*c*) Describe the concavity of the graph of f and find all inflection points (if any). (*d*) Sketch the graph of f. Show any horizontal or vertical asymptotes.

23.20 In the graph of $y = f(x)$ in Fig. 23-15: (*a*) find all x such that $f'(x) > 0$; (*b*) find all x such that $f''(x) > 0$.

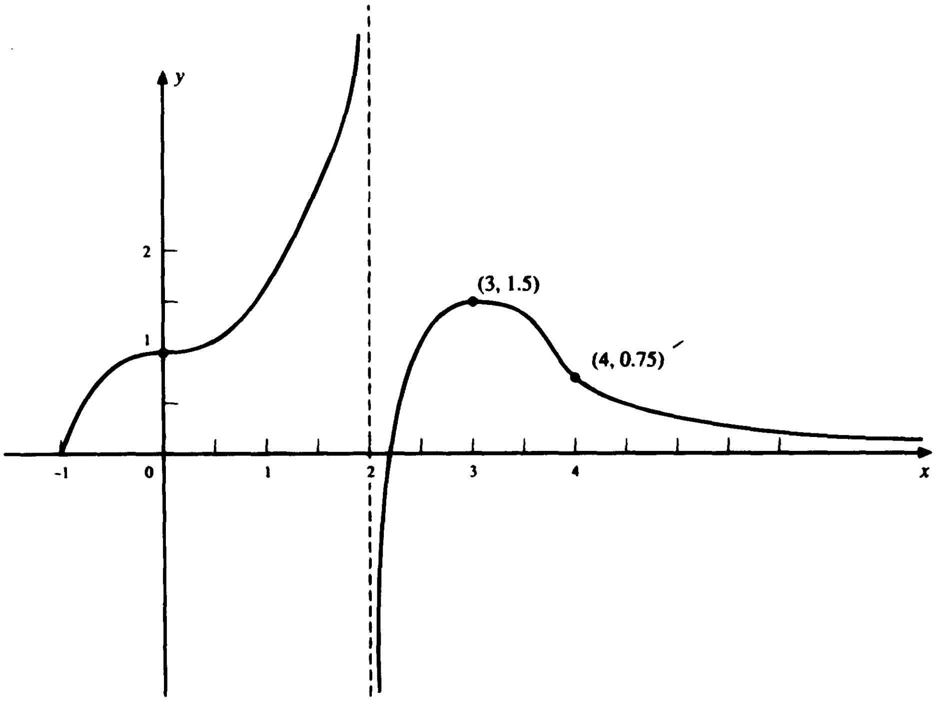


Fig. 23-15

Chapter 24

More Maximum and Minimum Problems

Until now we have been able to find the absolute maxima and minima of differentiable functions only on *closed* intervals (see Section 14.2). The following result often enables us to handle cases where the function is defined on a half-open interval, open interval, infinite interval, or the set of all real numbers. Remember that, in general, there is no guarantee that a function has an absolute maximum or an absolute minimum on such domains.

Theorem 24.1: Let f be a continuous function on an interval $\mathscr{I}$, with a *single* relative extremum within $\mathscr{I}$. Then this relative extremum is also an absolute extremum on $\mathscr{I}$.

Intuitive Argument: Refer to Fig. 24-1. Suppose that f has a relative maximum at c and no other relative extremum inside $\mathscr{I}$. Take any other number d in $\mathscr{I}$. The curve moves downward on both sides of c. Hence, if the value $f(d)$ were greater than $f(c)$, then, at some point u between c and d, the curve would have to change direction and start moving upward. But then f would have a relative minimum at u, contradicting our assumption. The result for a relative minimum follows by applying to $-f$ the result just obtained for a relative maximum.

For a rigorous proof, see Problem 24.20.

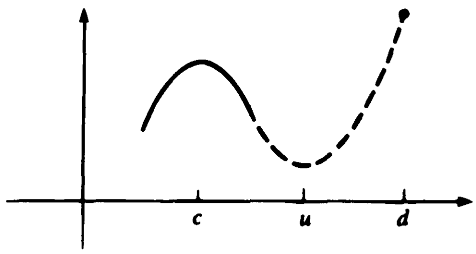


Fig. 24-1

EXAMPLES

(*a*) Find the shortest distance from the point $P(1, 0)$ to the parabola $x = y^2$ [see Fig. 24-2(*a*)].

