Chapter 29

Antiderivatives

29.1 DEFINITION AND NOTATION

Definition: An *antiderivative* of a function f is a function whose derivative is f.

EXAMPLES

(*a*) x^2 is an antiderivative of $2x$, since $D_x(x^2) = 2x$.

(*b*) $x^4/4$ is an antiderivative of x^3, since $D_x(x^4/4) = x^3$.

(*c*) $3x^3 - 4x^2 + 5$ is an antiderivative of $9x^2 - 8x$, since $D_x(3x^3 - 4x^2 + 5) = 9x^2 - 8x$.

(*d*) $x^2 + 3$ is an antiderivative of $2x$, since $D_x(x^2 + 3) = 2x$.

(*e*) $\sin x$ is an antiderivative of $\cos x$, since $D_x(\sin x) = \cos x$.

Examples (*a*) and (*d*) show that a function can have more than one antiderivative. This is true for all functions. If $g(x)$ is an antiderivative of $f(x)$, then $g(x) + C$ is also an antiderivative of $f(x)$, where C is any constant. The reason is that $D_x(C) = 0$, whence

$$D_x(g(x) + C) = D_x(g(x))$$

Let us find the relationship between any two antiderivatives of a function.

Theorem 29.1: If $F'(x) = 0$ for all x in an interval $\mathscr{I}$, then $F(x)$ is a constant on $\mathscr{I}$.

The assumption $F'(x) = 0$ tells us that the graph of F always has a horizontal tangent. It is then obvious that the graph of F must be a horizontal straight line; that is, $F(x)$ is constant. For a rigorous proof, see Problem 29.4.

Corollary 29.2: If $g'(x) = h'(x)$ for all x in an interval $\mathscr{I}$, then there is a constant C such that $g(x) = h(x) + C$ for all x in $\mathscr{I}$.

Indeed,

$$D_x(g(x) - h(x)) = g'(x) - h'(x) = 0$$

whence, by Theorem 29.1, $g(x) - h(x) = C$, or $g(x) = h(x) + C$.

According to Corollary 29.2, any two antiderivatives of a given function differ only by a constant. Thus, if we know one antiderivative of a function, we know them all.

NOTATION $\int f(x)\,dx$ stands for *any* antiderivative of f. Thus,

$$D_x\left(\int f(x)\,dx\right) = f(x)$$

OTHER TERMINOLOGY Sometimes the term *indefinite integral* is used instead of antiderivative, and the process of finding antiderivatives is termed *integration*. In the expression $\int f(x)\,dx$, $f(x)$ is called the *integrand*. The motive for this nomenclature will become clear in Chapter 31.

EXAMPLES

(*a*) $\int x^2\,dx = \frac{x^3}{3} + C$. Since $D_x(x^3/3) = x^2$, we know that $x^3/3$ is an antiderivative of x^2. By Corollary 29.2, any other antiderivative of x^2 is of the form $(x^3/3) + C$, where C is a constant.

(*b*) $\int \cos x\,dx = \sin x + C$

(*c*) $\int \sin x\,dx = -\cos x + C$

(*d*) $\int \sec^2 x\,dx = \tan x + C$

(*e*) $\int 0\,dx = C$

(*f*) $\int 1\,dx = x + C$

29.2 RULES FOR ANTIDERIVATIVES

The rules for derivatives—in particular, the sum-or-difference rule and the chain rule—yield corresponding rules for antiderivatives.

RULE 1. $\int a\,dx = ax + C$ for any constant a.

EXAMPLE

$$\int 3\,dx = 3x + C$$

RULE 2. $\int x^r\,dx = \frac{x^{r+1}}{r+1} + C$ for any rational number r other than $r = -1$.

NOTE The antiderivative of x^{-1} will be dealt with in Chapter 34.

Rule 2 follows from Theorem 15.4, according to which

$$D_x(x^{r+1}) = (r+1)x^r \qquad \text{or} \qquad D_x\left(\frac{x^{r+1}}{r+1}\right) = x^r$$

EXAMPLES

(*a*) $$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{3/2}}{\frac{3}{2}} + C = \frac{2}{3}x^{3/2} + C$$

(*b*) $$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2}x^{-2} + C = -\frac{1}{2x^2} + C$$

RULE 3. $\int af(x)\,dx = a\int f(x)\,dx$ for any constant a.

This follows from $D_x\left(a \cdot \int f(x)\,dx\right) = a \cdot D_x\left(\int f(x)\,dx\right) = af(x)$.

EXAMPLE
$$\int 5x^2\,dx = 5\int x^2\,dx = 5\left(\frac{x^3}{3}\right) + C = \frac{5x^3}{3} + C$$

RULE 4. (i) $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

(ii) $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

For $D_x\left(\int f(x)\,dx \pm \int g(x)\,dx\right) = D_x\left(\int f(x)\,dx\right) \pm D_x\left(\int g(x)\,dx\right) = f(x) \pm g(x)$.

EXAMPLE
$$\int (x^2 + x^3)\,dx = \int x^2\,dx + \int x^3\,dx = \frac{x^3}{3} + \frac{x^4}{4} + C$$

Notice that we find a specific antiderivative, $x^3/3 + x^4/4$, and then add the "arbitrary" constant C.

Rules 1 through 4 enable us to compute the antiderivative of any polynomial.

EXAMPLE
$$\int \left(3x^5 - \frac{1}{2}x^4 + 7x^2 + x - 3\right) dx = 3\left(\frac{x^6}{6}\right) - \frac{1}{2}\left(\frac{x^5}{5}\right) + 7\left(\frac{x^3}{3}\right) + \frac{x^2}{2} - 3x + C$$
$$= \frac{x^6}{2} - \frac{x^5}{10} + \frac{7}{3}x^3 + \frac{x^2}{2} - 3x + C$$

The next rule will prove to be extremely useful.

RULE 5 (*Quick Formula I*). $\int (g(x))^r g'(x)\,dx = \frac{(g(x))^{r+1}}{r+1} + C$

The power chain rule implies that

$$D_x\left(\frac{(g(x))^{r+1}}{r+1}\right) = \frac{1}{r+1} D_x(g(x)^{r+1}) = \frac{1}{r+1} \cdot (r+1)(g(x))^r g'(x) = (g(x))^r g'(x)$$

which yields quick formula I.

EXAMPLES

(*a*) $$\int \left(\frac{1}{2}x^2 + 5\right)^7 x\,dx = \frac{1}{8}\left(\frac{1}{2}x^2 + 5\right)^8 + C$$

(*b*) $$\int \sqrt{2x - 5}\,dx = \frac{1}{2}\int (2x-5)^{1/2}(2)\,dx = \frac{1}{2}\frac{(2x-5)^{3/2}}{\frac{3}{2}} + C = \frac{1}{3}(2x-5)^{3/2} + C$$

RULE 6 (*Substitution Method*). Deferring the general formulation and justification to Problem 29.18, we illustrate the method by three examples.

(i) Find $\int x^2 \cos x^3\,dx$. Let $x^3 = u$. Then, by Section 21.3, the differential of u is given by

$$du = D_x(x^3)\,dx = 3x^2\,dx \qquad \text{or} \qquad x^2\,dx = \frac{1}{3}\,du$$

Now substitute u for x^3 and $\frac{1}{3}du$ for $x^2\,dx$,

$$\int x^2 \cos x^3\,dx = \int \frac{1}{3}\cos u\,du = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin x^3 + C$$

(ii) Find $\int (x^2 + 3x - 5)^3(2x + 3)\,dx$. Let $u = x^2 + 3x - 5$, $du = (2x + 3)\,dx$. Then

$$\int (x^2 + 3x - 5)^3(2x + 3)\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{1}{4}(x^2 + 3x - 5)^4 + C$$

(iii) Find $\int \sin^2 x \cos x\,dx$. Let $u = \sin x$. Then $du = \cos x\,dx$, and

$$\int \sin^2 x \cos x\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$$

Notice that quick formula I (Rule 5) is a special case of Rule 6, corresponding to the substitution $u = g(x)$. The beauty of quick formula I is that, when it is applicable, it allows us to avoid the bother of going through the substitution process.

Solved Problems

29.1 Find the following antiderivatives:

(a) $\int (\sqrt[3]{x} - 5x^2)\,dx$ (b) $\int (4x + \sqrt{x^5} - 2)\,dx$

(c) $\int (x^2 - \sec^2 x)\,dx$ (d) $\int \frac{2\sqrt{x} + 3x^2}{x}\,dx$

(a)
$$\int (\sqrt[3]{x} - 5x^2)\,dx = \int (x^{1/3} - 5x^2)\,dx$$
$$= \frac{x^{4/3}}{\frac{4}{3}} - 5\left(\frac{x^3}{3}\right) + C \qquad \text{[by Rules 2 and 4]}$$
$$= \frac{3}{4}x^{4/3} - \frac{5}{3}x^3 + C$$

(b)
$$\int (4x + \sqrt{x^5} - 2)\,dx = \int (4x + x^{5/2} - 2)\,dx = 4\left(\frac{x^2}{2}\right) + \frac{x^{7/2}}{\frac{7}{2}} - 2x + C$$
$$= 2x^2 + \frac{2}{7}x^{7/2} - 2x + C$$

(c)
$$\int (x^2 - \sec^2 x)\,dx = \int x^2\,dx - \int \sec^2 x\,dx = \frac{x^3}{3} - \tan x + C$$

(d)
$$\int \frac{2\sqrt{x} + 3x^2}{x}\,dx = \int \left(\frac{2}{\sqrt{x}} + 3x\right)dx = 2\int x^{-1/2}\,dx + 3\int x\,dx \qquad \text{[by Rules 1 and 4]}$$
$$= 2\,\frac{x^{1/2}}{\frac{1}{2}} + 3\,\frac{x^2}{2} + C = 4\sqrt{x} + \frac{3}{2}x^2 + C$$

29.2 Find the following antiderivatives:

(a) $\int (2x^3 - x)^4(6x^2 - 1)\,dx$ (b) $\int \sqrt[3]{5x^2 - 1}\;x\,dx$

(*a*) Notice that $D_x(2x^3 - x) = 6x^2 - 1$. So, by quick formula I,

$$\int (2x^3 - x)^4(6x^2 - 1)\,dx = \frac{1}{5}(2x^3 - x)^5 + C$$

(*b*) Observe that $D_x(5x^2 - 1) = 10x$. Then, by Rule 1,

$$\int \sqrt[3]{5x^2 - 1}\,x\,dx = \int (5x^2 - 1)^{1/3}\,x\,dx = \frac{1}{10}\int (5x^2 - 1)^{1/3}\,10x\,dx$$
$$= \frac{1}{10}\frac{(5x^2 - 1)^{4/3}}{\frac{4}{3}} + C \qquad \text{[by quick formula I]}$$
$$= \frac{3}{40}(5x^2 - 1)^{4/3} + C = \frac{3}{40}(\sqrt[3]{5x^2 - 1})^4 + C$$
$$= \frac{3}{40}\sqrt[3]{(5x^2 - 1)^4} + C$$

(For manipulations of rational powers, review Section 15.2.)

29.3 Use the substitution method to evaluate:

(*a*) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ (*b*) $\displaystyle\int x\sec^2(3x^2 - 1)\,dx$ (*c*) $\displaystyle\int x^2\sqrt{x + 2}\,dx$

(*a*) Let $u = \sqrt{x}$. Then,

$$du = D_x(\sqrt{x})\,dx = D_x(x^{1/2})\,dx = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$$

Hence,
$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = 2\int \sin u\,du = -2\cos u + C = -2\cos\sqrt{x} + C$$

(*b*) Let $u = 3x^2 - 1$. Then $du = 6x\,dx$, and

$$\int x\sec^2(3x^2 - 1)\,dx = \frac{1}{6}\int \sec^2 u\,du = \frac{1}{6}\tan u + C = \frac{1}{6}\tan(3x^2 - 1) + C$$

(*c*) Let $u = x + 2$. Then $du = dx$ and $x = u - 2$. Hence,

$$\int x^2\sqrt{x + 2}\,dx = \int (u - 2)^2\sqrt{u}\,du = \int (u^2 - 4u + 4)u^{1/2}\,du$$
$$= \int (u^{5/2} - 4u^{3/2} + 4u^{1/2})\,du \qquad [\text{by } u^r u^s = u^{r+s}]$$
$$= \frac{2}{7}u^{7/2} - \frac{8}{5}u^{5/2} + \frac{8}{3}u^{3/2} + C = \frac{2}{7}(x + 2)^{7/2} - \frac{8}{5}(x + 2)^{5/2} + \frac{8}{3}(x + 2)^{3/2} + C$$

The substitution $u = \sqrt{x + 2}$ would also work.

29.4 Prove Theorem 29.1.

Let a and b be any two numbers in $\mathscr{I}$. By the mean-value theorem (Theorem 17.2), there is a number c between a and b, and therefore in $\mathscr{I}$, such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

But by hypothesis, $F'(c) = 0$; hence, $F(b) - F(a) = 0$, or $F(b) = F(a)$.

29.5 A rocket is shot straight up into the air with an initial velocity of 256 feet per second. (*a*) When does it reach its maximum height? (*b*) What is its maximum height? (*c*) When does it hit the ground again? (*d*) What is its speed when it hits the ground?

In free-fall problems, $v = \int a\,dt$ and $s = \int v\,dt$ because, by definition, $a = dv/dt$ and $v = ds/dt$. Since $a = -32$ feet per second per second (when *up* is positive),

$$v = \int -32\,dt = -32t + C_1$$

$$s = \int (-32t + C_1)\,dt = (-32)\frac{t^2}{2} + C_1 t + C_2 = -16t^2 + C_1 t + C_2$$

in which the values of C_1 and C_2 are determined by the initial conditions of the problem. In the present case, it is given that $v(0) = 256$ and $s(0) = 0$. Hence, $256 = 0 + C_1$ and $0 = 0 + 0 + C_2$, so that

$$v = -32t + 256 \qquad (1)$$

$$s = -16t^2 + 256t \qquad (2)$$

(*a*) For maximum height, $ds/dt = v = -32t + 256 = 0$. So,

$$t = \frac{256}{32} = 8 \text{ seconds}$$

when the maximum height is reached.

(*b*) Substituting $t = 8$ in (*2*),

$$s(8) = -16(8)^2 + 256(8) = -1024 + 2048 = 1024 \text{ feet}$$

(*c*) Setting $s = 0$ in (*2*),

$$-16t^2 + 256t = 0$$
$$-16t(t - 16) = 0$$
$$t = 0 \quad \text{or} \quad t = 16$$

The rocket leaves the ground at $t = 0$ and returns at $t = 16$.

(*d*) Substituting $t = 16$ in (*1*), $v(16) = -32(16) + 256 = -256$ feet per second. The speed is the magnitude of the velocity, 256 feet per second.

29.6 Find an equation of the curve passing through the point (2, 3) and having slope $3x^3 - 2x + 5$ at each point (x, y).

The slope is given by the derivative. So,

$$\frac{dy}{dx} = 3x^3 - 2x + 5$$

Hence,
$$y = \int (3x^3 - 2x + 5)\,dx = \frac{3}{4}x^4 - x^2 + 5x + C$$

Since (2, 3) is on the curve,

$$3 = \frac{3}{4}(2)^4 - (2)^2 + 5(2) + C = 12 - 4 + 10 + C = 18 + C$$

Thus, $C = -15$, and

$$y = \tfrac{3}{4}x^4 - x^2 + 5x - 15$$

Supplementary Problems

29.7 Find the following antiderivatives:

(a) $\int (2x^3 - 5x^2 + 3x + 1)\,dx$ (b) $\int \left(5 - \frac{1}{\sqrt{x}}\right) dx$ (c) $\int 2\sqrt[4]{x}\,dx$

(d) $\int 5\sqrt[3]{x^2}\,dx$ (e) $\int \frac{3}{x^4}\,dx$ (f) $\int (x^2 - 1)\sqrt{x}\,dx$

(g) $\int \left(\frac{1}{x^3} - \frac{1}{x^5}\right) dx$ (h) $\int \frac{3x^2 - 2x + 1}{\sqrt{x}}\,dx$ (i) $\int (3 \sin x + 5 \cos x)\,dx$

(j) $\int (7 \sec^2 x - \sec x \tan x)\,dx$ (k) $\int (\csc^2 x + 3x^2)\,dx$ (l) $\int x\sqrt{3x}\,dx$

(m) $\int \frac{1}{\sec x}\,dx$ (n) $\int \tan^2 x\,dx$ (o) $\int x(x^4 + 2)^2\,dx$

[*Hint:* Use Theorem 28.3 in (*n*).]

29.8 Evaluate the following antiderivatives by using Rule 5 or Rule 6. [In (*m*), $a \neq 0$.]

(a) $\int \sqrt{7x + 4}\,dx$ (b) $\int \frac{1}{\sqrt{x - 1}}\,dx$ (c) $\int (3x - 5)^{12}\,dx$

(d) $\int \sin(3x - 1)\,dx$ (e) $\int \sec^2 \frac{x}{2}\,dx$ (f) $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\,dx$

(g) $\int (4 - 2t^2)^7 t\,dt$ (h) $\int x^2 \sqrt[3]{x^3 + 5}\,dx$ (i) $\int \frac{x}{\sqrt{x + 1}}\,dx$

(j) $\int \sqrt[3]{x^2 - 2x + 1}\,dx$ (k) $\int (x^4 + 1)^{1/3} x^7\,dx$ (l) $\int \frac{x}{\sqrt{1 + 5x^2}}\,dx$

(m) $\int x\sqrt{ax + b}\,dx$ (n) $\int \frac{\cos 3x}{\sin^2 3x}\,dx$ (o) $\int \sqrt{1 - x}\,x^2\,dx$

(p) $\int (3x - 5)^{12} x\,dx$ (q) $\int (4 - 7t^2)^7 t\,dt$ (r) $\int \frac{\sin(1/x)\cos(1/x)}{x^2}\,dx$

(s) $\int \frac{1}{x^5} \sec^2 \frac{3}{x^4}\,dx$

29.9 A rocket is shot vertically upward from a tower 240 feet above the ground, with an initial velocity of 224 feet per second. (*a*) When will it attain its maximum height? (*b*) What will be its maximum height? (*c*) When will it strike the ground? (*d*) With what speed will it hit the ground?

29.10 (*Rectilinear Motion, Chapter 18*) A particle moves along the *x*-axis with acceleration $a = 2t - 3$ feet per second per second. At time $t = 0$, it is at the origin and moving with a speed of 4 feet per second in the positive direction. (*a*) Find a formula for its velocity v in terms of t. (*b*) Find a formula for its position x in terms of t. (*c*) When and where does the particle change direction? (*d*) At what times is the particle moving toward the left?

29.11 Rework Problem 29.10 if $a = t^2 - \frac{13}{3}$ feet per second per second.

29.12 A rocket shot straight up from ground level hits the ground 10 seconds later. (*a*) What was its initial velocity? (*b*) How high did it go?

29.13 A motorist applies the brakes on a car moving at 45 miles per hour on a straight road, and the brakes cause a constant deceleration of 22 feet per second per second. (*a*) In how many seconds will the car stop? (*b*) How many feet will the car have traveled after the time the brakes were applied? [*Hint:* Put the origin at the point where the brakes were initially applied, and let $t = 0$ at that time. Note that speed and deceleration involve different units of distance and time; change the speed to feet per second.]

29.14 A particle moving on a straight line has acceleration $a = 5 - 3t$, and its velocity is 7 at time $t = 2$. If $s(t)$ is the distance from the origin at time t, find $s(2) - s(1)$.

29.15 (*a*) Find the equation of a curve passing through the point (3, 2) and having slope $2x^2 - 5$ at point (x, y). (*b*) Find the equation of a curve passing through the point (0, 1) and having slope $12x + 1$ at point (x, y).

29.16 A ball rolls in a straight line, with an initial velocity of 10 feet per second. Friction causes the velocity to decrease at a constant rate of 4 feet per second per second until the ball stops. How far will the ball roll? [*Hint:* $a = -4$ and $v_0 = 10$.]

29.17 A particle moves on the x-axis with acceleration $a(t) = 2t - 2$ for $0 \le t \le 3$. The initial velocity v_0 at $t = 0$ is 0. (*a*) Find the velocity $v(t)$. (*b*) When is $v(t) < 0$? (*c*) When does the particle change direction? (*d*) Find the displacement between $t = 0$ and $t = 3$. (*Displacement* is the net change in position.) (*e*) Find the total distance traveled from $t = 0$ to $t = 3$.

29.18 Justify the following form of the substitution method (Rule 6):

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

where u is replaced by $g(x)$ after integration on the right. The "substitution" would be applied to the left-hand side by letting $u = g(x)$ and $du = g'(x)\,dx$. [*Hint:* By the chain rule,

$$D_x\left(\int f(u)\,du\right) = D_u\left(\int f(u)\,du\right) \cdot (du/dx) = f(u)(du/dx) = f(g(x))g'(x).]$$

Chapter 30

The Definite Integral

30.1 SIGMA NOTATION

The Greek capital letter Σ is used in mathematics to indicate repeated addition.

EXAMPLES

(*a*) $\sum_{i=1}^{99} i = 1 + 2 + 3 + \cdots + 99$

that is, the sum of the first 99 positive integers.

(*b*) $\sum_{i=1}^{6} (2i - 1) = 1 + 3 + 5 + 7 + 9 + 11$

that is, the sum of the first six odd positive integers.

(*c*) $\sum_{i=2}^{5} 3i = 6 + 9 + 12 + 15 = 3(2 + 3 + 4 + 5) = 3\sum_{i=2}^{5} i$

(*d*) $\sum_{j=1}^{15} j^2 = 1^2 + 2^2 + 3^2 + \cdots + 15^2 = 1 + 4 + 9 + \cdots + 225$

(*e*) $\sum_{j=1}^{5} \sin j\pi = \sin \pi + \sin 2\pi + \sin 3\pi + \sin 4\pi + \sin 5\pi$

In general, given a function f defined on the integers, and given integers k and $n \geq k$,

$$\sum_{i=k}^{n} f(i) = f(k) + f(k + 1) + \cdots + f(n)$$

30.2 AREA UNDER A CURVE

Let f be a function such that $f(x) \geq 0$ for all x in the closed interval $[a, b]$. Then its graph is a curve lying on or above the x-axis (see Fig. 30-1). We have an intuitive idea of the *area A* of the region $\mathscr{R}$ lying under the curve, above the x-axis, and between the vertical lines $x = a$ and $x = b$. Let us set up a procedure for finding the value of the area A.

Select points $x_1, x_2, \ldots, x_{n-1}$ inside $[a, b]$ (see Fig. 30-2). Let $x_0 = a$ and $x_n = b$,

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

These divide $[a, b]$ into the n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$. Let the lengths of these subintervals be $\Delta_1 x, \Delta_2 x, \ldots, \Delta_n x$, where

$$\Delta_i x \equiv x_i - x_{i-1}$$

Draw vertical lines $x = x_i$ from the x-axis up to the graph, thereby dividing the region $\mathscr{R}$ into n strips. If $\Delta_i A$ is the area of the ith strip, then

$$A = \sum_{i=1}^{n} \Delta_i A$$

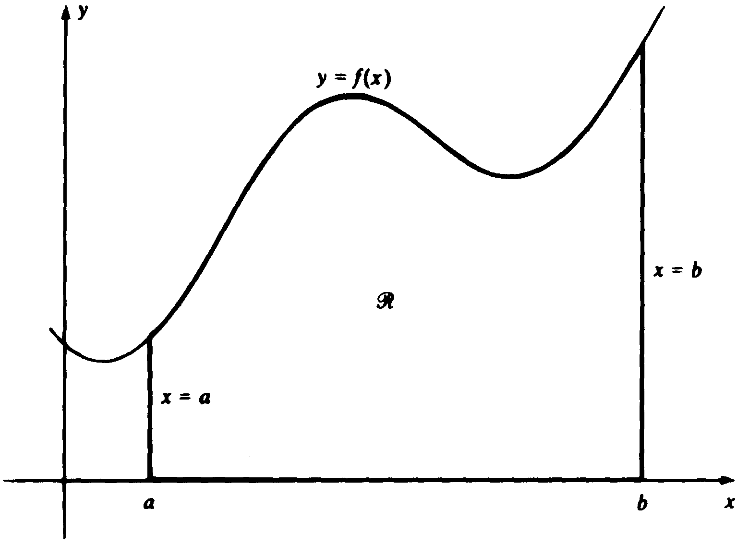


Fig. 30-1

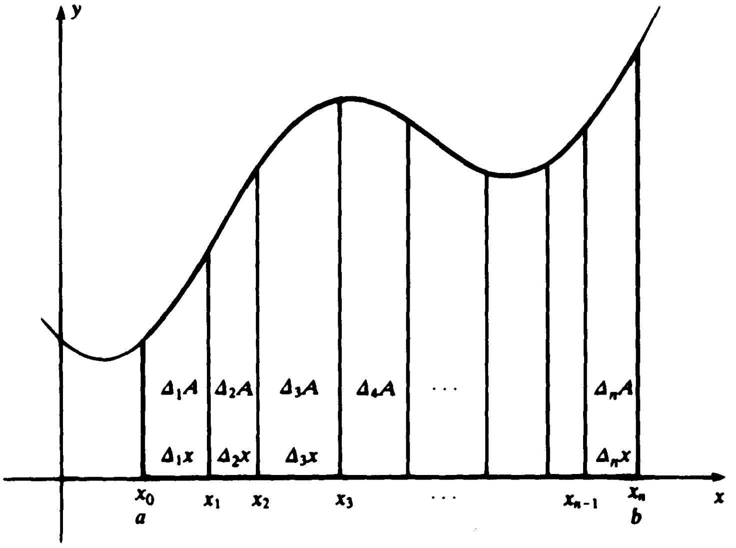


Fig. 30-2

Approximate the area $\Delta_i A$ as follows. Choose any point x_i^* in the ith subinterval $[x_{i-1}, x_i]$ and draw the vertical line segment from the point x_i^* up to the graph (see the dashed lines in Fig. 30-3); the length of this segment is $f(x_i^*)$. The rectangle with base $\Delta_i x$ and height $f(x_i^*)$ has area $f(x_i^*)\,\Delta_i x$, which is approximately the area $\Delta_i A$ of the ith strip. So, the total area A under the curve is approximately the sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta_i x = f(x_1^*)\,\Delta_1 x + f(x_2^*)\,\Delta_2 x + \cdots + f(x_n^*)\,\Delta_n x \qquad (30.1)$$

The approximation becomes better and better as we divide the interval $[a, b]$ into more and more subintervals and as we make the lengths of these subintervals smaller and smaller. If successive approximations get as close as one wishes to a specific number, then this number is denoted by

$$\int_a^b f(x)\,dx$$

and is called the *definite integral of f from a to b*. Such a number does not exist for all functions f, but it does exist, for example, when the function f is continuous on $[a, b]$.

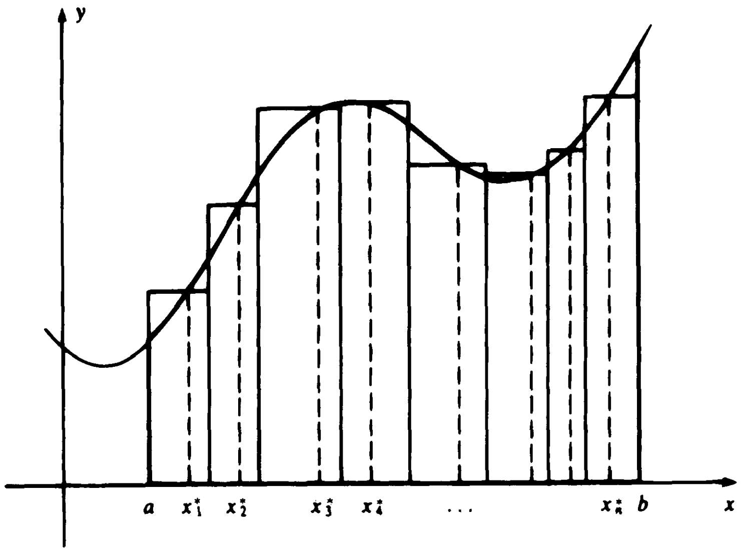


Fig. 30-3

EXAMPLE Approximating the definite integral by a *small* number n of rectangular areas does not usually give good numerical results. To see this, consider the function $f(x) = x^2$ on $[0, 1]$. Then $\int_0^1 x^2\,dx$ is the area under the parabola $y = x^2$, above the x-axis, between $x = 0$ and $x = 1$. Divide $[0, 1]$ into $n = 10$ equal subintervals by the points 0.1, 0.2, ..., 0.9 (see Fig. 30-4). Thus, each $\Delta_i x$ equals 1/10. In the ith subinterval, choose x_i^* to be the left-hand endpoint $(i - 1)/10$. Then,

$$\begin{aligned}\int_0^1 x^2\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta_i x &= \sum_{i=1}^{10}\left(\frac{i-1}{10}\right)^2\left(\frac{1}{10}\right) = \sum_{i=1}^{10}\frac{(i-1)^2}{100}\left(\frac{1}{10}\right)\\ &= \frac{1}{1000}\sum_{i=1}^{10}(i-1)^2 \qquad \text{[by example (}c\text{) above]}\\ &= \frac{1}{1000}(0 + 1 + 4 + \cdots + 81) = \frac{1}{1000}(285) = 0.285\end{aligned}$$

As will be shown in Problem 30.2, the exact value is

$$\int_0^1 x^2\,dx = \frac{1}{3} = 0.333\ldots$$

So the above approximation is not too good. In terms of Fig. 30-4, there is too much unfilled space between the curve and the tops of the rectangles.

Now, for an arbitrary (not necessarily nonnegative) function f on $[a, b]$, a sum of the form *(30.1)* can be defined, without any reference to the graph of f or to the notion of area. The precise epsilon–delta procedure of Problem 8.4(*a*) can be used to determine whether this sum approaches a limiting value as n approaches ∞ and as the maximum of the lengths $\Delta_i x$ approaches 0. If it does, the function f is said to

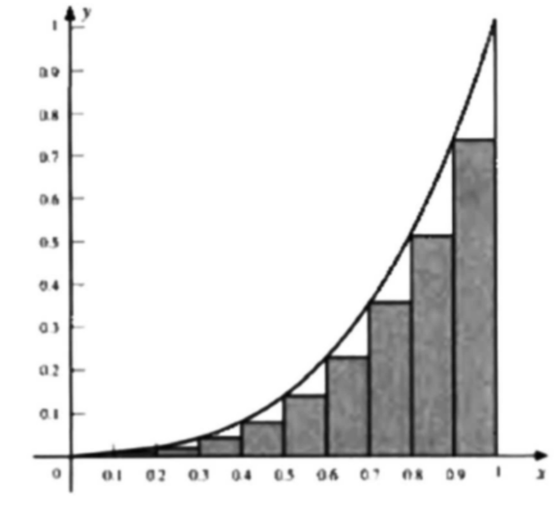


Fig. 30-4

be *integrable* on $[a, b]$, and the limit is called the *definite integral of f* on $[a, b]$ and is denoted by[1]

$$\int_a^b f(x)\,dx$$

In the following section, we shall state several properties of the definite integral, omitting any proof that depends on the precise definition in favor of the intuitive picture of the definite integral as an area [when $f(x) \geq 0$].

30.3 PROPERTIES OF THE DEFINITE INTEGRAL

Theorem 30.1: If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Theorem 30.2: $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ for any constant c.

Obviously, since the respective approximating sums enjoy this relationship [example (*c*) above], the limits enjoy it as well.

EXAMPLE Suppose that $f(x) \leq 0$ for all x in $[a, b]$. The graph of f—along with its mirror image, the graph of $-f$—is shown in Fig. 30-5. Since $-f(x) \geq 0$,

$$\int_a^b -f(x)\,dx = \text{area } B$$

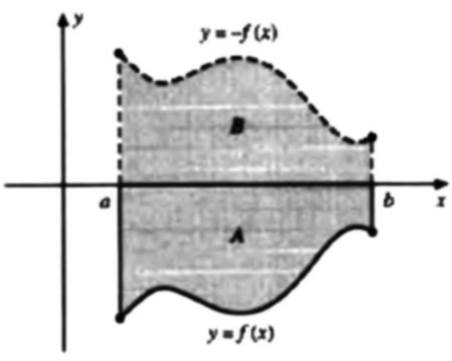


Fig. 30-5

[1] The definite integral is also called the *Riemann integral of f* on $[a, b]$ and the sums (*30.1*) are called *Riemann sums* for f on $[a, b]$.

But by symmetry, area B = area A; and by Theorem 30.2, with $c = -1$,

$$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$$

It follows that

$$\int_a^b f(x)\,dx = -(\text{area } A)$$

In other words, the definite integral of a nonpositive function is the *negative* of the area *above* the graph of the function and *below* the x-axis.

Theorem 30.3: If f and g are integrable on $[a, b]$, then so are $f + g$ and $f - g$, and

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Again, this property is implied by the corresponding property of the approximating sums,

$$\sum_{i=1}^{n} [P(i) \pm Q(i)] = \sum_{i=1}^{n} P(i) \pm \sum_{i=1}^{n} Q(i)$$

Theorem 30.4: If $a < c < b$ and if f is integrable on $[a, c]$ and on $[c, b]$, then f is integrable on $[a, b]$, and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

For $f(x) \geq 0$, the theorem is obvious: the area under the graph from a to b must be the sum of the areas from a to c and from c to b.

EXAMPLE Theorem 30.4 yields a geometric interpretation for the definite integral when the graph of f has the appearance shown in Fig. 30-6. Here,

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \int_{c_3}^{c_4} f(x)\,dx + \int_{c_4}^{b} f(x)\,dx \\ &= A_1 - A_2 + A_3 - A_4 + A_5\end{aligned}$$

That is, the definite integral may be considered a total area, in which areas *above* the x-axis are counted as *positive*, and areas *below* the x-axis are counted as *negative*. Thus, we can infer from Fig. 27-2(*b*) that

$$\int_0^{2\pi} \sin x\,dx = 0$$

because the positive area from 0 to π is just canceled by the negaive area from π to 2π.

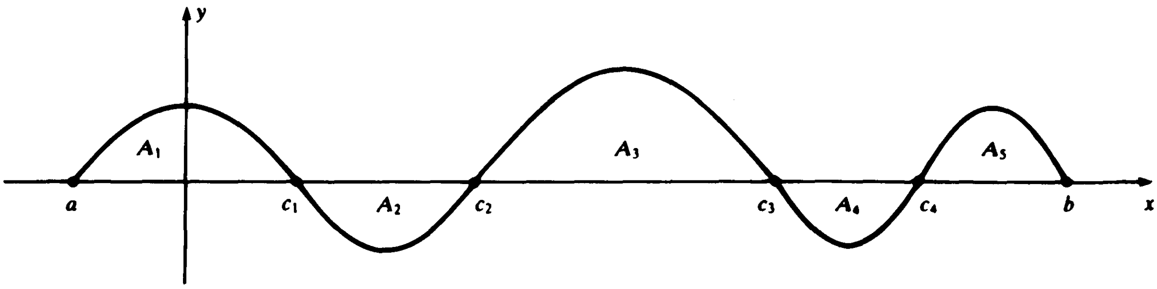


Fig. 30-6

Arbitrary Limits of Integration

In defining $\int_a^b f(x)\,dx$, we have assumed that the *limits of integration* a and b are such that $a < b$. Extend the definition as follows:

(1) $\int_a^a f(x)\,dx = 0$.

(2) If $a > b$, let $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (with the definite integral on the right falling under the original definition).

Under this extended definition, *interchanging the limits of integration in any definite integral reverses the algebraic sign of the integral.* Moreover, the equations of Theorems 30.2, 30.3, and 30.4 now hold for arbitrary limits of integration a, b, and c.

Solved Problems

30.1 Show that $\displaystyle\int_a^b 1\,dx = b - a$.

For any subdivision $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ of $[a, b]$, the approximating sum *(30.1)* is

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta_i x = \sum_{i=1}^{n} \Delta_i x \qquad [\text{since } f(x) = 1 \text{ for all } x]$$
$$= (x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) = x_n - x_0 = b - a$$

Since every approximating sum is equal to $b - a$,

$$\int_a^b 1\,dx = b - a$$

As an alternative, intuitive proof, note that $\int_a^b 1\,dx$ is equal to the area of a rectangle with base of length $b - a$ and height 1, since the graph of the constant function 1 is the line $y = 1$. This area is $(b - a)(1) = b - a$ (see Fig. 30-7).

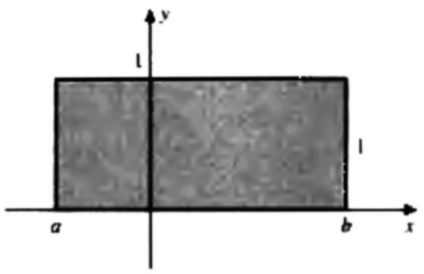


Fig. 30-7

30.2 Calculate $\displaystyle\int_0^1 x^2\,dx$. $\left[\text{You may assume the formula } 1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}, \text{ which is established in Problem 30.12}(a, ii).\right]$

Divide the interval $[0, 1]$ into n equal parts, as indicated in Fig. 30-8, making each $\Delta_i x = 1/n$. In the ith subinterval $[(i-1)/n, i/n]$, let x_i^* be the right endpoint i/n. Then *(30.1)* becomes

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta_i x = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2$$
$$= \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6} = \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$
$$= \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

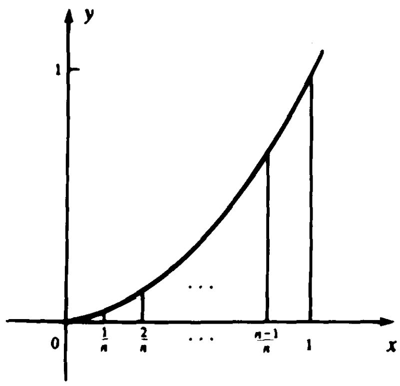


Fig. 30-8

We can make the subdivision finer and finer by letting n approach infinity. Then,

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{6}(1)(2) = \frac{1}{3}$$

This kind of direct calculation of a definite integral is possible only for the very simplest functions $f(x)$. A much more powerful method will be explained in Chapter 31.

30.3 Let $f(x)$ and $g(x)$ be integrable on $[a, b]$.

(*a*) If $f(x) \geq 0$ on $[a, b]$, show that

$$\int_a^b f(x)\,dx \geq 0$$

(*b*) If $f(x) \leq g(x)$ on $[a, b]$, show that

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

(*c*) If $m \leq f(x) \leq M$ on $[a, b]$, show that

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

(*a*) The definite integral, being the area under the graph of f, cannot be negative. More fundamentally, every approximating sum (*30.1*) is nonnegative, since $f(x_i^*) \geq 0$ and $\Delta_i x > 0$. Hence (as shown in Problem 9.10), the limiting value of the approximating sums is also nonnegative.

(*b*) Because $g(x) - f(x) \geq 0$ on $[a, b]$,

$$\int_a^b (g(x) - f(x))\,dx \geq 0 \qquad \text{[by (a)]}$$

$$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx \geq 0 \qquad \text{[by Theorem 30.3]}$$

$$\int_a^b g(x)\,dx \geq \int_a^b f(x)\,dx$$

(*c*)

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx \qquad \text{[by (b)]}$$

$$m\int_a^b 1\,dx \leq \int_a^b f(x)\,dx \leq M\int_a^b 1\,dx \qquad \text{[by Theorem 30.2]}$$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \qquad \text{[by Problem 30.1]}$$

Supplementary Problems

30.4 Evaluate:

(a) $\int_2^5 8\,dx$ (b) $\int_0^1 5x^2\,dx$ (c) $\int_0^1 (x^2+4)\,dx$

[*Hint:* Use Problems 30.1 and 30.2.]

30.5 For the function f graphed in Fig. 30-9, express $\int_0^5 f(x)\,dx$ in terms of the areas A_1, A_2, A_3.

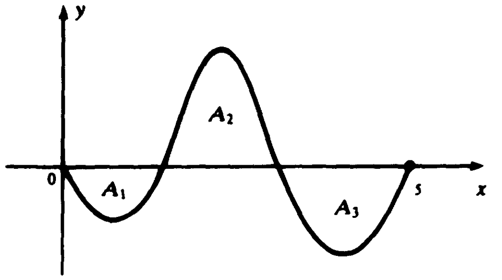


Fig. 30-9

30.6 (a) Show that $\int_0^b x\,dx = \frac{b^2}{2}$. You may assume the formula $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ proved in Problem 30.12(a). Check your result by using the standard formula for the area of a triangle. [*Hint:* Divide the interval $[0, b]$ into n equal subintervals, and choose $x_i^* = ib/n$, the right endpoint of the ith subinterval.]

(b) Show that $\int_a^b x\,dx = \frac{b^2 - a^2}{2}$. [*Hint:* Use (a) and Theorem 30.4.]

(c) Evaluate $\int_1^3 5x\,dx$. [*Hint:* Use Theorem 30.2 and (b).]

30.7 Show that the equation of Theorem 30.4,

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

holds for any numbers a, b, c, such that the two definite integrals on the left can be defined in the extended sense. [*Hint:* Consider all six arrangements of distinct a, b, c: $a < b < c$, $a < c < b$, $b < a < c$, $b < c < a$, $c < a < b$, $c < b < a$. Also consider the cases where two of the numbers are equal or all three are equal.]

30.8 Show that $1 \le \int_1^2 x^3\,dx \le 8$. [*Hint:* Use Problem 30.3(c).]

30.9 (a) Find $\int_0^2 \sqrt{4 - x^2}\,dx$ by using a formula of geometry. [*Hint:* What curve is the graph of $y = \sqrt{4 - x^2}$?]

(b) From part (a) infer that $0 \le \pi \le 4$. (Much closer estimates of π are obtainable this way.)

30.10 Evaluate:

(a) $\sum_{i=1}^{3} (3i - 1)$ (b) $\sum_{k=0}^{4} (3k^2 + 4)$

(c) $\sum_{j=0}^{3} \sin\frac{j\pi}{6}$ (d) $\sum_{n=1}^{5} f\left(\frac{1}{n}\right)$ if $f(x) = \frac{1}{x}$

30.11 If f is continuous on $[a, b]$, $f(x) \geq 0$ on $[a, b]$, and $f(x) > 0$ for some x in $[a, b]$, show that

$$\int_a^b f(x)\,dx > 0$$

[*Hint:* By continuity, $f(x) > K > 0$ on some closed interval inside $[a, b]$. Use Theorem 30.4 and Problem 30.3(*c*).]

30.12 (*a*) Use mathematical induction (see Problem 12.2) to prove:

(i) $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ (ii) $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(*b*) By looking at the cases when $n = 1, 2, 3, 4, 5$, guess a formula for $1^3 + 2^3 + \cdots + n^3$ and then prove it by mathematical induction. [*Hint:* Compare the values of formula (i) in part (*a*) for $n = 1, 2, 3, 4, 5$.]

30.13 If the graph of f between $x = 1$ and $x = 5$ is as shown in Fig. 30-10, evaluate $\int_1^5 f(x)\,dx$.

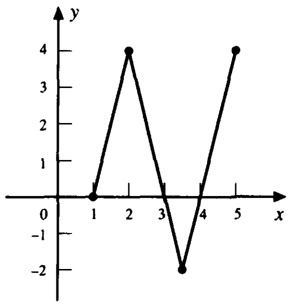


Fig. 30-10

30.14 Let $f(x) = 3x + 1$ for $0 \leq x \leq 1$. If the interval $[0, 1]$ is divided into five subintervals of equal length, what is the smallest corresponding Riemann sum (*30.1*)?

Chapter 31

The Fundamental Theorem of Calculus

31.1 CALCULATION OF THE DEFINITE INTEGRAL

We shall develop a simple method for calculating

$$\int_a^b f(x)\,dx$$

a method based on a profound and surprising connection between differentiation and integration. This connection, discovered by Isaac Newton and Gottfried von Leibniz, the co-inventors of calculus, is expressed in the following:

Theorem 31.1: Let f be continuous on $[a, b]$. Then, for x in $[a, b]$,

$$\int_a^x f(t)\,dt$$

is a function of x such that

$$D_x\left(\int_a^x f(t)\,dt\right) = f(x)$$

A proof may be found in Problem 31.5.

Now for the computation of the definite integral, let $F(x) = \int f(x)\,dx$ denote some known antiderivative of $f(x)$ (for x in $[a, b]$). According to Theorem 31.1, the function $\int_a^x f(t)\,dt$ is also an antiderivative of $f(x)$. Hence, by Corollary 29.2,

$$\int_a^x f(t)\,dt = F(x) + C$$

for some constant C. When $x = a$,

$$0 = \int_a^a f(t)\,dt = F(a) + C \qquad \text{or} \qquad C = -F(a)$$

Thus, when $x = b$,

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

and we have proved:

Theorem 31.2 (*Fundamental Theorem of Calculus*): Let f be continuous on $[a, b]$, and let $F(x) = \int f(x)\,dx$. Then,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

NOTATION The difference $F(b) - F(a)$ will often be denoted by $F(x)]_a^b$, and the fundamental theorem notated as

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

EXAMPLES

(*a*) Recall the complicated evaluation $\int_0^1 x^2\,dx = \frac{1}{3}$ in Problem 30.2. If, instead, we choose the antiderivative $x^3/3$ and apply the fundamental theorem,

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\bigg]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

(*b*) Let us find the area A under one arch of the curve $y = \sin x$; say, the arch from $x = 0$ to $x = \pi$. With $\int \sin x\,dx = -\cos x + \sqrt{5}$ the fundamental theorem gives

$$A = \int_0^\pi \sin x\,dx = (-\cos x + \sqrt{5})\bigg]_0^\pi = (-\cos \pi + \sqrt{5}) - (-\cos 0 + \sqrt{5})$$
$$= [-(-1) + \sqrt{5}] - (-1 + \sqrt{5}) = 1 + 1 + \sqrt{5} - \sqrt{5} = 2$$

Observe that the $\sqrt{5}$-terms canceled out in the calculation of A. Ordinarily, we pick the "simplest" antiderivative (here, $-\cos x$) for use in the fundamental theorem.