The distance from an arbitrary point $Q(x, y)$ on the parabola to the point $P(1, 0)$ is, by (*2.1*),

$$\begin{aligned} u &= \sqrt{(x-1)^2 + y^2} \\ &= \sqrt{(x-1)^2 + x} \qquad [y^2 = x \text{ at } Q] \\ &= \sqrt{x^2 - 2x + 1 + x} = \sqrt{x^2 - x + 1} \end{aligned}$$

But minimizing u is equivalent to minimizing $u^2 \equiv F(x) = x^2 - x + 1$ on the interval $[0, +\infty)$ (the value of x is restricted by the fact that $x = y^2 \geq 0$).

$$F'(x) = 2x - 1 \qquad F''(x) = 2$$

The only critical number is the solution of

$$F'(x) = 2x - 1 = 0 \qquad \text{or} \qquad x = \tfrac{1}{2}$$

Now $F''(\frac{1}{2}) = 2 > 0$. So by the second-derivative test, the function F has a relative minimum at $x = \frac{1}{2}$. Theorem 24.1 implies that this is an absolute minimum. When $x = \frac{1}{2}$,

$$y^2 = x = \frac{1}{2} \qquad \text{and} \qquad y = \pm\frac{1}{\sqrt{2}} = \pm\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

Thus, the points on the parabola closest to $(1, 0)$ are $(\frac{1}{2}, \sqrt{2}/2)$ and $(\frac{1}{2}, -\sqrt{2}/2)$.

(*b*) An open box (that is, a box without a top) is to be constructed with a square base [see Fig. 24-2(*b*)] and is required to have a volume of 48 cubic inches. The bottom of the box costs 3 cents per square inch, whereas the sides cost 2 cents per square inch. Find the dimensions that will minimize the cost of the box.

Let x be the side of the square bottom, and let h be the height. Then the cost of the bottom is $3x^2$ and the cost of each of the four sides is $2xh$, giving a total cost of

$$C = 3x^2 + 4(2xh) = 3x^2 + 8xh$$

The volume is $V = 48 = x^2h$. Hence, $h = 48/x^2$ and

$$C = 3x^2 + 8x\left(\frac{48}{x^2}\right) = 3x^2 + \frac{384}{x} = 3x^2 + 384x^{-1}$$

which is to be minimized on $(0, +\infty)$. Now

$$\frac{dC}{dx} = 6x - 384x^{-2} = 6x - \frac{384}{x^2}$$

and so the critical numbers are the solutions of

$$6x - \frac{384}{x^2} = 0$$

$$6x = \frac{384}{x^2}$$

$$x^3 = 64$$

$$x = 4$$

Now $$\frac{d^2C}{dx^2} = 6 - (-2)384x^{-3} = 6 + \frac{768}{x^3} > 0$$

for all positive x; in particular, for $x = 4$. By the second-derivative test, C has a relative minimum at $x = 4$. But since 4 is the only positive critical number and C is continuous on $(0, +\infty)$, Theorem 24.1 tells us that C has an absolute minimum at $x = 4$. When $x = 4$,

$$h = \frac{48}{x^2} = \frac{48}{16} = 3$$

So, the side of the base should be 4 inches and the height 3 inches.

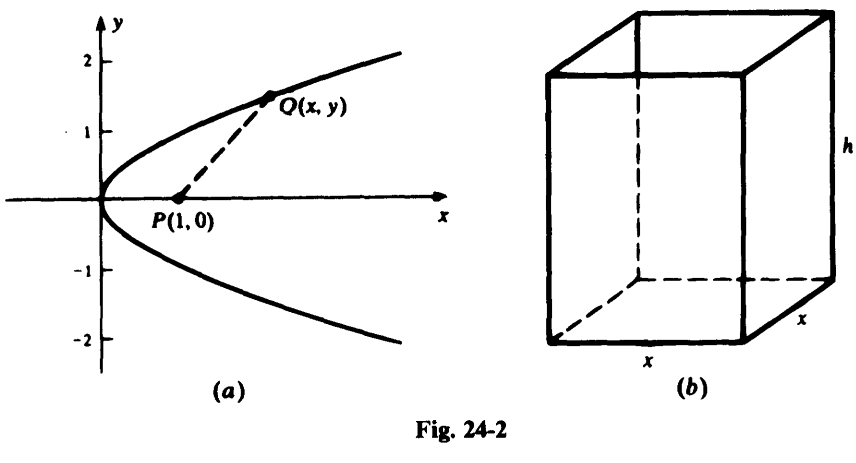


Fig. 24-2

Solved Problems

24.1 A farmer must fence in a rectangular field with one side along a stream; no fence is needed on that side. If the area must be 1800 square meters and the fencing cost \$2 per meter, what dimensions will minimize the cost?

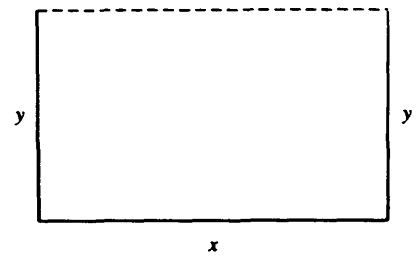
Let x and y be the lengths of the sides parallel and perpendicular to the stream, respectively. Then the cost C is

$$C = 2(x + 2y) = 2x + 4y$$

But $1800 = xy$, or $x = 1800/y$, so that

$$C = 2\left(\frac{1800}{y}\right) + 4y = \frac{3600}{y} + 4y = 3600y^{-1} + 4y$$

and
$$\frac{dC}{dy} = -3600y^{-2} + 4 = -\frac{3600}{y^2} + 4$$



We wish to minimize $C(y)$ for $y > 0$. So we look for positive critical numbers,

$$-\frac{3600}{y^2} + 4 = 0$$

$$4 = \frac{3600}{y^2}$$

$$y^2 = \frac{3600}{4} = 900$$

$$y = +30$$

Now $\dfrac{d^2C}{dy^2} = \dfrac{d}{dy}(-3600y^{-2} + 4) = 7200y^{-3} = \dfrac{7200}{y^3}$, which is positive at $y = +30$. Thus, by the second-derivative test, C has a relative minimum at $y = 30$. Since $y = 30$ is the only positive critical number, there can be no other relative extremum in the interval $(0, +\infty)$. Therefore, C has an absolute minimum at $y = 30$, by Theorem 24.1. When $y = 30$ meters,

$$x = \frac{1800}{y} = \frac{1800}{30} = 60 \text{ meters}$$

24.2 If $c_1, c_2, \ldots, c_n$ are the results of n measurements of an unknown quantity, a method for estimating the value of that quantity is to find the number x that minimizes the function

$$f(x) = (x - c_1)^2 + (x - c_2)^2 + \cdots + (x - c_n)^2$$

This method is called the *least-squares principle*. Find the value of x determined by the least-squares principle.

$$f'(x) = 2(x - c_1) + 2(x - c_2) + \cdots + 2(x - c_n)$$

To find the critical numbers,

$$2(x - c_1) + 2(x - c_2) + \cdots + 2(x - c_n) = 0$$
$$(x - c_1) + (x - c_2) + \cdots + (x - c_n) = 0$$
$$nx - (c_1 + c_2 + \cdots + c_n) = 0$$
$$nx = c_1 + c_2 + \cdots + c_n$$
$$x = \frac{c_1 + c_2 + \cdots + c_n}{n}$$

As $f''(x) = 2 + 2 + \cdots + 2 = 2n > 0$, we have, by the second-derivative test, a relative minimum of f at the unique critical number. By Theorem 24.1, this relative minimum is also an absolute minimum on the set of all real x. Thus, the least-squares principle prescribes the *average of the n measurements.*

24.3 Let $f(x) = \dfrac{4x^2 - 3}{x - 1}$ for $0 \le x < 1$. Find the absolute extrema, if any, of f on [0, 1),

$$f'(x) = \frac{(x-1)D_x(4x^2-3) - (4x^2-3)D_x(x-1)}{(x-1)^2} = \frac{(x-1)(8x) - (4x^2-3)(1)}{(x-1)^2}$$
$$= \frac{8x^2 - 8x - 4x^2 + 3}{(x-1)^2} = \frac{4x^2 - 8x + 3}{(x-1)^2} = \frac{(2x-3)(2x-1)}{(x-1)^2}$$

To find the critical numbers, set $f'(x) = 0$,

$$\frac{(2x-3)(2x-1)}{(x-1)^2} = 0$$
$$(2x-3)(2x-1) = 0$$
$$2x - 3 = 0 \quad \text{or} \quad 2x - 1 = 0$$
$$x = \tfrac{3}{2} \quad \text{or} \quad x = \tfrac{1}{2}$$

So the only critical number in [0, 1) is $x = \frac{1}{2}$.

Let us apply the first-derivative test (Theorem 23.4),

$$f'(x) = \frac{(2x-3)(2x-1)}{(x-1)^2} = \frac{2(x-\frac{3}{2}) \cdot 2(x-\frac{1}{2})}{(x-1)^2} = \frac{4(x-\frac{3}{2})(x-\frac{1}{2})}{(x-1)^2}$$

For x immediately to the left of $\frac{1}{2}$, $x - \frac{1}{2} < 0$ and $x - \frac{3}{2} < 0$, and so, $f'(x) > 0$. For x immediately to the right of $\frac{1}{2}$, $x - \frac{1}{2} > 0$ and $x - \frac{3}{2} < 0$, and so, $f'(x) < 0$. Thus, we have the case $\{+, -\}$, and f has a relative maximum at $x = \frac{1}{2}$. (The second-derivative test could have been used instead.) The function f has no absolute minimum on [0, 1). Its graph has the line $x = 1$ as a vertical asymptote, since $\lim_{x\to 1^-} f(x) = \lim_{x \to 1^-} \dfrac{4x^2-3}{x-1} = -\infty$ (see Fig. 24-3).[1]