31.2 AVERAGE VALUE OF A FUNCTION

The *average* or *mean* of two numbers a_1 and a_2 is

$$\frac{a_1 + a_2}{2}$$

For n numbers $a_1, a_2, \ldots, a_n$, the average is

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

Now consider the function f defined on an interval $[a, b]$. Since f may assume infinitely many values, we cannot directly use the above definition to talk about the average of all the values of f. However, let us divide the interval $[a, b]$ into n equal subintervals, each of length

$$\Delta x = \frac{b-a}{n}$$

Choose an arbitrary point x_i^* in the ith subinterval. Then the average of the n numbers $f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$ is

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{1}{n}\sum_{i=1}^{n} f(x_i^*)$$

If n is large, this value should be a good estimate of the intuitive idea of the "average value of f on $[a, b]$." But,

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i^*) = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\,\Delta x \qquad \left[\text{since } \frac{1}{n} = \frac{1}{b-a}\,\Delta x\right]$$

As n approaches infinity, the sum on the right approaches $\int_a^b f(x)\,dx$ (by definition of the definite integral), and we are led to:

Definition: The average value of f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

EXAMPLES

(*a*) The average value V of $\sin x$ on $[0, \pi]$ is

$$V = \frac{1}{\pi - 0}\int_0^\pi \sin x\,dx = \frac{1}{\pi}(2) \qquad \text{[by example (}b\text{) above]}$$
$$= \frac{2}{\pi} \approx 0.64$$

(*b*) The average value V of x^3 on $[0, 1]$ is

$$V = \frac{1}{1-0}\int_0^1 x^3\,dx = \int_0^1 x^3\,dx$$

Now $\int x^3\,dx = x^4/4$. Hence, by the fundamental theorem,

$$V = \int_0^1 x^3\,dx = \frac{x^4}{4}\bigg]_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}$$

With the mean value of a function defined in this fashion, we have the following useful

Theorem 31.3 (*Mean-Value Theorem for Integrals*): If a function f is continuous on $[a, b]$, it assumes its mean value in $[a, b]$; that is,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

for some c such that $a \le c \le b$.

For the proof, see Problem 31.4. Note that, by contrast, the average of a finite set of numbers $a_1, a_2, \ldots, a_n$ in general does not coincide with any of the a_i.

31.3 CHANGE OF VARIABLE IN A DEFINITE INTEGRAL

To evaluate a definite integral by the fundamental theorem, an antiderivative $\int f(x)\,dx$ is required. It was seen in Chapter 29 that the substitution of a new variable u may be useful in finding $\int f(x)\,dx$. When the substitution is made in the definite integral too, the limits of integration a and b must be replaced by the corresponding values of u.

EXAMPLE Let us compute

$$\int_0^1 \sqrt{5x+4}\,dx$$

Let $u = 5x + 4$; then $du = 5\,dx$. Consider the limits of integration: when $x = 0$, $u = 4$; when $x = 1$, $u = 9$. Therefore,

$$\int_0^1 \sqrt{5x+4}\,dx = \int_4^9 \sqrt{u}\,\frac{1}{5}\,du = \frac{1}{5}\int_4^9 u^{1/2}\,du = \frac{1}{5}\left(\frac{2}{3}u^{3/2}\right)\bigg]_4^9$$
$$= \frac{2}{15}(9^{3/2} - 4^{3/2}) = \frac{2}{15}[(\sqrt{9})^3 - (\sqrt{4})^3]$$
$$= \frac{2}{15}(3^3 - 2^3) = \frac{2}{15}(27 - 8) = \frac{2}{15}(19) = \frac{38}{15}$$

See Problem 31.6 for a justification of this procedure.

Solved Problems

31.1 Calculate the area A under the parabola $y = x^2 + 2x$ and above the x-axis, between $x = 0$ and $x = 1$.

Since $x^2 + 2x \geq 0$ for $x \geq 0$, we know that the graph of $y = x^2 + 2x$ is on or above the x-axis between $x = 0$ and $x = 1$. Hence, the area A is given by the definite integral

$$\int_0^1 (x^2 + 2x)\, dx$$

Evaluating by the fundamental theorem,

$$A = \int_0^1 (x^2 + 2x)\, dx = \left(\frac{x^3}{3} + x^2\right)\Big]_0^1 = \left(\frac{1^3}{3} + 1^2\right) - \left(\frac{0^3}{3} + 0^2\right) = \frac{1}{3} + 1 = \frac{4}{3}$$

31.2 Compute $\int_a^{a+2\pi} \sin x\, dx$. (Compare the example following Theorem 30.4, where $a = 0$.)

By the fundamental theorem,

$$\int_a^{a+2\pi} \sin x\, dx = -\cos x\Big]_a^{a+2\pi} = 0$$

since the cosine function has period 2π.

31.3 Compute the mean value V of $\sqrt{x}$ on [0, 4]. For what x in [0, 4] does the value occur (as guaranteed by Theorem 31.3)?

$$V = \frac{1}{4-0}\int_0^4 \sqrt{x}\, dx = \frac{1}{4}\int_0^4 x^{1/2}\, dx = \frac{1}{4}\left(\frac{2}{3}x^{3/2}\right)\Big]_0^4$$
$$= \frac{1}{6}(4^{3/2} - 0^{3/2}) = \frac{1}{6}[(\sqrt{4})^3 - 0] = \frac{1}{6}(2^3) = \frac{8}{6} = \frac{4}{3}$$

This average value, $\frac{4}{3}$, is the value of $\sqrt{x}$ when $x = (\frac{4}{3})^2 = \frac{16}{9}$. Note that $0 < \frac{16}{9} < 4$.

31.4 Prove the mean-value theorem for integrals (Theorem 31.3).

Write $$V \equiv \frac{1}{b-a}\int_a^b f(x)\, dx$$

Let m and M be the minimum and maximum values of f on $[a, b]$. (The existence of m and M is guaranteed by Theorem 14.2.) Thus, $m \leq f(x) \leq M$ for all x in $[a, b]$, so that Problem 30.3(c) gives

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a) \qquad \text{or} \qquad m \leq V \leq M$$

But then, by the intermediate-value theorem (Theorem 17.4), the value V is assumed by f somewhere in $[a, b]$.

31.5 Prove Theorem 31.1.

Write $$g(x) \equiv \int_a^x f(t)\, dt$$

Then, $$g(x+h) - g(x) = \int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt$$
$$= \int_a^x f(t)\, dt + \int_x^{x+h} f(t)\, dt - \int_a^x f(t)\, dt \qquad \text{[by Theorem 30.4]}$$
$$= \int_x^{x+h} f(t)\, dt$$

By the mean-value theorem for integrals, the last integral is equal to $hf(x^*)$ for some x^* between x and $x + h$. Hence,

$$\frac{g(x+h) - g(x)}{h} = f(x^*)$$

and
$$D_x\left(\int_a^x f(t)\, dt\right) = D_x(g(x)) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} f(x^*)$$

Now as $h \to 0$, $x + h \to x$, and so $x^* \to x$ (since x^* lies between x and $x + h$). Since f is continuous,

$$\lim_{h \to 0} f(x^*) = f(x)$$

and the proof is complete.

31.6 (*Change of Variables*) Consider $\int_a^b f(x)\, dx$. Let $x = g(u)$, where, as x varies from a to b, u increases or decreases from c to d. [See Fig. 31-1; in effect, we rule out $g'(u) = 0$ in $[c, d]$.] Show that

$$\int_a^b f(x)\, dx = \int_c^d f(g(u))g'(u)\, du$$

[The right-hand side is obtained by substituting $g(u)$ for x, $g'(u)\, du$ for dx, and changing the limits of integration from a and b to c and d.]

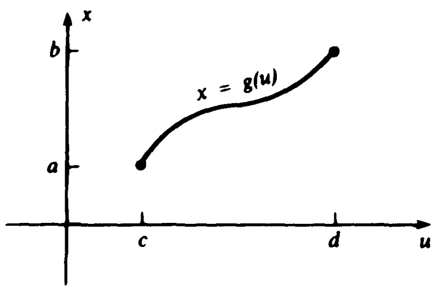


Fig. 31-1

Let
$$F(x) = \int f(x)\, dx \qquad \text{or} \qquad F'(x) = f(x)$$

The chain rule gives

$$D_u(F(g(u))) = F'(g(u))g'(u) = f(g(u))g'(u)$$

Hence,
$$\int f(g(u))g'(u)\, du = F(g(u))$$

By the fundamental theorem,

$$\int_c^d f(g(u))g'(u)\, du = F(g(u))\Big]_c^d = F(g(d)) - F(g(c))$$
$$= F(b) - F(a) = \int_a^b f(x)\, dx$$

31.7 Calculate
$$\int_0^1 \sqrt{x^2 + 1}\, x\, dx.$$

Let us find the antiderivative of $\sqrt{x^2+1}\,x$ by making the substitution $u = x^2 + 1$. Then, $du = 2x\,dx$, and

$$\int \sqrt{x^2+1}\,x\,dx = \int \sqrt{u}\,\frac{1}{2}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\,\frac{u^{3/2}}{\frac{3}{2}}$$
$$= \frac{1}{3}u^{3/2} = \frac{1}{3}(x^2+1)^{3/2} = \frac{1}{3}(\sqrt{x^2+1})^3$$

Hence, by the fundamental theorem,

$$\int_0^1 \sqrt{x^2+1}\,x\,dx = \frac{1}{3}(\sqrt{x^2+1})^3\Big]_0^1 = \frac{1}{3}((\sqrt{1^2+1})^3 - (\sqrt{0^2+1})^3)$$
$$= \frac{1}{3}((\sqrt{2})^3 - (\sqrt{1})^3) = \frac{1}{3}(2\sqrt{2} - 1)$$

ALGEBRA $(\sqrt{2})^3 = (\sqrt{2})^2 \cdot \sqrt{2} = 2\sqrt{2}$ and $(\sqrt{1})^3 = 1^3 = 1$

Alternate Method: Make the same substitution as above, but directly in the definite integral, changing the limits of integration accordingly. When $x = 0$, $u = 0^2 + 1 = 1$; when $x = 1$, $u = 1^2 + 1 = 2$. Thus, the first line of the computation above yields

$$\int_0^1 \sqrt{x^2+1}\,x\,dx = \frac{1}{2}\int_1^2 u^{1/2}\,du = \frac{1}{2}\,\frac{u^{3/2}}{\frac{3}{2}}\Big]_1^2 = \frac{1}{3}u^{3/2}\Big]_1^2$$
$$= \frac{1}{3}((\sqrt{2})^3 - (\sqrt{1})^3) = \frac{1}{3}(2\sqrt{2} - 1)$$

31.8 (*a*) If f is an even function (Section 7.3), show that, for any $a > 0$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

(*b*) If f is an odd function (Section 7.3), show that, for any $a > 0$,

$$\int_{-a}^{a} f(x)\,dx = 0$$

If $u = -x$, then $du = -dx$. Hence, for any integrable function $f(x)$,

$$\int_{-a}^{0} f(x)\,dx = \int_a^0 f(-u)(-du) = -\int_a^0 f(-u)\,du = \int_0^a f(-u)\,du$$

NOTATION Renaming the variable in a definite integral does not affect the value of the integral:

$$\int_a^b g(x)\,dx = \int_a^b g(t)\,dt = \int_a^b g(\theta)\,d\theta = \cdots$$

Thus, changing u to x,

$$\int_{-a}^{0} f(x)\,dx = \int_0^a f(-x)\,dx \qquad (1)$$

and so

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \qquad \text{[by Theorem 30.4]}$$
$$= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \qquad \text{[by (1)]}$$
$$= \int_0^a (f(x) + f(-x))\,dx \qquad \text{[by Theorem 30.3]}$$

(a) For an even function, $f(x) + f(-x) = 2f(x)$, whence,

$$\int_{-a}^{a} f(x)\,dx = \int_0^a 2f(x)\,dx = 2\int_0^a f(x)\,dx$$

(b) For an odd function, $f(x) + f(-x) = 0$, whence,

$$\int_{-a}^{a} f(x)\,dx = \int_0^a 0\,dx = 0\int_0^a dx = 0$$

NOTATION One usually writes

$$\int_a^b dx \quad \text{instead of} \quad \int_a^b 1\,dx$$

31.9 (a) Let $f(x) \geq 0$ on $[a, b]$, and let $[a, b]$ be divided into n equal parts, of length $\Delta x = (b - a)/n$, by means of points $x_1, x_2, \ldots, x_{n-1}$ [see Fig. 31-2(a)]. Show that

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right) \qquad \textit{trapezoidal rule}$$

(b) Use the trapezoidal rule, with $n = 10$, to approximate

$$\int_0^1 x^2\,dx \quad (= 0.333\ldots)$$

(a) The area in the strip over the interval $[x_{i-1}, x_i]$ is approximately the area of trapezoid $ABCD$ in Fig. 31-2(b), which is

$$\frac{\Delta x}{2}[f(x_{i-1}) + f(x_i)]$$

GEOMETRY The area of a trapezoid of height h and bases b_1 and b_2 is

$$\frac{1}{2}h(b_1 + b_2)$$

---

where we understand $x_0 = a$, $x_n = b$. The area under the curve is then approximated by the sum of the trapezoidal areas,

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\{(f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n))\}$$

$$= \frac{\Delta x}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right)$$

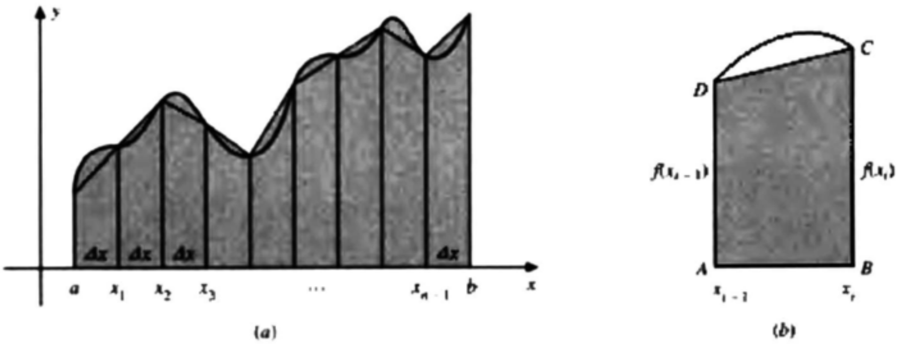


Fig. 31-2

(*b*) By the trapezoidal rule, with $n = 10$, $a = 0$, $b = 1$, $\Delta x = 1/10$, $x_i = i/10$,

$$\int_0^1 x^2\,dx \approx \frac{1}{20}\left(0^2 + 2\sum_{i=1}^{9}\frac{i^2}{100} + 1^2\right) = \frac{1}{20}\left(\frac{2}{100}\sum_{i=1}^{9} i^2 + 1\right)$$

$$= \frac{1}{20}\left(\frac{2}{100}(285) + 1\right) \quad \text{[by arithmetic or Problem 30.12(}a\text{, ii)]}$$

$$= \frac{285}{1000} + \frac{1}{20} = 0.285 + 0.050 = 0.335$$

whereas the exact value is 0.333 ....[1]

Supplementary Problems

31.10 Use the fundamental theorem to compute the following definite integrals:

(*a*) $\int_{-1}^{3}(3x^2 - 2x + 1)\,dx$ (*b*) $\int_0^{\pi/4}\cos x\,dx$ (*c*) $\int_0^{\pi/3}\sec^2 x\,dx$

(*d*) $\int_1^{16} x^{3/2}\,dx$ (*e*) $\int_4^5\left(\frac{2}{\sqrt{x}} - x\right)dx$ (*f*) $\int_0^1\sqrt{x^2 - 6x + 9}\,dx$

31.11 Calculate the areas under the graphs of the following functions, above the x-axis and between the two indicated values a and b of x. [In part (*g*), the area *below* the x-axis is counted negative.]

(*a*) $f(x) = \sin x \quad \left(a = \frac{\pi}{6}, b = \frac{\pi}{3}\right)$ (*b*) $f(x) = x^2 + 4x \quad (a = 0, b = 3)$

(*c*) $f(x) = \frac{1}{\sqrt[3]{x}} \quad (a = 1, b = 8)$ (*d*) $f(x) = \sqrt{4x + 1} \quad (a = 0, b = 2)$

(*e*) $f(x) = x^2 - 3x \quad (a = 3, b = 5)$ (*f*) $f(x) = \sin^2 x \cos x \quad \left(a = 0, b = \frac{\pi}{2}\right)$

(*g*) $f(x) = x^2(x^3 - 2) \quad (a = 1, b = 2)$ (*h*) $f(x) = 4x - x^2 \quad (a = 0, b = 3)$

31.12 Compute the following definite integrals:

(*a*) $\int_0^{\pi/2}\cos x \sin x\,dx$ (*b*) $\int_0^{\pi/4}\tan x \sec^2 x\,dx$ (*c*) $\int_{-1}^{1}\sqrt{3x^2 - 2x + 3}\,(3x - 1)\,dx$

(*d*) $\int_0^{\pi/2}\sqrt{\sin x + 1}\cos x\,dx$ (*e*) $\int_{-1}^{2}\sqrt{x + 2}\,x^2\,dx$ (*f*) $\int_2^5\sqrt{x^3 - 4}\,x^5\,dx$

(*g*) $\int_3^{15}\sqrt[3]{x^2 - 9}\,x^3\,dx$ (*h*) $\int_0^1\frac{x}{(2x^2 + 1)^3}\,dx$ (*i*) $\int_0^8\frac{x}{(x + 1)^{3/2}}\,dx$

(*j*) $\int_{-1}^{2}|x - 1|\,dx$ (*k*) $\int_1^2\frac{x^7 - 2x + 1}{4x^3}\,dx$ (*l*) $\int_{-2}^{0}(x + 2)\sqrt{x + 3}\,dx$

(*m*) $\int_2^5\sqrt{x^3 - 4}\,x^5\,dx$ (*n*) $\int_0^{\pi/8}\sec^2 2x \tan^3 2x\,dx$

[*Hint:* Apply Theorem 30.4 to part (*j*).]

---

[1] When f has a continuous second derivative, it can be shown that the error in approximating $\int_a^b f(x)\,dx$ by the trapezoidal rule is at most $((b - a)/12n^2)M$, where M is the maximum of $|f''(x)|$ on $[a, b]$ and n is the number of subintervals.

31.13 Compute the average value of each of the following functions on the given interval:

(*a*) $f(x) = \sqrt[3]{x}$ on $[0, 1]$ (*b*) $f(x) = \sec^2 x$ on $\left[0, \frac{\pi}{4}\right]$

(*c*) $f(x) = x^2 - 2x - 1$ on $[-2, 3]$ (*d*) $f(x) = \sin x + \cos x$ on $[0, \pi]$

31.14 Verify the mean-value theorem for integrals in the following cases:

(*a*) $f(x) = x + 2$ on $[1, 2]$ (*b*) $f(x) = x^3$ on $[0, 1]$ (*c*) $f(x) = x^2 + 5$ on $[0, 3]$

31.15 Evaluate by the change-of-variable technique:

(*a*) $\int_{1/2}^{3} \sqrt{2x + 3}\, x^2\, dx$ (*b*) $\int_{0}^{\pi/2} \sin^5 x \cos x\, dx$

31.16 Using only geometric reasoning, calculate the average value of $f(x) = \sqrt{2x - x^2}$ on $[0, 2]$. [*Hint:* If $y = f(x)$, then $(x - 1)^2 + y^2 = 1$. Draw the graph.]

31.17 If, in a period of time T, an object moves along the x-axis from x_1 to x_2, calculate its average velocity. [*Hint:* $\int v\, dt = x$.]

31.18 Find:

(*a*) $D_x\left(\int_2^x \sqrt{5 + 7t^2}\, dt\right)$ (*b*) $D_x\left(\int_x^1 \sin^3 t\, dt\right)$ (*c*) $D_x\left(\int_{-x}^{x} \sqrt[3]{t^4 + 1}\, dt\right)$

[*Hint:* In part (*c*), use Problem 31.8(*a*).]

31.19 Evaluate $\int_{-3}^{3} x^2 \sin x\, dx$.

31.20 (*a*) Find $D_x\left(\int_1^{3x^2} \sqrt{t^5 + 1}\, dt\right)$. $\left[\textit{Hint:}\right.$ With $u = 3x^2$, the chain rule yields $D_x\left(\int_1^u \sqrt{t^5 + 1}\, dt\right) = D_u\left(\int_1^u \sqrt{t^5 + 1}\, dt\right) \cdot \frac{du}{dx}$, and Theorem 31.1 applies on the right side.$\left.\right]$

(*b*) Find a formula for $D_x\left(\int_a^{h(x)} f(t)\, dt\right)$.

(*c*) Evaluate $D_x\left(\int_0^{3x} \sqrt{t}\, dt\right)$ and $D_x\left(\int_{5x}^{1} \left(\frac{t}{t^3} + 1\right) dt\right)$.

31.21 Solve for b: $\int_1^b x^{n-1}\, dx = \frac{2}{n}$.

31.22 If $\int_3^5 f(x - k)\, dx = 1$, compute

$$\int_{3-k}^{5-k} f(x)\, dx$$

[*Hint:* Let $x = u - k$.]

31.23 If $f(x) = \begin{cases} \sin x & \text{for } x < 0 \\ 3x^2 & \text{for } x \geq 0 \end{cases}$, find $\int_{-\pi/2}^{1} f(x)\,dx$.

31.24 Given that $2x^2 - 8 = \int_a^x f(t)\,dt$, find: (*a*) a formula for $f(x)$; (*b*) the value of a.

31.25 Define $H(x) \equiv \int_1^x \frac{1}{1+t^2}\,dt$.

(*a*) Find $H(1)$ (*b*) Find $H'(1)$ (*c*) Show that $H(4) - H(2) < \frac{2}{5}$

31.26 If the average value of $f(x) = x^3 + bx - 2$ on $[0, 2]$ is 4, find b.

31.27 Find $\lim\limits_{h \to 0} \left(\frac{1}{h} \int_2^{2+h} \sqrt[3]{x^2 + 2}\,dx \right)$.

31.28 If g is continuous, which of the following integrals are equal?

(*a*) $\int_a^b g(x)\,dx$ (*b*) $\int_{a+1}^{b+1} g(x-1)\,dx$ (*c*) $\int_0^{b-a} g(x+a)\,dx$

31.29 The region above the x-axis and under the curve $y = \sin x$, between $x = 0$ and $x = \pi$, is divided into two parts by the line $x = c$. The area of the left part is $\frac{1}{3}$ the area of the right part. Find c.