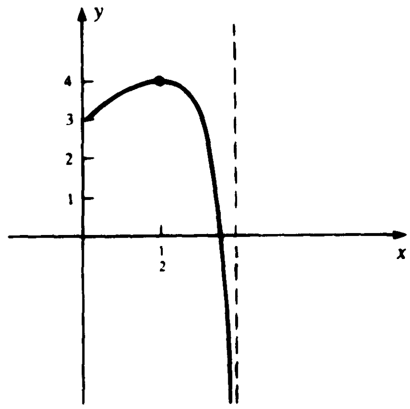


Fig. 24-3

[1] Note that $\dfrac{4x^2-3}{x-1} = 4(x+1) + \dfrac{1}{x-1} \to -\infty$ as $x \to 1^-$.

Supplementary Problems

24.4 A rectangular field is to be fenced in so that the resulting area is 100 square meters. Find the dimensions of that field (if any) for which the perimeter is: (*a*) a maximum; (*b*) a minimum.

24.5 Find the point(s) on the parabola $2x = y^2$ closest to the point (1, 0).

24.6 Find the point(s) on the hyperbola $x^2 - y^2 = 2$ closest to the point (0, 1).

24.7 A closed box with a square base is to contain 252 cubic feet. The bottom costs \$5 per square foot, the top costs \$2 per square foot, and the sides cost \$3 per square foot. Find the dimensions that will minimize the cost.

24.8 Find the absolute maxima and minima (if any) of $f(x) = \dfrac{x^2 + 4}{x - 2}$ on the interval [0, 2).

24.9 A printed page must contain 60 square centimeters of printed material. There are to be margins of 5 centimeters on either side, and margins of 3 centimeters each on the top and the bottom. How long should the printed lines be in order to minimize the amount of paper used?

24.10 A farmer wishes to fence in a rectangular field of 10 000 square feet. The north–south fences will cost \$1.50 per foot, whereas the east–west fences will cost \$6.00 per foot. Find the dimensions of the field that will minimize the cost.

24.11 (*a*) Sketch the graph of $y = \dfrac{1}{1 + x^2}$.

(*b*) Find the point on the graph where the tangent line has the greatest slope.

24.12 (*a*) Find the dimensions of the closed cylindrical can [see Fig. 24-4(*a*)] that will have a capacity of k volume units and used the minimum amount of material. Find the ratio of the height h to the radius r of the top and bottom. (The volume is $V = \pi r^2 h$, and the lateral surface area is $S = 2\pi rh$.)

(*b*) If the bottom and the top of the can have to be cut from square pieces of metal and the rest of these squares is wasted [see Fig. 24-4(*b*)], find the dimensions that will minimize the amount of material used, and find the ratio of the height to the radius.

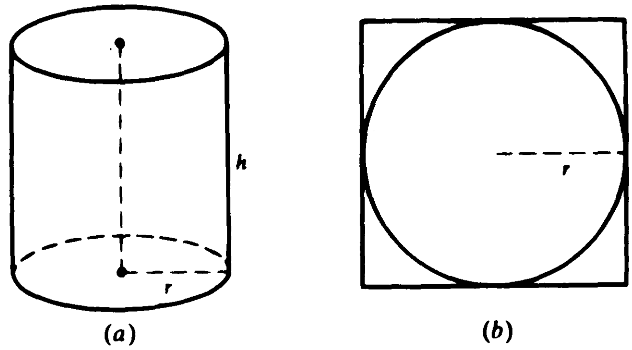


Fig. 24-4

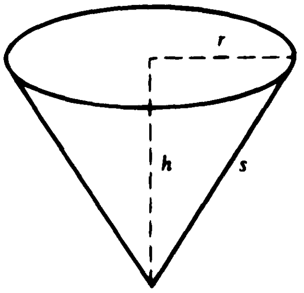


Fig. 24-5

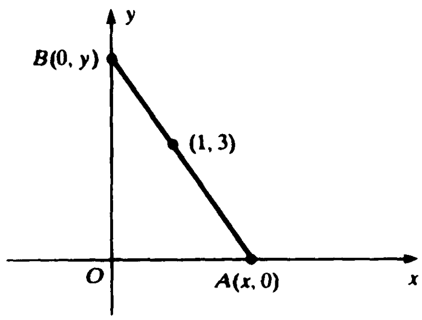


Fig. 24-6

24.13 A thin-walled cone-shaped cup is to hold 36π cubic inches of water when full. What dimensions will minimize the amount of material needed for the cup? (The volume is $V = \frac{1}{3}\pi r^2 h$ and the surface area is $A = \pi rs$; see Fig. 24-5.)

24.14 (*a*) Find the absolute extrema on $[0, +\infty)$ (if they exist) of $f(x) = \dfrac{x}{(x^2+1)^{3/2}}$. (*b*) Sketch the graph of f.

24.15 A rectangular bin, open at the top, is required to contain a volume of 128 cubic meters. If the bottom is to be a square, at a cost of \$2 per square meter, whereas the sides cost \$0.50 per square meter, what dimensions will minimize the cost?

24.16 The selling price P of an item is $100 - 0.02x$ dollars, where x is the number of items produced per day. If the cost C of producing and selling x items is $40x + 15\,000$ dollars per day, how many items should be produced and sold every day in order to maximize the profit?

24.17 Consider all lines through the point (1, 3) and intersecting the positive x-axis at $A(x, 0)$ and the positive y-axis at $B(0, y)$ (see Fig. 24-6). Find the line that makes the area of $\triangle BOA$ a minimum.

24.18 Conisder the function $f(x) = \dfrac{1}{2}x^2 + \dfrac{k}{x}$. (*a*) For what value of k will f have a relative minimum at $x = -2$? (*b*) For the value of k found in part (*a*), sketch the graph of f. (*c*) For what value(s) of k will f have an absolute minimum?

24.19 Find the point(s) on the graph of $3x^2 + 10xy + 3y^2 = 9$ closest to the origin. [*Hint:* Minimize $x^2 + y^2$, making use of implicit differentiation.]

24.20 Fill in the gaps in the following proof of Theorem 24.1. Assume that f is continuous on an interval $\mathscr{I}$. Let f have a relative maximum at c in $\mathscr{I}$, but no other relative extremum in $\mathscr{I}$. We must show that f has an absolute maximum on $\mathscr{I}$ at c. Assume, to the contrary, that $d \neq c$ is a point in $\mathscr{I}$ with $f(c) < f(d)$. On the closed interval $\mathscr{I}$ with endpoints c and d, f has an absolute minimum at some point u. Since f has a relative maximum at c, u is different from c and, therefore, $f(u) < f(c)$. Hence, $u \neq d$. So, u is in the interior of $\mathscr{I}$, whence f has a *relative* minimum at $u \neq c$.

24.21 Prove the following theorem, similar to Theorem 24.1: If the graph of f is concave upward (downward) over an interval $\mathscr{I}$, then any relative minimum (maximum) of f in $\mathscr{I}$ is an absolute minimum (maximum) on $\mathscr{I}$. [*Hint:* Consider the relationship of the graph of f to the tangent line at the relative extremum.)

24.22 Find the absolute extrema (if any) of $f(x) = x^{2/5} - \frac{1}{7}x^{7/5}$ on $(-1, 1]$.

Chapter 25

Angle Measure

25.1 ARC LENGTH AND RADIAN MEASURE

Figure 25-1(*a*) illustrates the traditional system of angle measure. A complete rotation is divided into 360 equal parts, and the measure assigned to each part is called a *degree*. In modern mathematics and science, it is useful to define a different unit of angle measure.

Definition: Consider a circle with a radius of one unit [see Fig. 25-1(*b*)]. Let the center be C, and let CA and CB be two radii for which the intercepted arc $\widehat{AB}$ of the circle has length 1. Then the central angle ACB is taken to be the unit of measure, *one radian*.

Let X be the number of degrees in $\measuredangle ACB$ of radian measure 1. Then the ratio of X to 360° (a complete rotation) is the same as the ratio of $\widehat{AB}$ to the entire circumference, 2π. Since $\widehat{AB} = 1$,

$$\frac{X}{360} = \frac{1}{2\pi} \qquad \text{or} \qquad X = \frac{360}{2\pi} = \frac{180}{\pi}$$

Thus,

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \tag{25.1}$$

If we approximate π as 3.14, then 1 radian is about 57.3 degrees. If we multiply (*25.1*) by $\pi/180$, we obtain