31.30 Find the value(s) of k for which

$$\int_0^2 x^k\,dx = \int_0^2 (2-x)^k\,dx$$

31.31 The velocity v of an object moving on the x-axis is $\cos 3t$. It is at the origin at $t = 0$. (*a*) Find a formula for the position x at any time t. (*b*) Find the average value of the position x over the interval $0 \leq t \leq \pi/3$. (*c*) For what values of t in $[0, \pi/3]$ is the object moving to the right? (*d*) What are the maximum and minimum x-coordinates of the object?

31.32 An object moves on a straight line with velocity $v = 3t - 1$, where v is measured in meters per second. How far does the object move in the period of $0 \leq t \leq 2$ seconds? [*Hint:* Apply the fundamental theorem.]

31.33 Evaluate:

(*a*) $\lim\limits_{n \to +\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n\pi}{n} \right)$

(*b*) $\lim\limits_{n \to +\infty} \left\{ \sec^2\left(\frac{\pi}{4n}\right) + \sec^2\left(2\frac{\pi}{4n}\right) + \cdots + \sec^2\left((n-1)\frac{\pi}{4n}\right) + 2 \right\} \frac{\pi}{4n}$

31.34 (*Midpoint Rule*) In a Riemann sum (*30.1*), $\sum_{i=1}^{n} f(x_i^*)\,\Delta_i x$, if we choose x_i^* to be the midpoint of the ith subinterval, then the resulting sum is said to be obtained by the *midpoint rule*. Use the midpoint rule to approximate $\int_0^1 x^2\,dx$, using a division into five equal subintervals, and compare with the exact result obtained by the fundamental theorem.

31.35 (*Simpson's Rule*) If we divide $[a, b]$ into n equal subintervals by means of the points $a = x_0, x_1, x_2, \ldots, x_n = b$, and n is even, then the approximation to $\int_a^b f(x)\,dx$ given by

$$\frac{b-a}{3n}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\right)$$

is said to be obtained by *Simpson's rule*. Aside from the first and last terms, the coefficients consist of alternating 4's and 2's. (The underlying idea is to use parabolas as approximating arcs instead of line segments as in the trapezoidal rule.)[2] Apply Simpson's rule to approximate $\int_0^\pi \sin x\,dx$, with $n = 4$, and compare the result with the exact answer obtained by the fundamental theorem.

31.36 Consider the integral $\int_0^1 x^3\,dx$.

(*a*) Use the trapezoidal rule [Problem 31.9(*a*)], with $n = 10$, to approximate the integral, and compare the result with the exact answer obtained by the fundamental theorem. [*Hint:* You may assume the formula $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$.]

(*b*) GC Approximate the integral by the midpoint rule, with $n = 10$.

(*c*) GC Approximate the integral by Simpson's rule, with $n = 10$.

---

[2] Simpson's rule is usually much more accurate than the midpoint rule or the trapezoidal rule. If f has a continuous fourth derivative on $[a, b]$, then the error in approximating $\int_a^b f(x)\,dx$ by Simpson's rule is at most $((b-a)^5/180n^4)M_4$, where M_4 is the maximum of $|f^{(4)}(x)|$ on $[a, b]$ and n is the number of subintervals.

Chapter 32

Applications of Integration I: Area and Arc Length

32.1 AREA BETWEEN A CURVE AND THE y-AXIS

We have learned how to find the area of a region like that shown in Fig. 32-1. Now let us consider what happens when x and y are interchanged.

EXAMPLES

(*a*) The graph of $x = y^2 + 1$ is a parabola, with its "nose" at (1, 0) and the positive x-axis as its axis of symmetry (see Fig. 32-2). Consider the region $\mathscr{R}$ consisting of all points to the left of this graph, to the right of the y-axis, and between $y = -1$ and $y = 2$. If we apply the reasoning used to calculate the area of a region like that shown in Fig. 32-1, but with x and y interchanged, we must integrate "along the y-axis." Thus, the area of $\mathscr{R}$ is given by the definite integral

$$\int_{-1}^{2} (y^2 + 1)\, dy$$

The fundamental theorem gives

$$\int_{-1}^{2} (y^2 + 1)\, dy = \left(\frac{y^3}{3} + y\right)\Bigg]_{-1}^{2} = \left(\frac{2^3}{3} + 2\right) - \left(\frac{(-1)^3}{3} + (-1)\right)$$
$$= \left(\frac{8}{3} + 2\right) - \left(-\frac{1}{3} - 1\right) = \frac{9}{3} + 3 = 3 + 3 = 6$$

(*b*) Find the area of the region above the line $y = x - 3$ in the first quadrant and below the line $y = 4$ (the shaded region of Fig. 32-3). Thinking of x as a function of y, namely, $x = y + 3$, we can express the area as

$$\int_{0}^{4} (y + 3)\, dy = \left(\frac{y^2}{2} + 3y\right)\Bigg]_{0}^{4}$$
$$= \left(\frac{4^2}{2} + 3(4)\right) - \left(\frac{0^2}{2} + 3(0)\right) = \frac{16}{2} + 12 = 20$$

Check this result by computing the area of trapezoid $OBCD$ by the geometrical formula given in Problem 31.9.

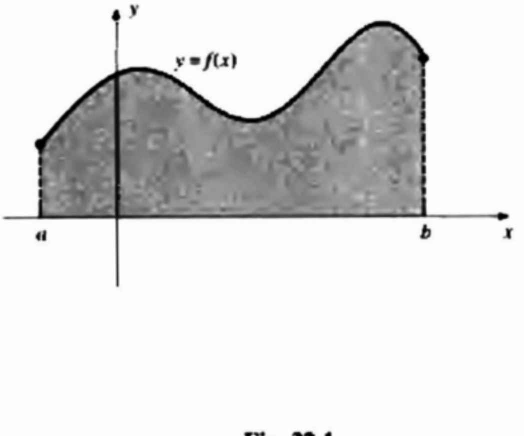


Fig. 32-1

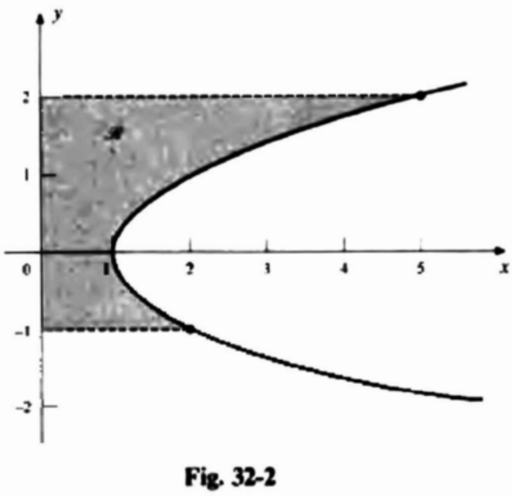


Fig. 32-2

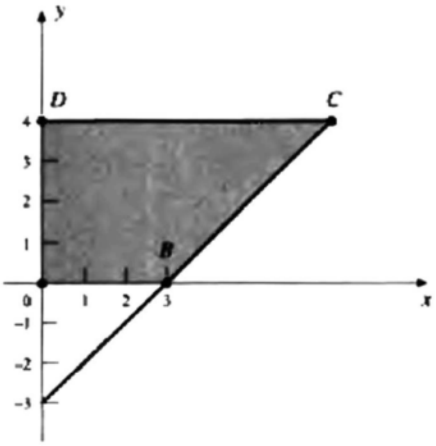


Fig. 32-3

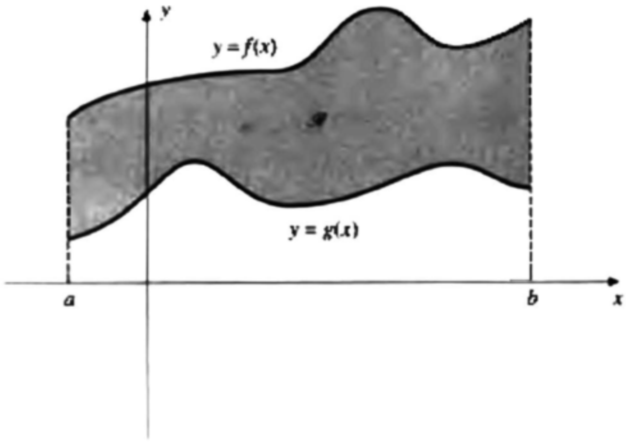


Fig. 32-4

32.2 AREA BETWEEN TWO CURVES

Assume that $0 \le g(x) \le f(x)$ for x in $[a, b]$. Let us find the area A of the region $\mathscr{R}$ consisting of all points between the graphs of $y = g(x)$ and $y = f(x)$, and between $x = a$ and $x = b$. As may be seen from Fig. 32-4, A is the area under the upper curve $y = f(x)$ minus the area under the lower curve $y = g(x)$; that is,

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx \tag{32.1}$$

EXAMPLE Figure 32-5 shows the region $\mathscr{R}$ under the line $y = \frac{1}{2}x + 2$, above the parabola $y = x^2$, and between the y-axis and $x = 1$. Its area is

$$\begin{aligned}\int_0^1 \left(\left(\frac{1}{2}x + 2\right) - x^2\right) dx &= \left(\frac{x^2}{4} + 2x - \frac{x^3}{3}\right)\Bigg]_0^1 \\ &= \left(\frac{1^2}{4} + 2(1) - \frac{1^3}{3}\right) - \left(\frac{0^2}{4} + 2(0) - \frac{0^3}{3}\right) \\ &= \frac{1}{4} + 2 - \frac{1}{3} = \frac{9}{4} - \frac{1}{3} = \frac{27 - 4}{12} = \frac{23}{12}\end{aligned}$$

Formula (32.1) is still valid when the condition on the two functions is relaxed to

$$g(x) \le f(x)$$

that is, when the curves are allowed to lie partly or totally below the x-axis, as in Fig. 32-6. See Problem 32.3 for a proof of this statement.

Another application of (32.1) is in finding the area of a region enclosed by two curves.

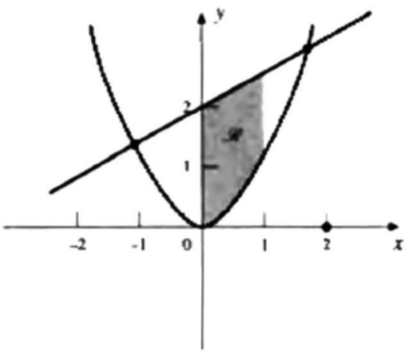


Fig. 32-5

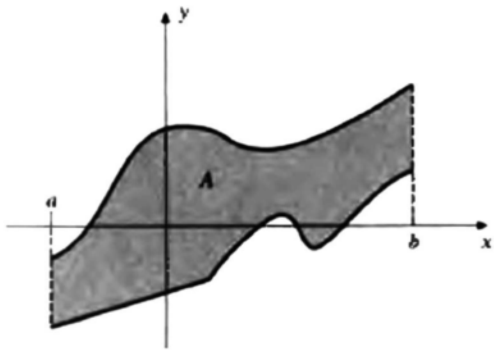


Fig. 32-6

EXAMPLE Find the area of the region bounded by the parabola $y = x^2$ and the line $y = x + 2$ (see Fig. 32-7).

The limits of integration a and b in *(32.1)* must be the x-coordinates of the intersection points P and Q, respectively. These are found by solving simultaneously the equations of the curves $y = x^2$ and $y = x + 2$. Thus,

$$x^2 = x + 2 \quad \text{or} \quad x^2 - x - 2 = 0 \quad \text{or} \quad (x-2)(x+1) = 0$$

whence, $x = a = -1$ and $x = b = 2$. Thus,

$$\begin{aligned} A &= \int_{-1}^{2} [(x+2) - x^2]\, dx = \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\Bigg]_{-1}^{2} \\ &= \left(\frac{2^2}{2} + 2(2) - \frac{2^3}{3}\right) - \left(\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right) \\ &= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{3}{2} + 6 - \frac{9}{3} \\ &= \frac{3}{2} + 6 - 3 = \frac{3}{2} + 3 = \frac{3+6}{2} = \frac{9}{2} \end{aligned}$$

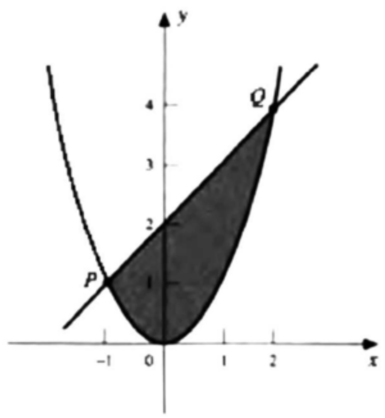


Fig. 32-7

32.3 ARC LENGTH

Consider a differentiable (not just continuous) function f on a closed interval $[a, b]$. The graph of f is a curve running from $(a, f(a))$ to $(b, f(b))$. We shall find a formula for the length L of this curve.

Divide $[a, b]$ into n equal parts, each of length Δx. To each x_i in this subdivision corresponds the point $P_i(x_i, f(x_i))$ on the curve (see Fig. 32-8). For large n, the sum $\overline{P_0P_1} + \overline{P_1P_2} + \cdots + \overline{P_{n-1}P_n} \equiv \sum_{i=1}^{n} \overline{P_{i-1}P_i}$ of the lengths of the line segments $P_{i-1}P_i$ is an approximation to the length of the curve. Now, by the distance formula *(2.1)*,

$$\overline{P_{i-1}P_i} = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

But $x_i - x_{i-1} = \Delta x$; also, by the mean-value theorem (Theorem 17.2),

$$f(x_i) - f(x_{i-1}) = (x_i - x_{i-1})f'(x_i^*) = (\Delta x)f'(x_i^*)$$

for some x_i^* in (x_{i-1}, x_i). Hence,

$$\begin{aligned} \overline{P_{i-1}P_i} &= \sqrt{(\Delta x)^2 + (\Delta x)^2(f'(x_i^*))^2} = \sqrt{\{1 + (f'(x_i^*))^2\}(\Delta x)^2} \\ &= \sqrt{1 + (f'(x_i^*))^2}\,\sqrt{(\Delta x)^2} = \sqrt{1 + (f'(x_i^*))^2}\,\Delta x \end{aligned}$$

and

$$\sum_{i=1}^{n} \overline{P_{i-1}P_i} = \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2}\,\Delta x$$

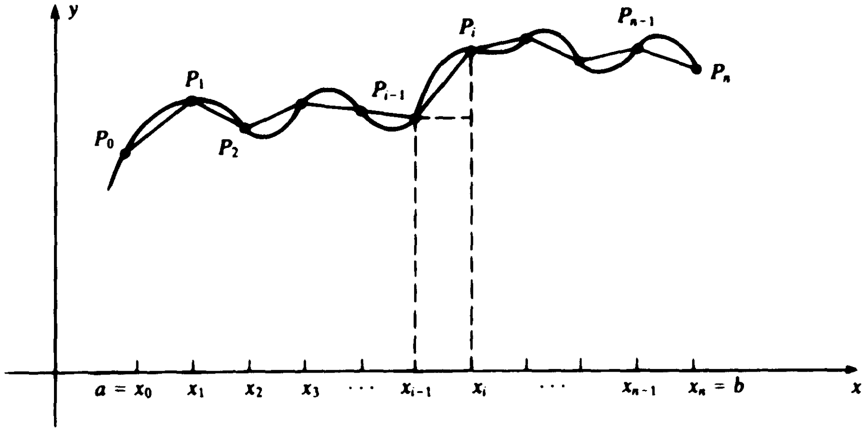


Fig. 32-8

The right-hand sum approximates the definite integral

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + (y')^2}\, dx$$

Therefore, letting $n \to \infty$, we obtain

$$L = \int_a^b \sqrt{1 + (y')^2}\, dx \qquad \textit{arc-length formula} \tag{32.2}$$

EXAMPLE Find the arc length of the graph of $y = x^{3/2}$ from (1, 1) to (4, 8).
We have

$$y' = \frac{3}{2} x^{1/2} \quad \text{and} \quad (y')^2 = \frac{9}{4} x$$

Hence, by the arc-length formula,

$$L = \int_1^4 \sqrt{1 + \frac{9}{4} x}\, dx$$

Let $$u = 1 + \frac{9}{4} x \qquad du = \frac{9}{4} dx \qquad dx = \frac{4}{9} du$$

When $x = 1$, $u = \frac{13}{4}$; when $x = 4$, $u = 10$. Thus,

$$L = \int_{13/4}^{10} \sqrt{u}\, \frac{4}{9}\, du = \frac{4}{9} \int_{13/4}^{10} u^{1/2}\, du = \frac{4}{9} \left(\frac{2}{3} u^{3/2}\right)\Bigg]_{13/4}^{10}$$
$$= \frac{8}{27}\left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right) = \frac{8}{27}\left((\sqrt{10})^3 - \left(\frac{\sqrt{13}}{2}\right)^3\right)$$
$$= \frac{8}{27}\left(10\sqrt{10} - \frac{13\sqrt{13}}{8}\right) = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$$

where, in the next-to-last step, we have used the identity $(\sqrt{c})^3 = (\sqrt{c})^2(\sqrt{c}) = c\sqrt{c}$.

Solved Problems

32.1 Find the area A of the region to the left of the parabola $x = -y^2 + 4$ and to the right of the y-axis.

The region is shown in Fig. 32-9. Notice that the parabola cuts the y-axis at $y = \pm 2$. (Set $x = 0$ in the equation of the curve.) Hence,

$$A = \int_{-2}^{2} (-y^2 + 4)\, dy = 2\int_0^2 (-y^2 + 4)\, dy \qquad \text{[by Problem 31.8(}a\text{)]}$$
$$= 2\left(-\frac{y^3}{3} + 4y\right)\Big]_0^2 = 2\left\{\left(-\frac{2^3}{3} + 4(2)\right) - \left(\frac{0^3}{3} + 4(0)\right)\right\}$$
$$= 2\left(-\frac{8}{3} + 8\right) = 2\left(-\frac{8}{3} + \frac{24}{3}\right) = 2\left(\frac{16}{3}\right) = \frac{32}{3}$$

32.2 Find the area of the region between the curves $y = x^3$ and $y = 2x$, between $x = 0$ and $x = 1$ (see Fig. 32-10).

For $0 \le x \le 1$,

$$2x - x^3 = x(2 - x^2) = x(\sqrt{2} + x)(\sqrt{2} - x) \ge 0$$

since all three factors are nonnegative. Thus, $y = x^3$ is the lower curve, and $y = 2x$ is the upper curve. By (32.1),

$$A = \int_0^1 (2x - x^3)\, dx = \left(x^2 - \frac{x^4}{4}\right)\Big]_0^1 = \left(1^2 - \frac{1^4}{4}\right) - \left(0^2 - \frac{0^4}{4}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

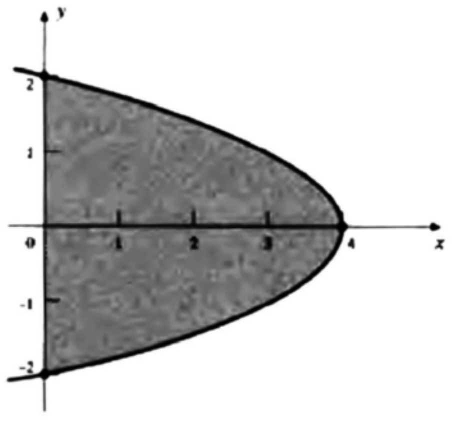


Fig. 32-9

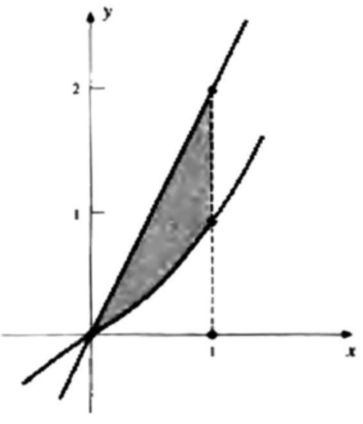
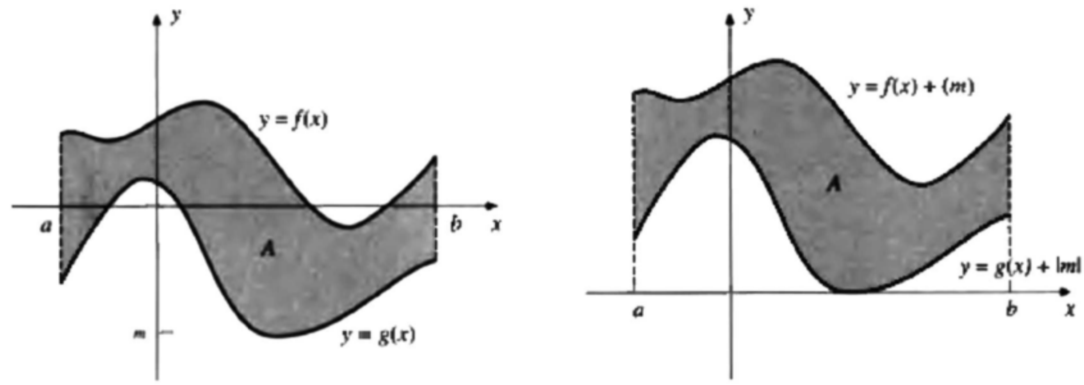


Fig. 32-10

32.3 Prove that the formula for the area $A = \int_a^b (f(x) - g(x))\, dx$ holds whenever $g(x) \le f(x)$ on $[a, b]$.

Let $m < 0$ be the absolute minimum of g on $[a, b]$ [see Fig. 32-11(*a*)]. (If $m \ge 0$, both curves lie above or on the x-axis, and this case is already known.) "Raise" both curves by $|m|$ units; the new graphs, shown in Fig. 32-11(*b*), are on or above the x-axis and include the same area A as the original graphs. Thus, by (32.1),

$$A = \int_a^b \{(f(x) + |m|) - (g(x) + |m|)\}\, dx$$
$$= \int_a^b (f(x) - g(x) + 0)\, dx = \int_a^b (f(x) - g(x))\, dx$$



(a) (b)

Fig. 32-11

32.4 Find the area A between the parabolas $y = x^2 - 1$ and $y = -(x^2 - 1)$.

From the symmetry of Fig. 32-12 it is clear that A will be equal to four times the area of the shaded region,

$$A = 4\int_0^1 -(x^2-1)\,dx = 4\int_0^1 (1-x^2)\,dx = 4\left(x - \frac{x^3}{3}\right)\Big]_0^1$$

$$= 4\left(\left(1-\frac{1^3}{3}\right) - \left(0 - \frac{0^3}{3}\right)\right) = 4\left(\frac{2}{3}\right) = \frac{8}{3}$$