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians} \tag{25.2}$$

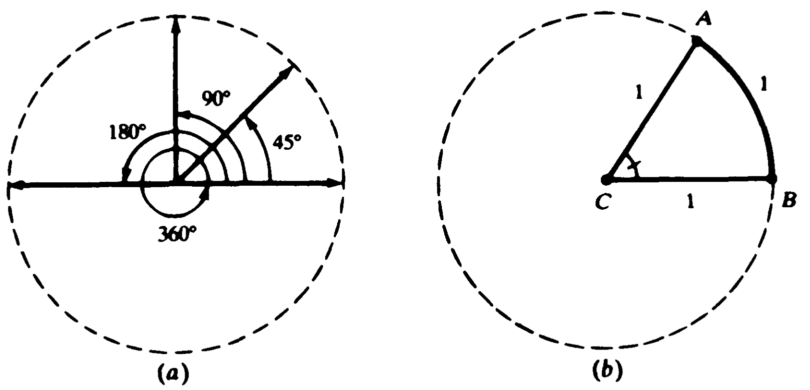


Fig. 25-1

EXAMPLE Let us find the radian measures of some "common" angles given in degrees. Clearly, the null angle is 0 in either measure. For an angle of 30°, (*25.2*) gives

$$30° = 30\left(\frac{\pi}{180} \text{ radians}\right) = \frac{\pi}{6} \text{ radians}$$

for an angle of 45°,

$$45° = 45\left(\frac{\pi}{180} \text{ radians}\right) = \frac{\pi}{4} \text{ radians}$$

and so on, generating Table 25-1. This table should be memorized by the student, who will often be going back and forth between degrees and radians.

Table 25-1

Degrees	Radians
0	0
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$
180	π
270	$\frac{3\pi}{2}$
360	2π

Consider now a circle of radius r with center O (see Fig. 25-2). Let $\measuredangle DOE$ contain θ radians and let s be the length of arc $\widehat{DE}$. The ratio of θ to the number 2π of radians in a complete rotation is the same as the ratio of s to the entire circumference $2\pi r$, $\theta/2\pi = s/2\pi r$. Hence,

$$s = r\theta \tag{25.3}$$

gives the basic relationship between the arc length, the radius, and the radian measure of the central angle.

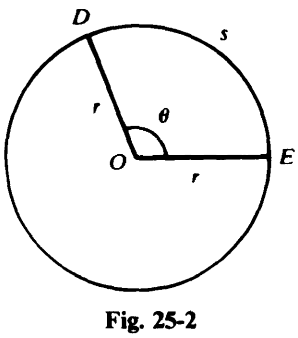


Fig. 25-2

25.2 DIRECTED ANGLES

Angles can be classified as positive or negative, according to the direction of the rotation that generates them. In Fig. 25-3(a), we shall agree that the directed angle AOB is taken to be *positive* when it is obtained by rotating the arrow OA *counterclockwise* toward arrow OB. On the other hand, the directed angle AOB in Fig. 25-3(b) is taken to be *negative* if it is obtained by rotating arrow OA *clockwise* toward arrow OB. Some examples of directed angles and their radian measures are shown in Fig. 25-4.

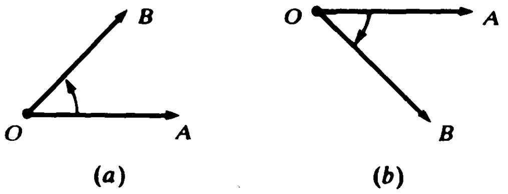


Fig. 25-3

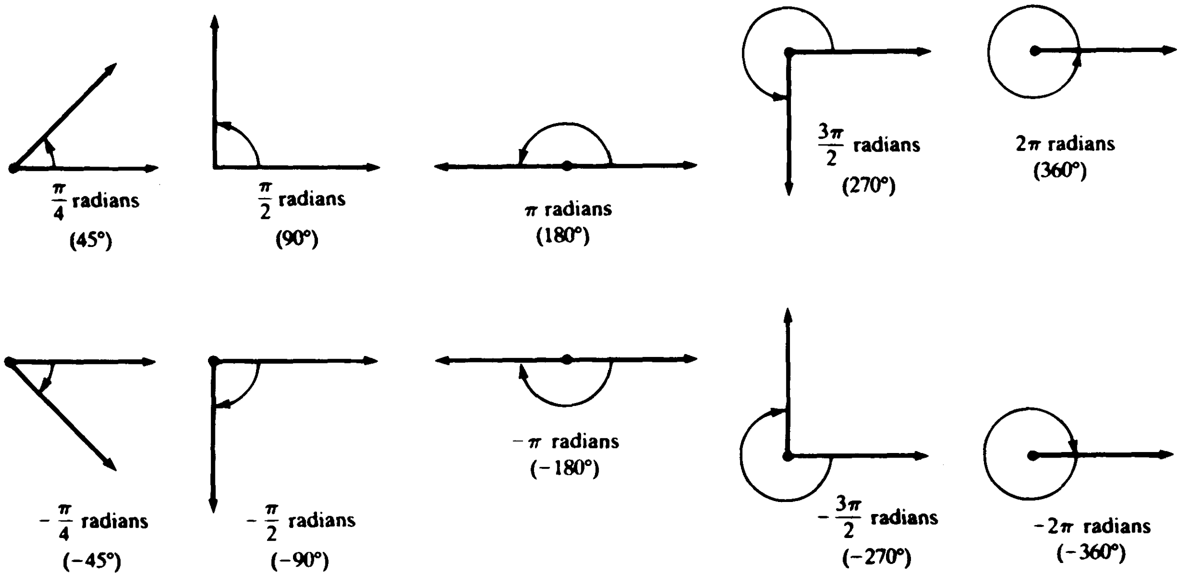


Fig. 25-4

Some directed angles corresponding to more than one complete rotation are shown in Fig. 25-5. It is apparent that directed angles whose radian measures differ by an integral multiple of 2π (e.g., the first and the last angles in Fig. 25-5) represent identical configurations of the two arrows. We shall say that such angles "have the same sides."

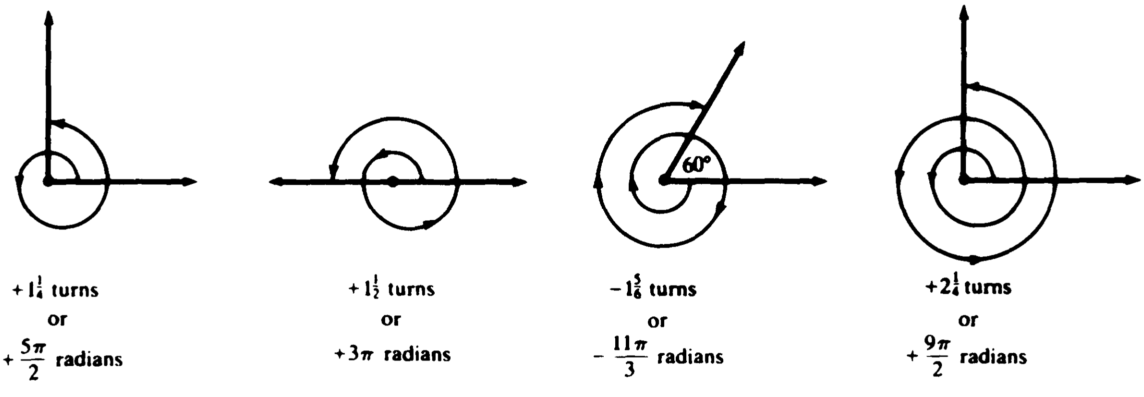


Fig. 25-5

Solved Problems

25.1 Express in radians an angle of: (*a*) 72°; (*b*) 150°.

Use (*25.2*).

(*a*) $72^\circ = 72\left(\frac{\pi}{180}\text{ radians}\right) = \frac{2\cdot 36}{5\cdot 36}(\pi\text{ radians}) = \frac{2\pi}{5}\text{ radians}$

(*b*) $150^\circ = \frac{150}{180}\pi = \frac{5(30)}{6(30)}\pi = \frac{5\pi}{6}\text{ radians}$

25.2 Express in degrees an angle of: (*a*) $5\pi/12$ radians; (*b*) 0.3π radians; (*c*) 3 radians.

Use (*25.1*).

(*a*) $\frac{5\pi}{12}\text{ radians} = \frac{5\pi}{12}\left(\frac{180}{\pi}\text{ degrees}\right) = \frac{5}{12}\times 180^\circ = 75^\circ$

(*b*) $0.3\pi\text{ radians} = \frac{0.3\pi}{\pi}\times 180^\circ = 54^\circ$

(*c*) $3\text{ radians} = \frac{3}{\pi}\times 180^\circ = \left(\frac{540}{\pi}\right)^\circ \approx 172^\circ$

Since $\pi \approx 3.14$.

25.3 (*a*) In a circle of radius 5 centimeters, what arc length along the circumference is intercepted by a central angle of $\pi/3$ radians?

(*b*) In a circle of radius 12 feet, what arc length along the circumference is intercepted by a central angle of 30°?

Use (*25.3*): $s = r\theta$.

(*a*) $s = 5\times\frac{\pi}{3} = \frac{5\pi}{3}$ centimeters

(*b*) The central angle must be changed to radian measure. By Table 25-1,

$$30^\circ = \frac{\pi}{6}\text{ radians} \quad\text{and so}\quad s = 12\times\frac{\pi}{6} = 2\pi\text{ feet}$$

25.4 The minute hand of an ancient tower clock is 5 feet long. How much time has elapsed when the tip has traveled through an arc of 188.4 inches?

In the formula $\theta = s/r$, s and r must be expressed in the same length unit. Choosing feet, we have $s = 188.4/12 = 15.7$ feet and $r = 5$ feet. Hence,

$$\theta = \frac{15.7}{5} = 3.14\text{ radians}$$

This is very nearly π radians, which is a half-revolution, or 30 minutes of time.