32.5 Find the area between the parabola $x = y^2$ and the line $y = 3x - 2$ (see Fig. 32-13).

Find the intersection points. $x = y^2$ and $y = 3x - 2$ imply

$$y = 3y^2 - 2$$
$$3y^2 - y - 2 = 0$$
$$(3y+2)(y-1) = 0$$
$$3y + 2 = 0 \quad \text{or} \quad y - 1 = 0$$
$$y = -\frac{2}{3} \quad \text{or} \quad y = 1$$

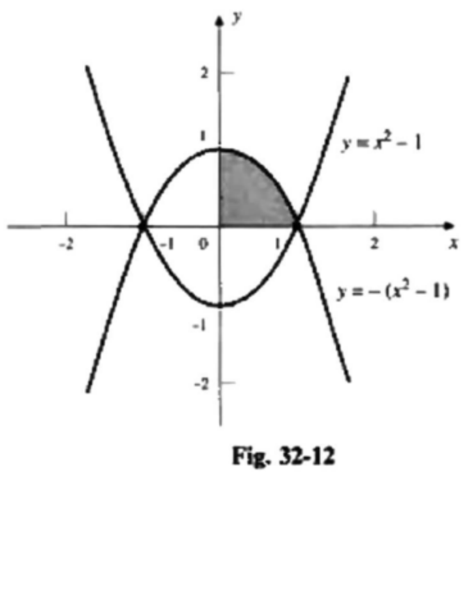


Fig. 32-12

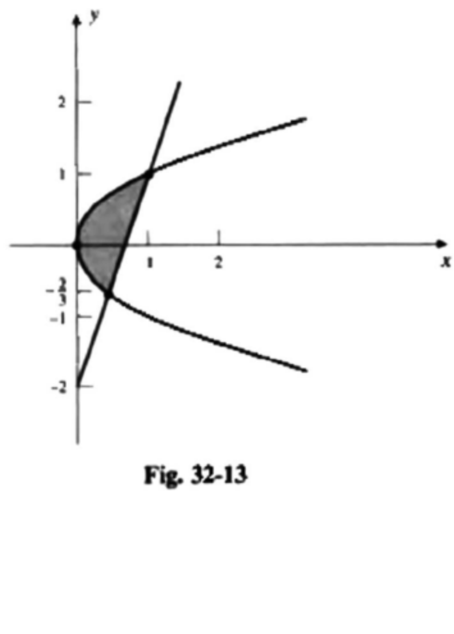


Fig. 32-13

Notice that we cannot find the area by integrating "along the x-axis" (unless we break the region into two parts). Integration along the y-axis is called for (which requires only the ordinates of the intersection points),

$$A = \int_{-2/3}^{1} \left(\frac{y+2}{3} - y^2\right) dy = \int_{-2/3}^{1} \left(\frac{y}{3} + \frac{2}{3} - y^2\right) dy$$

Here the "upper" curve is the line $y = 3x - 2$. We had to solve this equation for x in terms of y, obtaining $x = (y + 2)/3$. Evaluating by the fundamental theorem,

$$\begin{aligned} A &= \left(\frac{y^2}{6} + \frac{2}{3}y - \frac{y^3}{3}\right)\Big]_{-2/3}^{1} = \left(\frac{1}{6} + \frac{2}{3} - \frac{1}{3}\right) - \left(\frac{1}{6}\left(\frac{4}{9}\right) + \frac{2}{3}\left(-\frac{2}{3}\right) - \frac{1}{3}\left(-\frac{2}{3}\right)^3\right) \\ &= \left(\frac{1}{6} + \frac{1}{3}\right) - \left(\frac{2}{27} - \frac{4}{9} + \frac{8}{81}\right) = \frac{1}{2} - \left(\frac{6}{81} - \frac{36}{81} + \frac{8}{81}\right) = \frac{1}{2} - \left(\frac{-22}{81}\right) \\ &= \frac{1}{2} + \frac{22}{81} = \frac{81 + 44}{162} = \frac{125}{162} \end{aligned}$$

32.6 Find the length of the curve $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ from $x = 1$ to $x = 2$.

$$y = \frac{x^3}{6} + \frac{1}{2}x^{-1} \qquad y' = \frac{x^2}{2} - \frac{1}{2}x^{-2} = \frac{x^2}{2} - \frac{1}{2x^2}$$

Then,

$$\begin{aligned} (y')^2 &= \frac{x^4}{4} - \frac{1}{2} + \frac{1}{x^4} \\ 1 + (y')^2 &= \frac{x^4}{4} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2 \\ \sqrt{1 + (y')^2} &= \frac{x^2}{2} + \frac{1}{2x^2} = \frac{x^2}{2} + \frac{1}{2}x^{-2} \end{aligned}$$

Hence, the arc-length formula gives

$$\begin{aligned} L &= \frac{1}{2}\int_1^2 (x^2 + x^{-2})\,dx = \frac{1}{2}\left(\frac{x^3}{3} - x^{-1}\right)\Big]_1^2 \\ &= \frac{1}{2}\left(\left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right)\right) = \frac{1}{2}\left(\frac{7}{3} + \frac{1}{2}\right) = \frac{1}{2}\left(\frac{17}{6}\right) = \frac{17}{12} \end{aligned}$$

Supplementary Problems

32.7 Sketch and find the area of: (*a*) the region to the left of the parabola $x = 2y^2$, to the right of the y-axis, and between $y = 1$ and $y = 3$; (*b*) the region above the line $y = 3x - 2$, in the first quadrant, and below the line $y = 4$; (*c*) the region between the curve $y = x^3$ and the lines $y = -x$ and $y = 1$.

32.8 Sketch the following regions and find their areas:

(*a*) The region between the curves $y = x^2$ and $y = x^3$.

(*b*) The region between the parabola $y = 4x^2$ and the line $y = 6x - 2$.

(*c*) The region between the curves $y = \sqrt{x}$, $y = 1$, and $x = 4$.

(*d*) The region under the curve $\sqrt{x} + \sqrt{y} = 1$ and in the first quadrant.

(*e*) The region between the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/4$.

(*f*) The region between the parabola $x = -y^2$ and the line $y = x + 6$.

(*g*) The region between the parabola $y = x^2 - x - 6$ and the line $y = -4$.

(*h*) The region between the curves $y = \sqrt{x}$ and $y = x^3$.

(*i*) The region in the first quadrant between the curves $4y + 3x = 7$ and $y = x^{-2}$.

(*j*) The region bounded by the parabolas $y = x^2$ and $y = -x^2 + 6x$.

(*k*) The region bounded by the parabola $x = y^2 + 2$ and the line $y = x - 8$.

(*l*) The region bounded by the parabolas $y = x^2 - x$ and $y = x - x^2$.

(*m*) The region in the first quadrant bounded by the curves $y = x^2$ and $y = x^4$.

(*n*) The region between the curve $y = x^3$ and the lines $y = -x$ and $y = 1$.

32.9 Find the lengths of the following curves:

(*a*) $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$ from $x = 1$ to $x = 2$.

(*b*) $y = 3x - 2$ from $x = 0$ to $x = 1$.

(*c*) $y = x^{2/3}$ from $x = 1$ to $x = 8$.

(*d*) $x^{2/3} + y^{2/3} = 4$ from $x = 1$ to $x = 8$.

(*e*) $y = \dfrac{x^5}{15} + \dfrac{1}{4x^3}$ from $x = 1$ to $x = 2$.

(*f*) $y = \dfrac{1}{3}\sqrt{x}(3 - x)$ from $x = 0$ to $x = 3$.

(*g*) $24xy = x^4 + 48$ from $x = 2$ to $x = 4$.

(*h*) $y = \dfrac{2}{3}(1 + x^2)^{3/2}$ from $x = 0$ to $x = 3$.

32.10 Use Simpson's rule with $n = 10$ to approximate the arc length of the curve $y = f(x)$ on the given interval.

(*a*) $y = x^2$ on $[0, 1]$ (*b*) $y = \sin x$ on $[0, \pi]$ (*c*) $y = x^3$ on $[0, 5]$

Chapter 33

Applications of Integration II: Volume

The volumes of certain kinds of solids can be calculated by means of definite integrals.

33.1 SOLIDS OF REVOLUTION

Disk and Ring Methods

Let f be a continuous function such that $f(x) \geq 0$ for $a \leq x \leq b$. Consider the region $\mathscr{R}$ under the graph of $y = f(x)$, above the x-axis, and between $x = a$ and $x = b$ (see Fig. 33-1). If $\mathscr{R}$ is revolved about the x-axis, the resulting solid is called a *solid of revolution*. The generating regions $\mathscr{R}$ for some familiar solids of revolution are shown in Fig. 33-2.

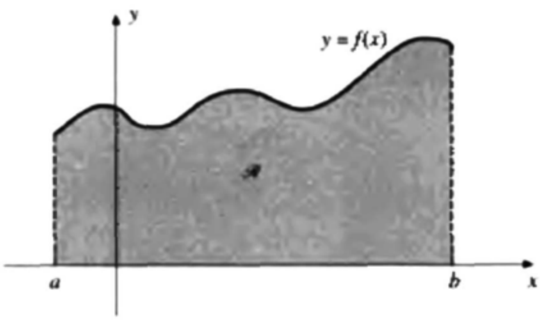


Fig. 33-1

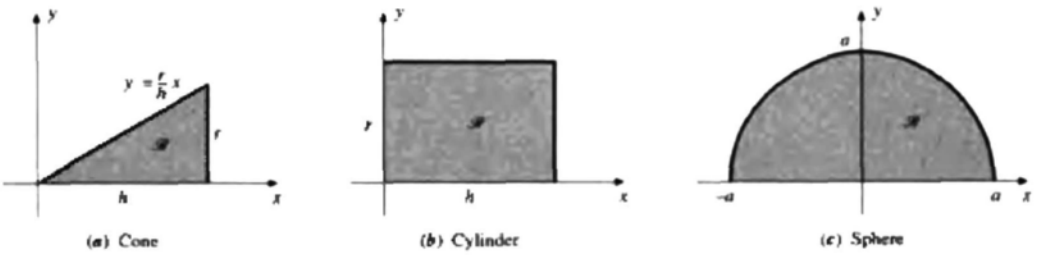


Fig. 33-2

Theorem 33.1: The volume V of the solid of revolution obtained by revolving the region of Fig. 33-1 about the x-axis is given by

$$V = \pi \int_a^b (f(x))^2\,dx = \pi \int_a^b y^2\,dx \qquad \textit{disk formula}$$

An argument for the disk formula is sketched in Problem 33.4.

If we interchange the roles of x and y and revolve the area "under" the graph of $x = g(y)$ about the y-axis, then the same reasoning leads to the disk formula

$$V = \pi \int_c^d (g(y))^2\,dy = \pi \int_c^d x^2\,dy$$

EXAMPLE Applying the disk formula to Fig. 33-2(*a*), we obtain

$$V = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2}{h^2} x^2\, dx$$
$$= \frac{\pi r^2}{h^2}\frac{x^3}{3}\Big]_0^h = \frac{\pi r^2}{h^2}\left(\frac{h^3}{3} - 0\right) = \frac{\pi r^2 h}{3}$$

which is the standard formula for the volume of a cone with height h and radius of base r.

Now let f and g be two functions such that $0 \le g(x) \le f(x)$ for $a \le x \le b$, and revolve the region $\mathscr{R}$ between the curves $y = f(x)$ and $y = g(x)$ about the x-axis (see Fig. 33-3). The resulting solid of revolution has a volume V which is the difference between the volume of the solid of revolution generated by the region under $y = f(x)$ and the volume of the solid of revolution generated by the region under $y = g(x)$. Hence, by Theorem 33.1,

$$V = \pi \int_a^b \{(f(x))^2 - (g(x))^2\}\, dx \qquad \textit{washer formula}^1$$

EXAMPLE Consider the region $\mathscr{R}$ bounded by the curves $y = \sqrt{x}$ and $y = x$ (see Fig. 33-4). The curves obviously intersect in the points (0, 0) and (1, 1). The bowl-shaped solid of revolution generated by revolving $\mathscr{R}$ about the x-axis has volume

$$V = \pi \int_0^1 ((\sqrt{x})^2 - x^2)\, dx = \pi \int_0^1 (x - x^2)\, dx = \pi\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big]_0^1$$
$$= \pi\left(\left(\frac{1}{2} - \frac{1}{3}\right) - 0\right) = \frac{\pi}{6}$$

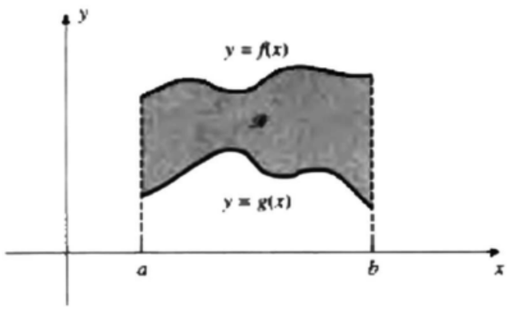


Fig. 33-3

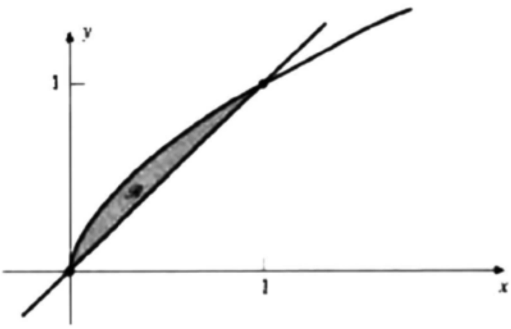


Fig. 33-4

Cylindrical Shell Method

Let f be a continuous function such that $f(x) \ge 0$ for $a \le x \le b$, where $a \ge 0$. As usual, let $\mathscr{R}$ be the region under the curve $y = f(x)$, above the x-axis, and between $x = a$ and $x = b$ (see Fig. 33-5). Now, however, revolve $\mathscr{R}$ about the y-axis. The resulting solid of revolution has volume

$$V = 2\pi \int_a^b x f(x)\, dx = 2\pi \int_a^b xy\, dx \qquad \textit{cylindrical shell formula}$$

For the basic idea behind this formula and its name, see Problem 33.5.

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[1] So termed because the cross section obtained by revolving a vertical segment has the shape of a plumber's washer.

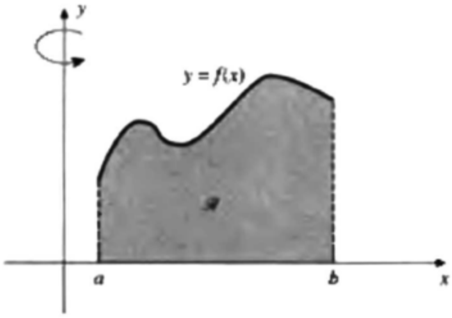

Fig. 33-5

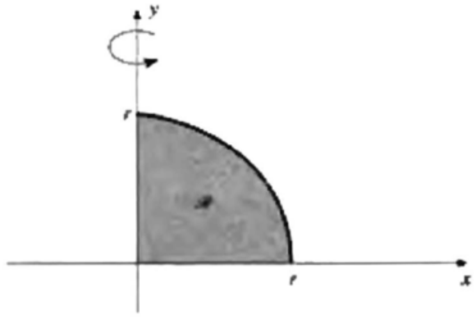

Fig. 33-6

EXAMPLE Consider the function $f(x) = \sqrt{r^2 - x^2}$ for $0 \le x \le r$. The graph of f is the part of the circle $x^2 + y^2 = r^2$ that lies in the first quadrant. Revolution about the y-axis of the region $\mathscr{R}$ under the graph of f (see Fig. 33-6) produces a solid hemisphere of radius r. By the cylindrical shell formula,

$$V = 2\pi \int_0^r x\sqrt{r^2 - x^2}\, dx$$

To evaluate V substitute $u = r^2 - x^2$. Then $du = -2x\, dx$, and the limits of integration $x = 0$ and $x = r$ become $u = r^2$ and $u = 0$, respectively,

$$V = 2\pi \int_{r^2}^0 u^{1/2}\left(-\frac{1}{2}\, du\right) = -\pi \int_{r^2}^0 u^{1/2}\, du = \pi \int_0^{r^2} u^{1/2}\, du$$
$$= \frac{2}{3}\pi u^{3/2}\Big]_0^{r^2} = \frac{2}{3}\pi(r^2)^{3/2} = \frac{2}{3}\pi r^3$$

(This result is more easily obtained by the disk formula $V = \pi \int_0^r x^2\, dy$. Try it.)

33.2 VOLUME BASED ON CROSS SECTIONS

Assume that a solid (not necessarily a solid of revolution) lies entirely between the plane perpendicular to the x-axis at $x = a$ and the plane perpendicular to the x-axis at $x = b$. For $a \le x \le b$, let the plane perpendicular to the x-axis at that value of x intersect the solid in a region of area $A(x)$, as indicated in Fig. 33-7. Then the volume V of the solid is given by

$$V = \int_a^b A(x)\, dx \qquad \textit{cross-section formula}$$

For a derivation, see Problem 33.6.

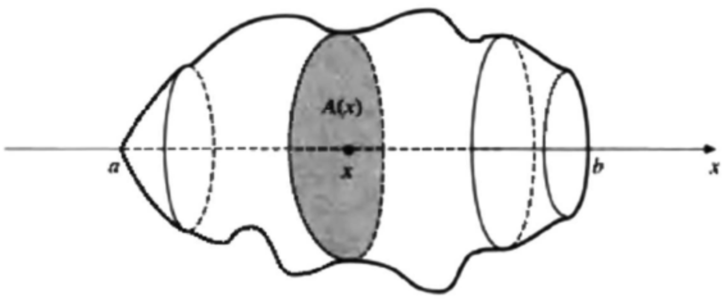

Fig. 33-7

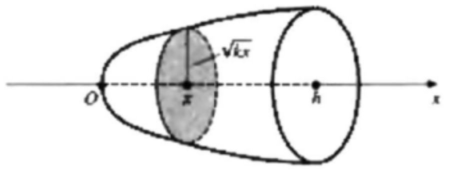

Fig. 33-8

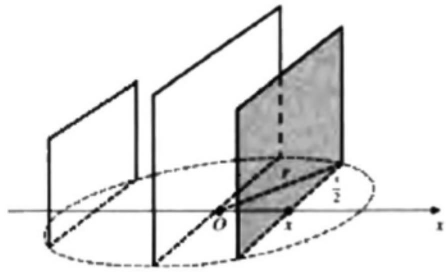

Fig. 33-9

EXAMPLES

(*a*) Assume that half of a salami of length h is such that a cross section perpendicular to the axis of the salami at a distance x from the end O is a circle of radius $\sqrt{kx}$ (see Fig. 33-8). Thus,

$$A(x) = \pi(\sqrt{kx})^2 = \pi kx$$

and the cross-section formula gives

$$V = \int_0^h \pi kx\, dx = \pi k \int_0^h x\, dx = \pi k \left.\frac{x^2}{2}\right]_0^h = \frac{\pi kh^2}{2}$$

Note that for this solid of revolution the disk formula would give the same expression for V.

(*b*) Assume that a solid has a base which is a circle of radius r. Assume that there is a diameter D such that all plane sections of the solid perpendicular to diameter D are squares (see Fig. 33-9). Find the volume.

Let the origin be the center of the circle and let the x-axis be the special diameter D. For a given value of x, with $-r \le x \le r$, the side $s(x)$ of the square cross section is obtained by applying the Pythagorean theorem to the right triangle with sides x, $s/2$, and r (see Fig. 33-9),

$$x^2 + \left(\frac{s}{2}\right)^2 = r^2$$

$$x^2 + \frac{s^2}{4} = r^2$$

$$s^2 = 4(r^2 - x^2) = A(x)$$

Then, by the cross-section formula,

$$V = \int_{-r}^{r} 4(r^2 - x^2)\, dx$$

$$= 2\int_0^r 4(r^2 - x^2)\, dx \qquad [\text{since } 4(r^2 - x^2) \text{ is an even function}]$$

$$= 8\left(r^2x - \frac{x^3}{3}\right)\Bigg]_0^r = 8\left\{\left(r^2(r) - \frac{r^3}{3}\right) - (0 - 0)\right\} = 8\left(\frac{2}{3}r^3\right) = \frac{16}{3}r^3$$

Solved Problems

33.1 Find the volume of the solid generated by revolving the given region about the given axis.

(*a*) The region under the parabola $y = x^2$, above the x-axis, between $x = 0$ and $x = 1$; about the x-axis.

(*b*) The same region as in part (*a*), but about the y-axis. The region is shown in Fig. 33-10.

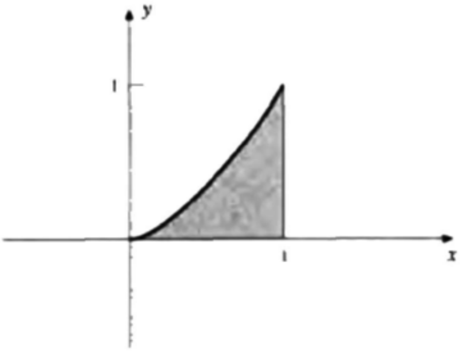

Fig. 33-10

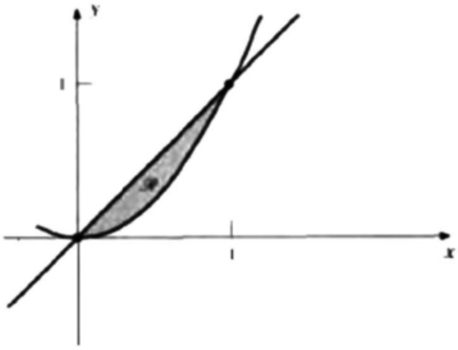

Fig. 33-11

(*a*) Use the disk formula,

$$V = \pi \int_0^1 (x^2)^2\,dx = \pi \int_0^1 x^4\,dx = \pi\left(\frac{x^5}{5}\right)\Big]_0^1 = \pi\left(\frac{1}{5}\right) = \frac{\pi}{5}$$

(*b*) Use the cylindrical shell formula,

$$V = 2\pi \int_0^1 x(x^2)\,dx = 2\pi \int_0^1 x^3\,dx = 2\pi\left(\frac{x^4}{4}\right)\Big]_0^1 = 2\pi\left(\frac{1}{4}\right) = \frac{\pi}{2}$$

33.2 Let $\mathscr{R}$ be the region between $y = x^2$ and $y = x$ (see Fig. 33-11). Find the volume of the solid obtained by revolving $\mathscr{R}$ around: (*a*) the x-axis; (*b*) the y-axis.

The curves intersect at (0, 0) and (1, 1).

(*a*) By the washer formula,

$$V = \pi \int_0^1 (x^2 - (x^2)^2)\,dx = \pi \int_0^1 (x^2 - x^4)\,dx = \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$$

(*b*) (*Method 1*) Use the washer formula along the y-axis,

$$V = \pi \int_0^1 ((\sqrt{y})^2 - y^2)\,dy = \pi \int_0^1 (y - y^2)\,dy = \pi\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{\pi}{6}$$

(*Method 2*) We can integrate along the x-axis and use the difference of two cylindrical shell formulas,

$$V = 2\pi\left(\int_0^1 x(x)\,dx - \int_0^1 x(x^2)\,dx\right) = 2\pi \int_0^1 (x^2 - x^3)\,dx$$
$$= 2\pi\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big]_0^1 = 2\pi\left(\left(\frac{1}{3} - \frac{1}{4}\right) - (0 - 0)\right) = 2\pi\left(\frac{1}{12}\right) = \frac{\pi}{6}$$

The formula used in method 2 can be formulated as follows:

$$V = 2\pi \int_a^b x(g(x) - f(x))\,dx \qquad \textit{difference of cylindrical shells}$$

where V is the volume of the solid obtained by revolving about the y-axis the region bounded above by $y = g(x)$, below by $y = f(x)$, and lying between $x = a$ and $x = b$, with $0 \le a < b$.

33.3 Find the volume of the solid whose base is a circle of radius r and such that every cross section perpendicular to a particular fixed diameter D is an equilateral triangle.

Let the center of the circular base be the origin, and let the x-axis be the diameter D. The area of the cross section at x is $A(x) = hs/2$ (see Fig. 33-12). Now, in the horizontal right triangle,

$$x^2 + \left(\frac{s}{2}\right)^2 = r^2 \qquad \text{or} \qquad \frac{s}{2} = \sqrt{r^2 - x^2}$$

and in the vertical right triangle,

$$h^2 + \left(\frac{s}{2}\right)^2 = s^2$$

$$h^2 + \frac{s^2}{4} = s^2$$

$$h^2 = 3\,\frac{s^2}{4}$$

$$h = \sqrt{3}\,\frac{s}{2} = \sqrt{3}\sqrt{r^2 - x^2}$$

Hence, $A(x) = \sqrt{3}(r^2 - x^2)$—an even function—and the cross-section formula gives

$$V = \sqrt{3}\int_{-r}^{r}(r^2 - x^2)\,dx = 2\sqrt{3}\int_{0}^{r}(r^2 - x^2)\,dx = 2\sqrt{3}\left(r^2x - \frac{x^3}{3}\right)\Bigg]_0^r$$

$$= 2\sqrt{3}\left(r^3 - \frac{r^3}{3}\right) = 2\sqrt{3}\left(\frac{2}{3}r^3\right) = \frac{4\sqrt{3}}{3}r^3$$

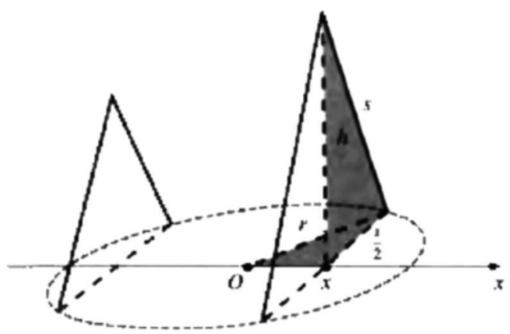

Fig. 33-12

33.4 Establish the disk formula $V = \pi\int_2^b (f(x))^2\,dx$.

We assume as valid the expression $\pi r^2 h$ for the volume of a cylinder of radius r and height h. Divide the interval $[a, b]$ into n equal subintervals, each of length $\Delta x = (b - a)/n$ (see Fig. 33-13). Consider the volume V_i obtained by revolving the region $\mathscr{R}_i$ above the ith subinterval about the x-axis. If m_i and M_i denote the absolute minimum and the absolute maximum of f on the i^{th} subinterval, it is plain that V_i must lie between the volume of a cylinder of radius m_i and height Δx and the volume of a cylinder of radius M_i and height Δx,

$$\pi m_i^2\,\Delta x \le V_i \le \pi M_i^2\,\Delta x \qquad \text{or} \qquad m_i^2 \le \frac{V_i}{\pi\,\Delta x} \le M_i^2$$

The intermediate-value theorem for the continuous function $(f(x))^2$ guarantees the existence of some point x_i^* in the i^{th} subinterval such that

$$\frac{V_i}{\pi\,\Delta x} = (f(x_i^*))^2 \qquad \text{or} \qquad V_i = \pi(f(x_i^*))^2\,\Delta x$$

Hence,
$$V = \sum_{i=1}^{n} V_i = \pi \sum_{i=1}^{n} (f(x_i^*))^2 \, \Delta x$$

Since this relation holds (for suitable numbers x_i^*) for arbitrary n, it must hold in the limit as $n \to \infty$,

$$V = \pi \lim_{n \to \infty} \left(\sum_{i=1}^{n} (f(x_i^*))^2 \, \Delta x \right) = \pi \int_a^b (f(x))^2 \, dx$$

which is the disk formula. The name derives from the use of cylindrical *disks* (of thickness Δx) to approximate the V_i.

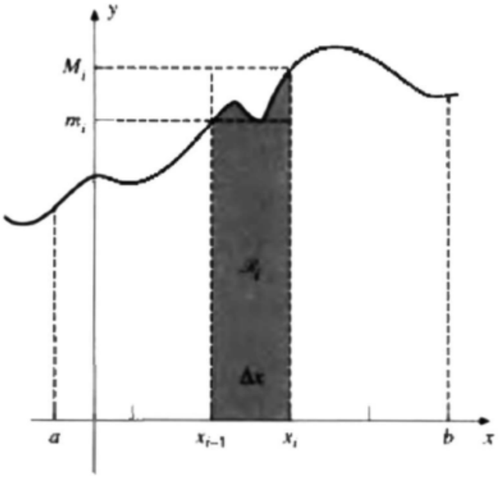

Fig. 33-13

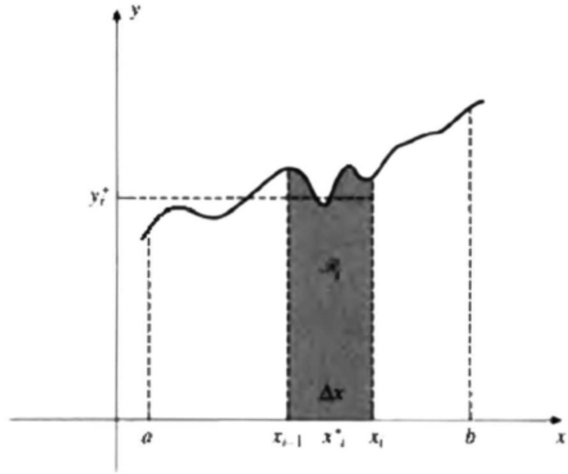

Fig. 33-14

33.5 Establish the cylindrical shell formula $V = 2\pi \int_a^b x f(x) \, dx$.

Divide the interval $[a, b]$ into n equal subintervals, each of length Δx. Let $\mathscr{R}_i$ be the region above the i^{th} subinterval (see Fig. 33-14). Let x_i^* be the midpoint of the i^{th} subinterval, $x_i^* = (x_{i-1} + x_i)/2$.

Now the solid obtained by revolving the region $\mathscr{R}_i$ about the y-axis is approximately the solid obtained by revolving the rectangle with base Δx and height $y_i^* = f(x_i^*)$. The latter solid is a *cylindrical shell*; that is, it is the difference between the cylinders obtained by rotating the rectangles with the same height $f(x_i^*)$ and with bases $[0, x_{i-1}]$ and $[0, x_i]$. Hence, it has volume

$$\begin{aligned}\pi x_i^2 f(x_i^*) - \pi x_{i-1}^2 f(x_i^*) &= \pi f(x_i^*)(x_i^2 - x_{i-1}^2) = \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \pi f(x_i^*)(2x_i^*)(\Delta x) = 2\pi x_i^* f(x_i^*) \, \Delta x\end{aligned}$$

Thus, the total V is approximated by

$$2\pi \sum_{i=1}^{n} x_i^* f(x_i^*) \, \Delta x$$

which in turn approximates the definite integral $2\pi \int_a^b x f(x) \, dx$.

33.6 Establish the cross-section formula $V = \int_a^b A(x) \, dx$.

Divide the interval $[a, b]$ into n equal subintervals $[x_{i-1}, x_i]$, each of length Δx. Choose a point x_i^* in $[x_{i-1}, x_i]$. If n is large, making Δx small, the piece of the solid between x_{i-1} and x_i will be very nearly a (noncircular) disk, of thickness Δx and base area $A(x_i^*)$ (see Fig. 33-15). This disk has volume $A(x_i^*)\,\Delta x$. Thus,

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x \to \int_a^b A(x)\,dx$$

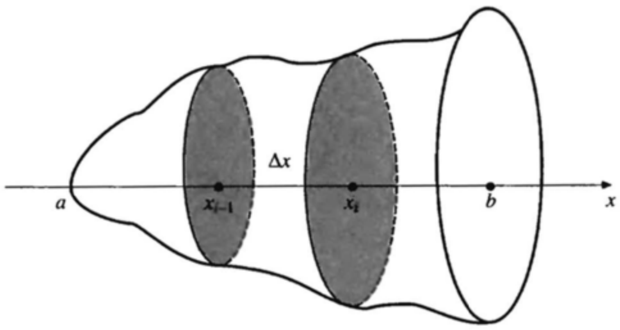

Fig. 33-15

33.7 (*Solids of Revolution about Lines Parallel to a Coordinate Axis*) If a region is revolved about a line parallel to a coordinate axis, we translate the line (and the region along with it) so that it goes over into the coordinate axis. The functions defining the boundary of the region have to be recalculated. The volume obtained by revolving the new region around the new line is equal to the desired volume.

(*a*) Consider the region $\mathscr{R}$ bounded above by the parabola $y = x^2$, below by the x-axis, and lying between $x = 0$ and $x = 1$ [see Fig. 33-16(*a*)]. Find the volume obtained by revolving $\mathscr{R}$ around the horizontal line $y = -1$.

(*b*) Find the volume obtained by revolving the region $\mathscr{R}$ of part (*a*) about the vertical line $x = -2$.

(*a*) Move $\mathscr{R}$ vertically upward by one unit to form a new region $\mathscr{R}^*$. The line $y = -1$ moves up to become the x-axis. $\mathscr{R}^*$ is bounded above by $y = x^2 + 1$ and below by the line $y = 1$. The volume we want is obtained by revolving $\mathscr{R}^*$ about the x-axis. The washer formula applies,

$$V = \pi \int_0^1 ((x^2+1)^2 - 1^2)\,dx = \pi \int_0^1 (x^4 + 2x^2)\,dx$$
$$= \pi\left(\frac{x^5}{5} + \frac{2}{3}x^3\right)\Big]_0^1 = \pi\left(\frac{1}{5} + \frac{2}{3}\right) = \frac{13\pi}{15}$$

(*b*) Move $\mathscr{R}$ two units to the right to form a new region $\mathscr{R}^*$ [see Fig. 33-16(*b*)]. The line $x = -2$ moves over to become the y-axis. $\mathscr{R}^*$ is bounded above by $y = (x-2)^2$ and below by the x-axis and lies

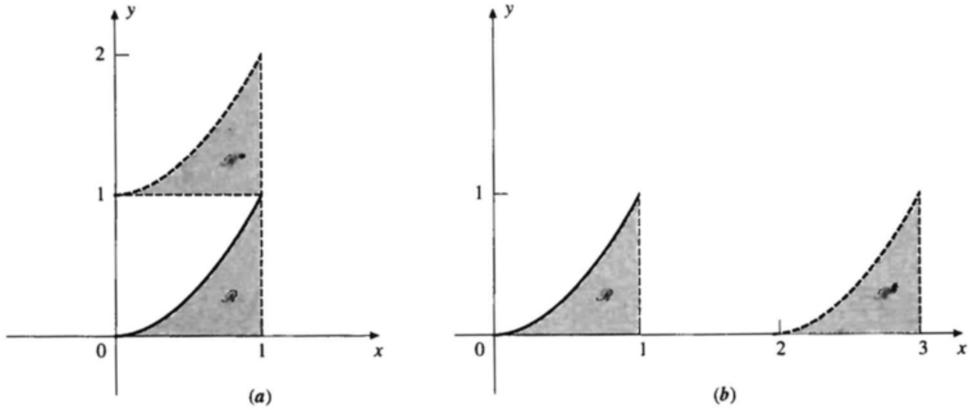

Fig. 33-16

between $x = 2$ and $x = 3$. The volume we want is obtained by revolving $\mathscr{R}^*$ about the y-axis. The cylindrical shell formula applies,

$$V = 2\pi \int_2^3 x(x-2)^2\,dx = 2\pi \int_2^3 (x^3 - 4x^2 + 4x)\,dx = 2\pi\left(\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right)\Big]_2^3$$
$$= 2\pi\left(\left(\frac{1}{4}(3)^4 - \frac{4}{3}(3)^3 + 2(3)^2\right) - \left(\frac{1}{4}(2)^4 - \frac{4}{3}(2)^3 + 2(2)^2\right)\right) = 2\pi\left(\frac{11}{12}\right) = \frac{11\pi}{6}$$

Supplementary Problems

Strategy: In calculating the volume of a solid of revolution we usually apply either the disk formula (or the washer formula) or the cylindrical shells formula (or the difference of cylindrical shells formula). To decide which formula to use:

(1) Decide along which axis you are going to integrate. This depends on the shape and position of the region $\mathscr{R}$ that is revolved.

(2) (i) Use the disk formula (or the washer formula) if the region $\mathscr{R}$ is revolved *perpendicular* to the axis of integration.

(ii) Use the cylindrical shells formula (or the difference of cylindrical shells formula) if the region $\mathscr{R}$ is revolved *parallel* to the axis of integration.

33.8 Find the volume of the solid generated by revolving the given region about the given axis.

(*a*) The region above the curve $y = x^3$, under the line $y = 1$, and between $x = 0$ and $x = 1$; about the x-axis.

(*b*) The region of part (*a*); about the y-axis.

(*c*) The region below the line $y = 2x$, above the x-axis, and between $x = 0$ and $x = 1$; about the y-axis.

(*d*) The region between the parabolas $y = x^2$ and $x = y^2$; about either the x-axis or the y-axis.

(*e*) The region (see Fig. 33-17) inside the circle $x^2 + y^2 = r^2$, with $0 \le x \le a < r$; about the y-axis. (This gives the volume cut from a sphere of radius r by a pipe of radius a whose axis is a diameter of the sphere.)

(*f*) The region (see Fig. 33-18) inside the circle $x^2 + y^2 = r^2$, with $x \ge 0$ and $y \ge 0$, and above the line $y = a$, where $0 \le a < r$; about the y-axis. (This gives the volume of a polar cap of a sphere.)

(*g*) The region bounded by $y = 1 + x^2$ and $y = 5$; about the x-axis.

(*h*) The region (see Fig. 33-19) inside the circle $x^2 + (y-b)^2 = a^2$, with $0 < a < b$, about the x-axis. [*Hint:* When you obtain an integral of the form $\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx$ notice that this is the area of a semicircle of radius a.] This problem gives the volume of a doughnut-shaped solid.

(*i*) The region bounded by $x^2 = 4y$ and $y = x/2$; about the y-axis.

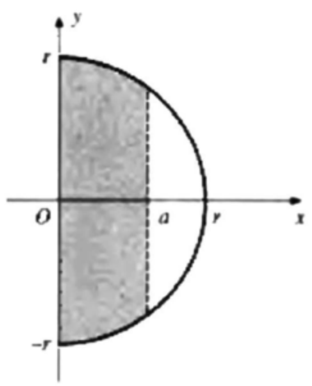

Fig. 33-17

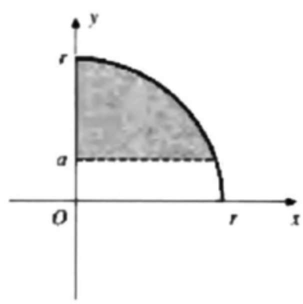

Fig. 33-18

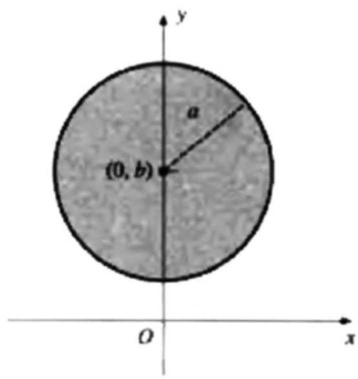

Fig. 33-19

(j) The region bounded by $y = 4/x$ and $y = (x - 3)^2$; about the x-axis. (Notice that the curves intersect when $x = 1$ and $x = 4$. What is special about the intersection at $x = 1$?)

(k) The region of part (j); about the y-axis.

(l) The region bounded by $xy = 1$, $x = 1$, $x = 3$, $y = 0$; about the x-axis.

(m) The region of part (l); about the y-axis.

33.9 Use the cross-section formula to find the volume of the following solids.

(a) The solid has a base which is a circle of radius r. Each cross section perpendicular to a fixed diameter of the circle is an isosceles triangle with altitude equal to one-half of its base.

(b) The solid is a wedge, cut from a perfectly round tree of radius r by two planes, one perpendicular to the axis of the tree and the other intersecting the first plane at an angle of 30° along a diameter (see Fig. 33-20).

(c) A square pyramid with a height of h units and a base of side r units. [*Hint:* Locate the x-axis as in Fig. 33-21. By similar right triangles,

$$\frac{d}{e} = \frac{h - x}{h}$$

and

$$\frac{A(x)}{r^2} = \left(\frac{d}{e}\right)^2$$

which determines $A(x)$.]

(d) The tetrahedron (see Fig. 33-22) formed by three mutually perpendicular faces and three mutually perpendicular edges of lengths a, b, c. [*Hint:* Another pyramid; proceed as in part (c).]

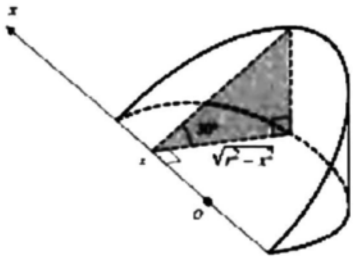

Fig. 33-20

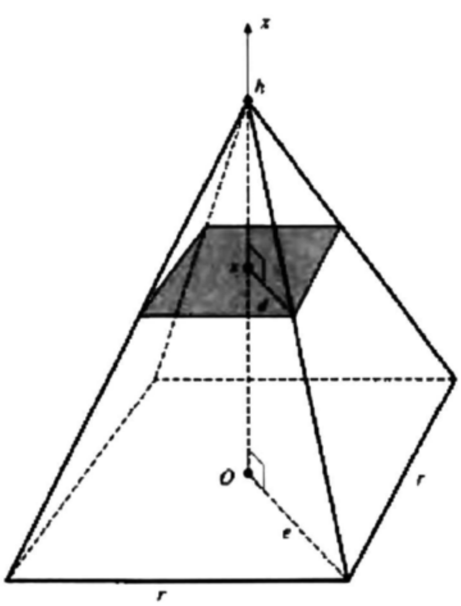

Fig. 33-21

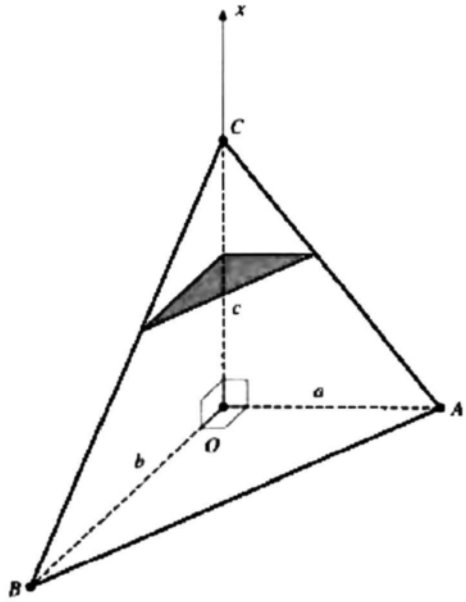

Fig. 33-22

33.10 (*a*) Let $\mathscr{R}$ be the region between $x = 0$ and $x = 1$ and bounded by the curves $y = x^3$ and $y = 2x$. Find the volume of the solid obtained by revolving $\mathscr{R}$ about the y-axis. (*b*) Let $\mathscr{R}$ be the region between the curves $y = 2x - x^2$ and $y = \frac{1}{2}x$. Find the volume of the solid obtained by revolving $\mathscr{R}$ about the y-axis.

33.11 Let $\mathscr{R}$ be the region in the first quadrant bounded by $y = x^3 + x$, $x = 2$, and the x-axis. (*a*) Find the volume of the solid obtained by revolving $\mathscr{R}$ about the line $y = -3$. (*b*) Find the volume of the solid obtained by revolving $\mathscr{R}$ about the line $x = -1$.

33.12 Let $\mathscr{R}$ be the region in the first quadrant bounded by $x = 4 - y^2$ and $y^2 = 4 - 2x$. (*a*) Sketch $\mathscr{R}$. (*b*) Find the volume of the solid obtained by revolving $\mathscr{R}$ about: (i) the x-axis; (ii) the y-axis.

33.13 Let $\mathscr{R}$ be the region in the second quadrant bounded by $y = 2x^2$, $y = x^2 + x + 2$, and the y-axis. (*a*) Sketch $\mathscr{R}$. (*b*) Find the volume of the solid obtained by revolving $\mathscr{R}$ about the y-axis.

33.14 Let $\mathscr{R}$ be the region in the second quadrant bounded by $y = 1 + x^2$ and $y = 10$. (*a*) Sketch $\mathscr{R}$. Then find the volume of the solid obtained by revolving $\mathscr{R}$ about: (*b*) the x-axis; (*c*) the y-axis; (*d*) the line $y = -1$; (*e*) the line $x = 1$.

Chapter 38

Integration by Parts

In this chapter, we shall learn one of the most useful techniques for finding antiderivatives. Let f and g be differentiable functions. The product rule tells us that

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$$

or, in terms of antiderivatives,

$$f(x)g(x) = \int (f(x)g'(x) + g(x)f'(x))\,dx = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

The substitutions $u = f(x)$ and $v = g(x)$ transform this into[1]

$$uv = \int u\,dv + \int v\,du$$

from which we obtain

$$\int u\,dv = uv - \int v\,du \qquad \textit{integration by parts}$$

The idea behind integration by parts is to replace a "difficult" integration $\int u\,dv$ by an "easy" integration $\int v\,du$.

EXAMPLES

(*a*) Find $\int xe^x\,dx$. This will have the form $\int u\,dv$ if we choose

$$u = x \qquad \text{and} \qquad dv = e^x\,dx$$

Since $dv = v'(x)\,dx$ and $dv = e^x\,dx$, we must have $v'(x) = e^x$. Hence,

$$v = \int e^x\,dx = e^x + C$$

and we take the simplest case, $C = 0$, making $v = e^x$.