25.5 What (positive) angles between 0 and 2π radians have the same sides as angles with the following measures?

(*a*) $\frac{9\pi}{4}$ radians (*b*) 390° (*c*) $-\frac{\pi}{2}$ radians (*d*) -3π radians

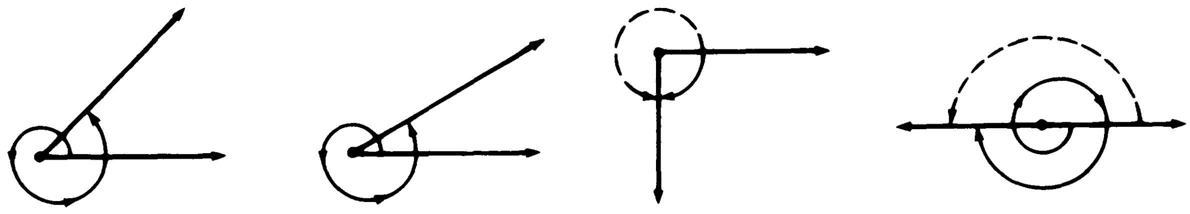
(*a*)
$$\frac{9\pi}{4} = \left(2 + \frac{1}{4}\right)\pi = 2\pi + \frac{\pi}{4}$$

Hence, $9\pi/4$ radians determines a counterclockwise complete rotation (2π radians) plus a counterclockwise rotation of $\pi/4$ radians (45°) [see Fig. 25-6(*a*)]. The "reduced angle"; that is, the angle with measure in $[0, 2\pi)$ and having the same sides as the given angle, therefore is $\pi/4$ radians.

(*b*) $390° = 360° + 30° = (2\pi \text{ radians}) + (\pi/6 \text{ radians})$ [see Fig. 25-6(*b*)]. The reduced angle is $\pi/6$ radians (or 30°).

(*c*) A clockwise rotation of $\pi/2$ radians (90°) is equivalent to a counterclockwise rotation of $2\pi - \pi/2 = 3\pi/2$ radians [see Fig. 25-6(*c*)]. Thus, the reduced angle is $3\pi/2$ radians.

(*d*) Adding a suitable multiple of 2π to the given angle, we have $-3\pi + 4\pi = +\pi$ radians; that is, a clockwise rotation of 3π radians is equivalent to a counterclockwise rotation of π radians [see Fig. 25-6(*d*)]. The reduced angle is π radians.



(*a*) (*b*) (*c*) (*d*)

Fig. 25-6

Supplementary Problems

25.6 Convert the following degree measures of angles into radian measures: (*a*) 36°; (*b*) 15°; (*c*) 2°; (*d*) $(90/\pi)°$; (*e*) 144°.

25.7 Convert the following radian measures of angles into degree measures: (*a*) 2 radians; (*b*) $\pi/5$ radians; (*c*) $7\pi/12$ radians; (*d*) $5\pi/4$ radians; (*e*) $7\pi/6$ radians.

25.8 If a bug moves a distance of 3π centimeters along a circular arc and if this arc subtends a central angle of 45°, what is the radius of the circle?

25.9 In each of the following cases, from the information about two of the quantities s (intercepted arc), r (radius), and θ (central angle), find the third quantity. (If only a number is given for θ, assume that it is the number of radians.) (*a*) $r = 10$, $\theta = \pi/5$; (*b*) $\theta = 60°$, $s = 11/21$; (*c*) $r = 1$, $s = \pi/4$; (*d*) $r = 2$, $s = 3$; (*e*) $r = 3$, $\theta = 90°$; (*f*) $\theta = 180°$, $s = 6.283\,18$; (*g*) $r = 10$, $\theta = 120°$.

25.10 If a central angle of a circle of radius r measures θ radians, find the area A of the sector of the circle determined by the central angle (see Fig. 25-7). [*Hint:* The area of the entire circle is πr^2.]

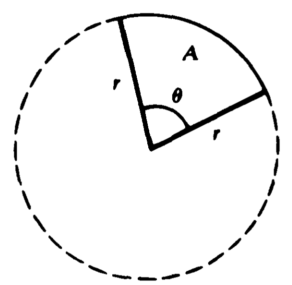


Fig. 25-7

25.11 Draw pictures of the rotations determining angles that measure: (*a*) 405°; (*b*) $11\pi/4$ radians; (*c*) $7\pi/2$ radians; (*d*) $-60°$; (*e*) $-\pi/6$ radians; (*f*) $-5\pi/2$ radians.

25.12 Reduce each angle in Problem 25.11 to the range of 0 to 2π radians.