Since $du = dx$, the integration by parts procedure assumes the following form:

$$\int u\,dv = uv - \int v\,du$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C = e^x(x-1) + C$$

[1] For example, $\int f(x)g'(x)\,dx = \int u\,dv$, where, in the result of the integration on the right, we replace u by $f(x)$ and v by $g(x)$. In fact, by the chain rule,

$$\frac{d}{dx}\left(\int u\,dv\right) = \frac{d}{dv}\left(\int u\,dv\right)\cdot\frac{dv}{dx} = u\cdot\frac{d}{dx}[g(x)] = f(x)g'(x) = \frac{d}{dx}\left(\int f(x)g'(x)\,dx\right)$$

Hence, $\int u\,dv = \int f(x)g'(x)\,dx$. Similarly, $\int v\,du = \int g(x)f'(x)\,dx$.

Integration by parts can be made easier to apply by setting up a box such as the following one for example (*a*) above:

$u = x$	$dv = e^x\,dx$
$du = dx$	$v = e^x$

In the first row, we put u and dv. In the second row, we write du and v. The result $uv - \int v\,du$ is obtained from the box by first multiplying the upper left corner u by the lower right corner v and then subtracting the integral $\int v\,du$ of the two entries in the second row.

Notice that everything depends on a wise choice of u and v. If we had instead picked $u = e^x$ and $dv = x\,dx$, then $v = \int x\,dx = x^2/2$ and we would have obtained

$$\int xe^x\,dx = e^x\frac{x^2}{2} - \int\frac{x^2}{2}e^x\,dx$$

which is true enough, but of little use in evaluating $\int xe^x\,dx$. We would have replaced the "difficult" integration $\int xe^x\,dx$ by the even more "difficult" integration $\int (x^2/2)e^x\,dx$.

(*b*) Find $\int x \ln x\,dx$. Let us try $u = \ln x$ and $dv = x\,dx$:

$u = \ln x$	$dv = x\,dx$
$du = \frac{1}{x}\,dx$	$v = \frac{x^2}{2}$

Then, $v = \int x\,dx = x^2/2$. Thus,

$$\int u\,dv = uv - \int v\,du$$

$$\int x\ln x\,dx = (\ln x)\frac{x^2}{2} - \int\frac{x^2}{2}\frac{1}{x}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\frac{x^2}{2} + C$$

$$= \frac{x^2}{4}(2\ln x - 1) + C$$

(*c*) Find $\int \ln x\,dx$. Let us try $u = \ln x$ and $dv = dx$:

$u = \ln x$	$dv = dx$
$du = \frac{1}{x}\,dx$	$v = x$

Then, $v = \int dx = x$. Thus,

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln x\,dx = x\ln x - \int x\frac{1}{x}\,dx$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + C$$

$$= x(\ln x - 1) + C$$

(*d*) Sometimes two integrations by parts are necessary. Consider $\int e^x \cos x\, dx$. Let $u = e^x$ and $dv = \cos x\, dx$:

$u = e^x$	$dv = \cos x\, dx$
$du = e^x\, dx$	$v = \sin x$

Thus,
$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx \tag{1}$$

Let us try to find $\int e^x \sin x\, dx$ by another integration by parts. Let $u = e^x$ and $dv = \sin x\, dx$:

$u = e^x$	$dv = \sin x\, dx$
$du = e^x\, dx$	$v = -\cos x$

Thus,
$$\int e^x \sin x\, dx = -e^x \cos x - \int (-e^x \cos x)\, dx$$
$$= -e^x \cos x + \int e^x \cos x\, dx$$

Substitute this expression for $\int e^x \sin x\, dx$ in (*1*) and solve the resulting equation for the desired antiderivative,

$$\int e^x \cos x\, dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x\, dx\right)$$
$$\int e^x \cos x\, dx = e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$$
$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$$
$$\int e^x \cos x\, dx = \frac{e^x(\sin x + \cos x)}{2} + C$$

Solved Problems

38.1 Find $\int xe^{-x}\, dx$.

Let

$u = x$	$dv = e^{-x}\, dx$
$du = dx$	$v = \int e^{-x}\, dx = -e^{-x}$

Integration by parts gives

$$\int xe^{-x}\, dx = -xe^{-x} - \int (-e^{-x})\, dx = -xe^{-x} + \int e^{-x}\, dx$$
$$= -xe^{-x} - e^{-x} + C = -e^{-x}(x + 1) + C$$

Another method would consist in making the change of variable $x = -t$ and using example (*a*) of this chapter.

38.2 (*a*) Establish the *reduction formula*

$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx \tag{2}$$

for $\int x^n e^x\,dx \quad (n = 1, 2, 3, \ldots)$.

(*b*) Compute $\int x^2 e^x\,dx$.

(*a*) Let

$u = x^n$	$dv = e^x\,dx$
$du = nx^{n-1}\,dx$	$v = e^x$

and integrate by parts,

$$\int x^n e^x\,dx = x^n e^x - \int e^x(nx^{n-1})\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$$

(*b*) For $n = 1$, (*2*) gives

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x = (x-1)e^x$$

as in example (*a*). We omit the arbitrary constant C until the end of the calculation. Now let $n = 2$ in (*2*),

$$\int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx = x^2e^x - 2((x-1)e^x)$$
$$= (x^2 - 2(x-1))e^x = (x^2 - 2x + 2)e^x + C$$

38.3 Find $\int \tan^{-1} x\,dx$.

Let

$u = \tan^{-1} x$	$dv = dx$
$du = \dfrac{1}{1+x^2}\,dx$	$v = x$

Hence,

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$$
$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$
$$= x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + C \qquad \text{[by quick formula II, Problem 34.5]}$$
$$= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C \qquad \text{[since } 1 + x^2 > 0\text{]}$$

38.4 Find $\int \cos^2 x\,dx$.

Let

$u = \cos x$	$dv = \cos x\,dx$
$du = -\sin x\,dx$	$v = \sin x$

Then,
$$\int \cos^2 x\,dx = \cos x \sin x - \int (\sin x)(-\sin x)\,dx$$
$$= \cos x \sin x + \int \sin^2 x\,dx$$
$$= \cos x \sin x + \int (1 - \cos^2 x)\,dx$$
$$= \cos x \sin x + \int 1\,dx - \int \cos^2 x\,dx$$

Solving this equation for $\int \cos^2 x\,dx$,

$$2\int \cos^2 x\,dx = \cos x \sin x + \int 1\,dx = \cos x \sin x + x$$
$$\int \cos^2 x\,dx = \frac{1}{2}(\cos x \sin x + x) + C$$

This result is more easily obtained by use of Theorem 26.8,

$$\int \cos^2 x\,dx = \frac{1}{2}\int (\cos 2x + 1)\,dx = \frac{1}{2}\left(\frac{\sin 2x}{2} + x\right) + C$$
$$= \frac{1}{2}\left(\frac{2\sin x \cos x}{2} + x\right) + C = \frac{1}{2}(\sin x \cos x + x) + C$$

38.5 Find $\int x \tan^{-1} x\,dx$.

Let

$u = \tan^{-1} x$	$dv = x\,dx$
$du = \dfrac{1}{1+x^2}\,dx$	$v = \dfrac{x^2}{2}$

Then,
$$\int x \tan^{-1} x\,dx = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

But
$$\frac{x^2}{1+x^2} = \frac{(1+x^2)-1}{1+x^2} = \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} = 1 - \frac{1}{1+x^2}$$

and so

$$\int \frac{x^2}{1+x^2}\,dx = \int 1\,dx - \int \frac{1}{1+x^2}\,dx = x - \tan^{-1} x + C$$

Hence,
$$\int x \tan^{-1} x\,dx = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + C_1$$
$$= \frac{1}{2}(x^2 \tan^{-1} x - (x - \tan^{-1} x)) + C_1$$
$$= \frac{1}{2}(x^2 \tan^{-1} x - x + \tan^{-1} x) + C_1$$
$$= \frac{1}{2}((\tan^{-1} x)(x^2 + 1) - x) + C_1$$

Supplementary Problems

38.6 Compute:

(*a*) $\int x^2 e^{-x}\,dx$ (*b*) $\int e^x \sin x\,dx$ (*c*) $\int x^3 e^x\,dx$ (*d*) $\int \sin^{-1} x\,dx$

(*e*) $\int x \cos x\,dx$ (*f*) $\int x^2 \sin x\,dx$ (*g*) $\int \cos(\ln x)\,dx$ (*h*) $\int x \cos(5x-1)\,dx$

(*i*) $\int e^{ax} \cos bx\,dx$ (*j*) $\int \sin^2 x\,dx$ (*k*) $\int \cos^3 x\,dx$ (*l*) $\int \cos^4 x\,dx$

(*m*) $\int x e^{3x}\,dx$ (*n*) $\int x \sec^2 x\,dx$ (*o*) $\int x \cos^2 x\,dx$ (*p*) $\int (\ln x)^2\,dx$

(*q*) $\int x \sin 2x\,dx$ (*r*) $\int x \sin x^2\,dx$ (*s*) $\int \frac{\ln x}{x^2}\,dx$ (*t*) $\int x^2 e^{3x}\,dx$

(*u*) $\int x^2 \tan^{-1} x\,dx$ (*v*) $\int \ln(x^2+1)\,dx$ (*w*) $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$ (*x*) $\int x^2 \ln x\,dx$

[*Hint:* Integration by parts is not a good method for part (*r*).]

38.7 Let $\mathcal{R}$ be the region bounded by the curve $y = \ln x$, the x-axis, and the line $x = e$. (*a*) Find the area of $\mathcal{R}$. (*b*) Find the volume generated by revolving $\mathcal{R}$ about (i) the x-axis; (ii) the y-axis.

38.8 Let $\mathcal{R}$ be the region bounded by the curve $y = x^{-1} \ln x$, the x-axis, and the line $x = e$. Find: (*a*) the area of $\mathcal{R}$; (*b*) the volume of the solid generated by revolving $\mathcal{R}$ about the y-axis; (*c*) the volume of the solid generated by revolving $\mathcal{R}$ about the x-axis. [*Hint:* In part (*c*), the volume integral, let $u = (\ln x)^2$, $v = -1/x$, and use Problem 38.6(*s*).]

38.9 Derive from Problem 38.8(*c*) the (good) bounds: $2.5 \le e \le 2.823$. [*Hint:* By Problem 30.3(*c*),

$$0 \le \int_1^e \left(\frac{\ln x}{x}\right)^2 dx \le \frac{e-1}{e^2} \qquad \text{or} \qquad 0 \le 2 - \frac{5}{e} \le \frac{e-1}{e^2}$$

The left-hand inequality gives $e \ge \frac{5}{2}$; the right-hand inequality gives $e \le (3 + \sqrt{7})/2$.]

38.10 Let $\mathcal{R}$ be the region under one arch of the curve $y = \sin x$, above the x-axis, and between $x = 0$ and $x = \pi$. Find the volume of the solid generated by revolving $\mathcal{R}$ about: (*a*) the x-axis; (*b*) the y-axis.

38.11 If n is a positive integer, find:

(*a*) $\int_0^{2\pi} x \cos nx\,dx$ (*b*) $\int_0^{2\pi} x \sin nx\,dx$

38.12 For $n = 2, 3, 4, \ldots$, find reduction formulas for:

(*a*) $\int \cos^n x\,dx$ (*b*) $\int \sin^n x\,dx$

(*c*) Use these formulas to check the answers to Problems 38.4, 38.6(*k*), 38.6(*l*), and 38.6(*j*).

38.13 (*a*) Find a reduction formula for $\int \sec^n x\,dx$ ($n = 2, 3, 4, \ldots$). (*b*) Use this formula, together with Problem 34.7, to compute: (i) $\int \sec^3 x\,dx$; (ii) $\int \sec^4 x\,dx$.

Chapter 39

Trigonometric Integrands and Trigonometric Substitutions

39.1 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

We already know the antiderivatives of some simple combinations of the basic trigonometric functions. In particular, we have derived all the formulas given in the second column of Appendix B. Let us now look at more complicated cases.

EXAMPLES

(*a*) Consider $\int \sin^k x \cos^n x \, dx$, where the nonnegative integers k and n are *not both even*. If, say, k is odd ($k = 2j + 1$), rewrite the integral as

$$\int \sin^{2j} x \cos^n x \sin x \, dx = \int (\sin^2 x)^j \cos^n x \sin x \, dx$$
$$= \int (1 - \cos^2 x)^j \cos^n x \sin x \, dx$$

Now the change of variable $u = \cos x$, $du = -\sin x \, dx$, produces a polynomial integrand. (For n odd, the substitution $u = \sin x$ would be made instead.) For instance,

$$\int \cos^2 x \sin^5 x \, dx = \int \cos^2 x \sin^4 x \sin x \, dx$$
$$= \int \cos^2 x (1 - \cos^2 x)^2 \sin x \, dx$$
$$= \int u^2(1 - u^2)^2(-1) \, du = -\int u^2(1 - 2u^2 + u^4) \, du$$
$$= -\int (u^2 - 2u^4 + u^6) \, du = -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C$$
$$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

(*b*) Consider the same antiderivative as in part (*a*), but with k and n *both even*; say, $k = 2p$ and $n = 2q$. Then, in view of the half-angle identities

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \text{and} \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

we can write

$$\int \sin^k x \cos^n x \, dx = \int (\sin^2 x)^p (\cos^2 x)^q \, dx$$
$$= \int \left(\frac{1 - \cos 2x}{2}\right)^p \left(\frac{1 + \cos 2x}{2}\right)^q dx$$
$$= \frac{1}{2^{p+q}} \int (1 - \cos 2x)^p (1 + \cos 2x)^q \, dx$$

When the binomials are multiplied out, the integrand will appear as a polynomial in cos $2x$,

$$1 + (q - p)(\cos 2x) + \cdots \pm (\cos 2x)^{p+q}$$

and so

$$\int \sin^k x \cos^n x \, dx = \frac{1}{2^{p+q}}\left[\int 1 \, dx + (q - p)\int (\cos 2x)\, dx + \cdots \pm \int (\cos 2x)^{p+q}\, dx\right] \tag{1}$$

On the right-hand side of (*1*) are antiderivatives of *odd powers* of cos $2x$, which may be evaluated by the method of example (*a*), and antiderivatives of *even powers* of cos $2x$, to which the half-angle formula may be applied again. Thus, if the sixth power were present, we would write

$$\int (\cos 2x)^6 \, dx = \int (\cos^2 2x)^3 \, dx = \int \left(\frac{1 + \cos 4x}{2}\right)^3 dx$$

and expand the polynomial in cos $4x$, and so forth. Eventually the process must end in a final answer, as is shown in the following specific case:

$$\begin{aligned}
\int \cos^2 x \sin^4 x \, dx &= \int (\cos^2 x)(\sin^2 x)^2 \, dx \\
&= \int \left(\frac{1 + \cos 2x}{2}\right)\left(\frac{1 - \cos 2x}{2}\right)^2 dx \\
&= \int \left(\frac{1 + \cos 2x}{2}\right)\left(\frac{1 - 2\cos 2x + \cos^2 2x}{4}\right) dx \\
&= \frac{1}{8}\int (1(1 - 2\cos 2x + \cos^2 2x) + (\cos 2x)(1 - 2\cos 2x + \cos^2 2x))\, dx \\
&= \frac{1}{8}\int (1 - 2\cos 2x + \cos^2 2x + \cos 2x - 2\cos^2 2x + \cos^3 2x)\, dx \\
&= \frac{1}{8}\int (1 - \cos 2x - \cos^2 2x + \cos^3 2x)\, dx \\
&= \frac{1}{8}\left(\int 1\, dx - \int \cos 2x \, dx - \int \cos^2 2x \, dx + \int \cos^3 2x \, dx\right) \\
&= \frac{1}{8}\left(x - \frac{\sin 2x}{2} - \int \frac{1 + \cos 4x}{2}\, dx + \int (\cos 2x)(1 - \sin^2 2x)\, dx\right) \\
&= \frac{1}{8}\left(x - \frac{\sin 2x}{2} - \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right) + \int \cos 2x \, dx - \frac{1}{2}\int u^2 \, du\right) \qquad [\text{letting } u = \sin 2x] \\
&= \frac{1}{8}\left(x - \frac{\sin 2x}{2} - \frac{x}{2} - \frac{\sin 4x}{8} + \frac{\sin 2x}{2} - \frac{1}{2}\frac{\sin^3 2x}{3}\right) + C \\
&= \frac{1}{8}\left(\frac{x}{2} - \frac{\sin 4x}{8} - \frac{\sin^3 2x}{6}\right) + C \\
&= \frac{x}{16} - \frac{\sin 4x}{64} - \frac{\sin^3 2x}{48} + C
\end{aligned}$$

(*c*) From Problem 34.6(*a*), we know how to integrate the first power of tan x,

$$\int \tan x \, dx = \ln |\sec x| + C$$

Higher powers are handled by means of a reduction formula. We have, for $n = 2, 3, \ldots$,

$$\begin{aligned}
\int \tan^n x \, dx &= \int \tan^{n-2} x(\tan^2 x)\, dx = \int \tan^{n-2} x(\sec^2 x - 1)\, dx \\
&= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx \\
&= \int u^{n-2}\, du - \int \tan^{n-2} x \, dx \qquad [\text{let } u = \tan x] \\
&= \frac{\tan^{n-1} x}{n - 1} - \int \tan^{n-2} x \, dx
\end{aligned} \tag{39.1}$$

Similarly, from

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

and the reduction formula of Problem 38.13(*a*), we can integrate all powers of sec *x*.

(*d*) Antiderivatives of the forms $\int \sin Ax \cos Bx \, dx$, $\int \sin Ax \sin Bx \, dx$, $\int \cos Ax \cos Bx \, dx$ can be computed by using the identities

$$\sin Ax \cos Bx = \frac{1}{2}(\sin (A+B)x + \sin (A-B)x)$$

$$\sin Ax \sin Bx = \frac{1}{2}(\cos (A-B)x - \cos (A+B)x)$$

$$\cos Ax \cos Bx = \frac{1}{2}(\cos (A-B)x + \cos (A+B)x)$$

For instance,

$$\int \sin 8x \sin 3x \, dx = \int \frac{1}{2}(\cos 5x - \cos 11x)\, dx = \frac{1}{2}\left(\frac{\sin 5x}{5} - \frac{\sin 11x}{11}\right) + C$$

39.2 TRIGONOMETRIC SUBSTITUTIONS

To find the antiderivative of a function involving such expressions as $\sqrt{a^2 + x^2}$ or $\sqrt{a^2 - x^2}$ or $\sqrt{x^2 - a^2}$, it is often helpful to substitute a trigonometric function for *x*.

EXAMPLES

(*a*) Evaluate $\int \sqrt{x^2 + 2}\, dx$.

None of the methods already available is of any use here. Let us make the substitution $x = \sqrt{2} \tan \theta$, where $-\pi/2 < \theta < \pi/2$. Equivalently, $\theta = \tan^{-1}(x/\sqrt{2})$. Figure 39-1 illustrates the relationship between *x* and θ, with θ interpreted as an angle. We have $dx = \sqrt{2}\sec^2 \theta \, d\theta$ and, from Fig. 39-1,

$$\frac{\sqrt{x^2+2}}{\sqrt{2}} = \sec \theta \qquad \text{or} \qquad \sqrt{x^2+2} = \sqrt{2}\sec \theta$$

where $\sec \theta > 0$ (since $-\pi/2 < \theta < \pi/2$). Thus,

$$\begin{aligned}
\int \sqrt{x^2+2}\, dx &= \int (\sqrt{2}\sec\theta)(\sqrt{2}\sec^2\theta)\, d\theta = 2\int \sec^3\theta \, d\theta \\
&= \sec\theta \tan\theta + \ln|\sec\theta + \tan\theta| + C \qquad \text{[by Problem 38.13(}b\text{)]} \\
&= \frac{\sqrt{x^2+2}}{\sqrt{2}}\frac{x}{\sqrt{2}} + \ln\left|\frac{\sqrt{x^2+2}}{\sqrt{2}} + \frac{x}{\sqrt{2}}\right| + C \\
&= \frac{x\sqrt{x^2+2}}{2} + \ln\frac{|\sqrt{x^2+2}+x|}{\sqrt{2}} + C \qquad \left[\text{since } \left|\frac{a}{b}\right| = \frac{|a|}{|b|}\right] \\
&= \frac{x\sqrt{x^2+2}}{2} + \ln|\sqrt{x^2+2}+x| - \ln\sqrt{2} + C \\
&= \frac{x\sqrt{x^2+2}}{2} + \ln|\sqrt{x^2+2}+x| + C_1
\end{aligned}$$

Note how the constant $-\ln\sqrt{2}$ was absorbed in the constant term in the last step. The absolute value signs in the logarithm may be dropped, since $\sqrt{x^2+2} + x > 0$ for all *x*. This follows from the fact that $\sqrt{x^2+2} > \sqrt{x^2} = |x| \geq -x$.

This example illustrates the following general rule: *If* $\sqrt{x^2 + a^2}$ *occurs in an integrand, try the substitution* $x = a \tan\theta$ *with* $-(\pi/2) < \theta < (\pi/2)$.

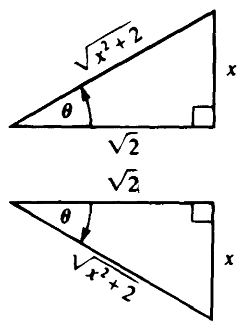

Fig. 39-1

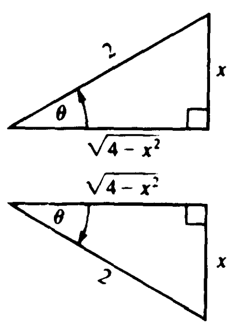

Fig. 39-2

(*b*) Evaluate $\int \frac{\sqrt{4-x^2}}{x}\,dx$.

Make the substitution $x = 2\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$; that is, $\theta = \sin^{-1}(x/2)$. The angle interpretation of θ is shown in Fig. 39-2. Now $dx = 2\cos\theta\,d\theta$ and

$$\frac{\sqrt{4-x^2}}{x} = \frac{\sqrt{4-4\sin^2\theta}}{2\sin\theta} = \frac{2\sqrt{1-\sin^2\theta}}{2\sin\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

Note that $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$, since $\cos\theta \ge 0$ when $-\pi/2 \le \theta \le \pi/2$. Thus,

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x}\,dx &= \int (\cot\theta)(2\cos\theta)\,d\theta = 2\int \left(\frac{\cos\theta}{\sin\theta}\right)(\cos\theta)\,d\theta \\ &= 2\int \frac{\cos^2\theta}{\sin\theta}\,d\theta = 2\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta = 2\left(\int \csc\theta\,d\theta - \int \sin\theta\,d\theta\right) \\ &= 2(\ln|\csc\theta - \cot\theta| + \cos\theta) + C \\ &= 2\left(\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \frac{\sqrt{4-x^2}}{2}\right) + C \\ &= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C\end{aligned}$$

This example illustrates the following general rule: *If $\sqrt{a^2-x^2}$ occurs in an integrand, try the substitution $x = a\sin\theta$ with $-\pi/2 \le \theta \le \pi/2$.*

(*c*) Evaluate $\int \frac{\sqrt{x^2-4}}{x^3}\,dx$.

Let $x = 2\sec\theta$, where $0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$; that is, $\theta = \sec^{-1}(x/2)$. Then $dx = 2\sec\theta\tan\theta\,d\theta$ and

$$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = 2\sqrt{\sec^2\theta - 1} = 2\sqrt{\tan^2\theta} = 2|\tan\theta| = 2\tan\theta$$

Note that $\tan\theta > 0$, since θ is in the first or third quadrant. Then

$$\begin{aligned}\int \frac{\sqrt{x^2-4}}{x^3}\,dx &= \int \frac{2\tan\theta}{8\sec^3\theta}(2\sec\theta\tan\theta)\,d\theta \\ &= \frac{1}{2}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{2}\int \frac{\sin^2\theta}{\cos^2\theta\sec^2\theta}\,d\theta = \frac{1}{2}\int \sin^2\theta\,d\theta \\ &= \frac{1}{4}(\theta - \sin\theta\cos\theta) + C = \frac{1}{4}\left(\sec^{-1}\frac{x}{2} - \frac{\sqrt{x^2-4}}{x}\frac{2}{x}\right) + C \\ &= \frac{1}{4}\left(\sec^{-1}\frac{x}{2} - \frac{2\sqrt{x^2-4}}{x^2}\right) + C\end{aligned}$$

The general rule illustrated by this example is: *If $\sqrt{x^2-a^2}$ occurs in an integrand, try the substitution $x = a\sec\theta$, with $0 \le \theta < (\pi/2)$ or $\pi \le \theta < 3\pi/2$.*

Solved Problems

39.1 Find $\int \sin^3 x \cos^2 x\, dx$.

The exponent of $\sin x$ is odd. So, let $u = \cos x$. Then, $du = -\sin x\, dx$, and

$$\begin{aligned}\int \sin^3 x \cos^2 x\, dx &= \int \sin^2 x \cos^2 x \sin x\, dx\\ &= \int (1-\cos^2 x)\cos^2 x \sin x\, dx\\ &= -\int (1-u^2)u^2\, du = \int (u^4 - u^2)\, du\\ &= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C\end{aligned}$$

39.2 Find $\int \cos^4 x \sin^4 x\, dx$.

The exponents are both even; in addition, they are equal. This allows an improvement on the method of Section 39.1, example (*b*).

$$\begin{aligned}\int \cos^4 x \sin^4 x\, dx &= \int \left(\frac{\sin 2x}{2}\right)^4 dx = \frac{1}{16}\int \sin^4 2x\, dx\\ &= \frac{1}{16}\int \left(\frac{1-\cos 4x}{2}\right)^2 dx\\ &= \frac{1}{64}\int (1 - 2\cos 4x + \cos^2 4x)\, dx\\ &= \frac{1}{64}\left(\int 1\, dx - \frac{1}{2}\int \cos u\, du + \frac{1}{4}\int \cos^2 u\, du\right) \qquad [\text{letting } u = 4x]\\ &= \frac{1}{64}\left(x - \frac{1}{2}\sin u + \frac{1}{8}(u + \sin u \cos u)\right)\\ &= \frac{1}{64}\left(x - \frac{1}{2}\sin 4x + \frac{x}{2} + \frac{\sin 4x \cos 4x}{8}\right) + C\\ &= \frac{1}{64}\left(\frac{3x}{2} - \frac{\sin 4x}{2} + \frac{\sin 8x}{16}\right) + C\\ &= \frac{1}{128}\left(3x - \sin 4x + \frac{\sin 8x}{8}\right) + C\end{aligned}$$

39.3 Find: (*a*) $\int \cos^5 x\, dx$; (*b*) $\int \sin^4 x\, dx$.

(*a*)
$$\begin{aligned}\int \cos^5 x\, dx &= \int \cos^4 x(\cos x)\, dx = \int (\cos^2 x)^2(\cos x)\, dx\\ &= \int (1-\sin^2 x)^2(\cos x)\, dx = \int (1 - 2\sin^2 x + \sin^4 x)(\cos x)\, dx\end{aligned}$$

Let $u = \sin x$. Then $du = \cos x\, dx$, and

$$\begin{aligned}\int \cos^5 x\, dx &= \int (1 - 2u^2 + u^4)\, du = u - \frac{2u^3}{3} + \frac{u^5}{5}\\ &= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C\end{aligned}$$

(*b*) This antiderivative was essentially obtained in Problem 39.2,

$$\begin{aligned}\int \sin^4 x\,dx &= 2\int \sin^4 (2u)\,du \qquad \left[\text{let } u = \frac{x}{2}\right]\\ &= 2\cdot\frac{16}{128}\left(3u - \sin 4u + \frac{\sin 8u}{8}\right) + C\\ &= \frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8}\right) + C\end{aligned}$$

39.4 Find $\int \tan^5 x\,dx$.

From the reduction formula *(39.1)*,

$$\int \tan^3 x\,dx = \frac{\tan^2 x}{2} - \int \tan x\,dx = \frac{\tan^2 x}{2} - \ln|\sec x|$$

$$\int \tan^5 x\,dx = \frac{\tan^4 x}{4} - \int \tan^3 x\,dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec x| + C$$

39.5 Show how to find $\int \tan^p x \sec^q x\,dx$: (*a*) when q is even; (*b*) when p is odd. (*c*) Illustrate both techniques with $\int \tan^3 x \sec^4 x\,dx$ and show that the two answers are equivalent.

(*a*) Let $q = 2r$ $(r = 1, 2, 3, \ldots)$. Then

$$\begin{aligned}\int \tan^p x \sec^{2r} x\,dx &= \int \tan^p x \sec^{2(r-1)} x(\sec^2 x)\,dx\\ &= \int \tan^p x(1 + \tan^2 x)^{r-1}(\sec^2 x)\,dx\end{aligned}$$

since $1 + \tan^2 x = \sec^2 x$. Now the substitution $u = \tan x$, $du = \sec^2 x\,dx$, produces a polynomial integrand.

(*b*) Let $p = 2s + 1$ $(s = 0, 1, 2, \ldots)$. Then,

$$\begin{aligned}\int \tan^{2s+1} x \sec^q x\,dx &= \int \tan^{2s} x \sec^{q-1} x(\sec x \tan x)\,dx\\ &= \int (\sec^2 x - 1)^s \sec^{q-1} x(\sec x \tan x)\,dx\end{aligned}$$

since $\tan^2 x = \sec^2 x - 1$. Now let $v = \sec x$, $dv = \sec x \tan x\,dx$, to obtain a polynomial integrand.

(*c*) By part (*a*),

$$\begin{aligned}\int \tan^3 x \sec^4 x\,dx &= \int \tan^3 x(1 + \tan^2 x)(\sec^2 x)\,dx\\ &= \int u^3(1 + u^2)\,du = \int (u^3 + u^5)\,du\\ &= \frac{u^4}{4} + \frac{u^6}{6} + C = \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C\end{aligned}$$

By part (*b*),

$$\begin{aligned}\int \tan^3 x \sec^4 x\,dx &= \int (\sec^2 x - 1)\sec^3 x(\sec x \tan x)\,dx\\ &= \int (v^2 - 1)v^3\,dv = \int (v^5 - v^3)\,dv\\ &= \frac{v^6}{6} - \frac{v^4}{4} + C = \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C\end{aligned}$$

Since $1 + u^2 = v^2$,

$$\frac{v^6}{6} - \frac{v^4}{4} = \frac{4v^6 - 6v^4}{24} = \frac{4(1+u^2)^3 - 6(1+u^2)^2}{24}$$

ALGEBRA	$(1+t)^3 = 1 + 3t + 3t^2 + t^3$

$$= \frac{4(1 + 3u^2 + 3u^4 + u^6) - 6(1 + 2u^2 + u^4)}{24}$$

$$= \frac{4 + 12u^2 + 12u^4 + 4u^6 - 6 - 12u^2 - 6u^4}{24}$$

$$= \frac{6u^4 + 4u^6 - 2}{24} = \frac{u^4}{4} + \frac{u^6}{6} - \frac{1}{12}$$

and so the two expressions for $\int \tan^3 x \sec^4 x\, dx$ are equivalent. (The $-\frac{1}{12}$ is soaked up by the arbitrary constant C.)

39.6 Find $\displaystyle\int \tan^2 x \sec x\, dx$.

Problem 39.5 is of no help here.

$$\int \tan^2 x \sec x\, dx = \int (\sec^2 x - 1) \sec x\, dx = \int (\sec^3 x - \sec x)\, dx$$

$$= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C \qquad \text{[by Problem 38.13(}b\text{)]}$$

$$= \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C$$

39.7 Prove the trigonometric identity $\sin Ax \cos Bx = \frac{1}{2}(\sin (A + B)x + \sin (A - B)x)$.

The sum and difference formulas of Theorem 26.6 give

$$\sin (A + B)x = \sin (Ax + Bx) = \sin Ax \cos Bx + \cos Ax \sin Bx$$
$$\sin (A - B)x = \sin (Ax - Bx) = \sin Ax \cos Bx - \cos Ax \sin Bx$$

and so, by addition, $\sin (A + B)x + \sin (A - B)x = 2 \sin Ax \cos Bx$.

39.8 Compute the value of $\displaystyle\int_0^{2\pi} \sin nx \cos kx\, dx$ for positive integers n and k.

Case 1: $n \neq k$. By Problem 39.7, with $A = n$ and $B = k$,

$$\int_0^{2\pi} \sin nx \cos kx\, dx = \frac{1}{2}\int_0^{2\pi} (\sin (n + k)x + \sin (n - k)x)\, dx$$

$$= -\frac{1}{2}\left(\frac{\cos (n+k)x}{n+k} + \frac{\cos (n-k)x}{n-k}\right)\Bigg]_0^{2\pi} = 0$$

because $\cos px$ is, for p an integer, a periodic function of period 2π.

Case 2: $n = k$. Then, by the double-angle formula for the sine function,

$$\int_0^{2\pi} \sin nx \cos nx\, dx = \frac{1}{2}\int_0^{2\pi} \sin 2nx\, dx = -\frac{1}{2}\left(\frac{\cos 2nx}{2n}\right)\Bigg]_0^{2\pi} = 0$$

39.9 Find $\dfrac{dx}{\sqrt{x^2+9}}$.

The presence of $\sqrt{x^2+9}$ suggests letting $x = 3\tan\theta$. Then $dx = 3\sec^2\theta\, d\theta$, and

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9} = \sqrt{9(\tan^2\theta+1)} = 3\sqrt{\sec^2\theta} = 3\sec\theta$$

So,
$$\int \frac{dx}{\sqrt{x^2+9}} = \int \frac{3\sec^2\theta}{3\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C$$
$$= \ln\left|\frac{\sqrt{x^2+9}}{3}+\frac{x}{3}\right| + C = \ln\left|\frac{\sqrt{x^2+9}+x}{3}\right| + C$$
$$= \ln|\sqrt{x^2+9}+x| + K = \ln(\sqrt{x^2+9}+x) + K$$

NOTE
$$\ln\left|\frac{\sqrt{x^2+9}+x}{3}\right| = \ln|\sqrt{x^2+9}+x| - \ln 3$$

and the constant $-\ln 3$ can be absorbed in the arbitrary constant K. Furthermore,

$$\sqrt{x^2+9}+x > 0$$

39.10 Find $\displaystyle\int \frac{dx}{x^2\sqrt{3-x^2}}$.

The presence of $\sqrt{3-x^2}$ suggests the substitution $x = \sqrt{3}\sin\theta$. Then, $dx = \sqrt{3}\cos\theta\,d\theta$.

$$\sqrt{3-x^2} = \sqrt{3-3\sin^2\theta} = \sqrt{3(1-\sin^2\theta)} = \sqrt{3}\sqrt{\cos^2\theta} = \sqrt{3}\cos\theta$$

and
$$\int \frac{dx}{x^2\sqrt{3-x^2}} = \int \frac{\sqrt{3}\cos\theta\,d\theta}{(3\sin^2\theta)(\sqrt{3}\cos\theta)} = \frac{1}{3}\int\frac{d\theta}{\sin^2\theta} = \frac{1}{3}\int \csc^2\theta\,d\theta$$
$$= -\frac{1}{3}\cot\theta + C$$

But
$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{3-x^2}/\sqrt{3}}{x/\sqrt{3}} = \frac{\sqrt{3-x^2}}{x}$$

Hence,
$$\int \frac{dx}{x^2\sqrt{3-x^2}} = -\frac{\sqrt{3-x^2}}{3x} + C$$

39.11 Find $\displaystyle\int \frac{x^2}{\sqrt{x^2-4}}\,dx$.

The occurrence of $\sqrt{x^2-4}$ suggests the substitution $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$.

$$\sqrt{x^2-4} = \sqrt{4\sec^2\theta-4} = \sqrt{4(\sec^2\theta-1)} = 2\sqrt{\tan^2\theta} = 2\tan\theta$$

and
$$\int \frac{x^2}{\sqrt{x^2-4}}\,dx = \int \frac{(4\sec^2\theta)(2\sec\theta\tan\theta)}{2\tan\theta}\,d\theta = 4\int \sec^3\theta\,d\theta$$
$$= 2(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|) + C \qquad \text{[by Problem 38.13(}b\text{)]}$$
$$= 2\left(\frac{x}{2}\frac{\sqrt{x^2-4}}{2} + \ln\left|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right|\right) + C$$
$$= \frac{x\sqrt{x^2-4}}{2} + 2\ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| + C$$
$$= \frac{x\sqrt{x^2-4}}{2} + 2\ln|x+\sqrt{x^2-4}| + K$$

where $K = C - 2\ln 2$ (compare Problem 39.9).

Supplementary Problems

39.12 Find the following antiderivatives:

(*a*) $\int \sin x \cos^2 x\,dx$ (*b*) $\int \cos^2 3x\,dx$ (*c*) $\int \sin^4 x \cos^5 x\,dx$ (*d*) $\int \cos^6 x\,dx$

(*e*) $\int \cos^6 x \sin^2 x\,dx$ (*f*) $\int \tan^2 \frac{x}{2}\,dx$ (*g*) $\int \tan^6 x\,dx$ (*h*) $\int \sec^5 x\,dx$

(*i*) $\int \tan^2 x \sec^4 x\,dx$ (*j*) $\int \tan^3 x \sec^3 x\,dx$ (*k*) $\int \tan^4 x \sec x\,dx$ (*l*) $\int \sin 2x \cos 2x\,dx$

(*m*) $\int \sin \pi x \cos 3\pi x\,dx$ (*n*) $\int \sin 5x \sin 7x\,dx$ (*o*) $\int \cos 4x \cos 9x\,dx$

39.13 Prove the following identities:

(*a*) $\sin Ax \sin Bx = \frac{1}{2}(\cos (A - B)x - \cos (A + B)x)$

(*b*) $\cos Ax \cos Bx = \frac{1}{2}(\cos (A - B)x + \cos (A + B)x)$

39.14 Calculate the following definite integrals, where the positive integers n and k are distinct:

(*a*) $\int_0^{2\pi} \sin nx \sin kx\,dx$ (*b*) $\int_0^{2\pi} \sin^2 nx\,dx$

39.15 Evaluate:

(*a*) $\int \frac{\sqrt{x^2 - 1}}{x}\,dx$ (*b*) $\int \frac{x^2}{\sqrt{4 - x^2}}\,dx$ (*c*) $\int \frac{\sqrt{1 + x^2}}{x}\,dx$ (*d*) $\int \frac{x}{\sqrt{2 - x^2}}\,dx$

(*e*) $\int \frac{dx}{x^2\sqrt{x^2 - 9}}$ (*f*) $\int \frac{dx}{(4 - x^2)^{3/2}}$ (*g*) $\int \frac{dx}{(x^2 + 9)^2}$ (*h*) $\int \frac{dx}{x\sqrt{16 - 9x^2}}$

(*i*) $\int x^2\sqrt{1 - x^2}\,dx$ (*j*) $\int e^{3x}\sqrt{1 - e^{2x}}\,dx$ (*k*) $\int \frac{dx}{(x^2 - 6x + 13)^2}$

[*Hint:* In part (*k*), complete the square.]

39.16 Find the arc length of the parabola $y = x^2$ from (0, 0) to (2, 4).

39.17 Find the arc length of the curve $y = \ln x$ from (1, 0) to (e, 1).

39.18 Find the arc length of the curve $y = e^x$ from (0, 1) to (1, e).

39.19 Find the arc length of the curve $y = \ln \cos x$ from (0, 0) to ($\pi/3$, $-\ln 2$).

39.20 Find the area enclosed by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Chapter 40

Integration of Rational Functions; The Method of Partial Fractions

This chapter will give a general method for evaluating indefinite integrals of the type

$$\int \frac{N(x)}{D(x)}\, dx$$

where $N(x)$ and $D(x)$ are polynomials. That is to say, we shall show how to find the antiderivative of any rational function $f(x) = N(x)/D(x)$ (see Section 9.3). Two assumptions will be made, neither of which is really restrictive: (i) the leading coefficient (the coefficient of the highest power of x) in $D(x)$ is $+1$; (ii) $N(x)$ is of lower degree than $D(x)$ [that is, $f(x)$ is a *proper* rational function].

EXAMPLES

(*a*) $$\frac{8x^4}{-\frac{1}{7}x^{10} + 3x - 11} = \frac{-7}{-7}\,\frac{8x^4}{-\frac{1}{7}x^{10} + 3x - 11} = \frac{-56x^4}{x^{10} - 21x + 77}$$

(*b*) Consider the improper rational function $f(x) = \dfrac{x^4 + 7x}{x^2 - 1}$. Long division (see Fig. 40-1) yields

$$f(x) = x^2 + 1 + \frac{7x + 1}{x^2 - 1}$$

Consequently,

$$\int f(x)\, dx = \int (x^2 + 1)\, dx + \int \frac{7x + 1}{x^2 - 1}\, dx = \frac{x^3}{3} + x + \int \frac{7x + 1}{x^2 - 1}\, dx$$

and the problem reduces to finding the antiderivative of a proper rational function.

$$\begin{array}{r|l} & x^2 + 1 \\ \hline x^2 - 1 & x^4 + 7x \\ & \underline{x^4 - x^2} \\ & \quad x^2 + 7x \\ & \quad \underline{x^2 - 1} \\ & \qquad 7x + 1 \end{array}$$

Fig. 40-1

The theorems that follow hold for polynomials with arbitrary real coefficients. However, for simplicity we shall illustrate them only with polynomials whose coefficients are integers.

Theorem 40.1: Any polynomial $D(x)$ with leading coefficient 1 can be expressed as the product of *linear factors*, of the form $x - a$, and *irreducible quadratic factors* (that cannot be factored further), of the form $x^2 + bx + c$, repetition of factors being allowed.

As explained in Section 7.4, the real roots of $D(x)$ determine its linear factors.

EXAMPLES

(*a*)
$$x^2 - 1 = (x - 1)(x + 1)$$

Here, the polynomial has two real roots (± 1) and, therefore, is a product of two linear factors.

(*b*)
$$x^3 + 2x^2 - 8x - 21 = (x - 3)(x^2 + 5x + 7)$$

The root $x = 3$, which generates the linear factor $x - 3$, was found by testing the divisors of 21. Division of $D(x)$ by $x - 3$ yielded the polynomial $x^2 + 5x + 7$. This polynomial is irreducible, since, by the quadratic formula, its roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2} = \frac{-5 \pm \sqrt{-3}}{2}$$

which are not real numbers.

Theorem 40.2 (*Partial Fractions Representation*): Any (proper) rational function $f(x) = N(x)/D(x)$ may be written as a sum of simpler, proper rational functions. Each summand has as denominator *one* of the linear or quadratic factors of $D(x)$, raised to some power.

By Theorem 40.2, $\int f(x)\,dx$ is given as a sum of simpler antiderivatives—antiderivatives which, in fact, can be found by the techniques already known to us.

It will now be shown how to construct the partial fractions representation and to integrate it term by term.

Case 1: *$D(x)$ is a product of nonrepeated linear factors.*

The partial fractions representation of $f(x)$ is

$$\frac{N(x)}{(x - a_1)(x - a_2)\cdots(x - a_n)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n}$$

The constant numerators $A_1, \ldots, A_n$ are evaluated as in the following example.

EXAMPLE
$$\frac{2x + 1}{(x + 1)(x - 1)} = \frac{A_1}{x + 1} + \frac{A_2}{x - 1}$$

Clear the denominators by multiplying both sides by $(x + 1)(x - 1)$,

$$(x + 1)(x - 1)\frac{2x + 1}{(x + 1)(x - 1)} = (x + 1)(x - 1)\frac{A_1}{x + 1} + (x + 1)(x - 1)\frac{A_2}{x - 1}$$
$$2x + 1 = A_1(x - 1) + A_2(x + 1) \tag{1}$$

In (*1*), substitute individually the roots of $D(x)$. With $x = -1$,

$$-1 = A_1(-2) + 0 \qquad \text{or} \qquad A_1 = \frac{1}{2}$$

and with $x = 1$,

$$3 = 0 + A_2(2) \qquad \text{or} \qquad A_2 = \frac{3}{2}$$

With all constants known, the antiderivative of $f(x)$ will be the sum of terms of the form

$$\int \frac{A}{x - a}\,dx = A \ln|x - a|$$

Case 2: *$D(x)$ is a product of linear factors, at least one of which is repeated.*

This is treated in the same manner as in Case 1, except that a repeated factor $(x-a)^k$ gives rise to a sum of the form

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_k}{(x-a)^k}$$

EXAMPLE

$$\frac{3x+1}{(x-1)^2(x-2)}=\frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{A_3}{x-2}$$

Multiply by $(x-1)^2(x-2)$,

$$3x+1=A_1(x-1)(x-2)+A_2(x-2)+A_3(x-1)^2 \tag{2}$$

Letting $x=1$,

$$4=0+A_2(-1)+0 \qquad \text{or} \qquad A_2=-4$$

Letting $x=2$,

$$7=0+0+A_3(1) \qquad \text{or} \qquad A_3=7$$

The remaining numerator, A_1, is determined by the condition that the coefficient of x^2 on the right side of (2) be zero (since it is zero on the left side). Thus,

$$A_1+A_3=0 \qquad \text{or} \qquad A_1=-A_3=-7$$

[More generally, we use all the roots of $D(x)$ to determine some of the A's, and then compare coefficients—of as many powers of x as necessary—to find the remaining A's.]

Now the antiderivatives of $f(x)$ will consist of terms of the form $\ln|x-a|$ plus at least one term of the form

$$\int\frac{A}{(x-a)^j}\,dx=\frac{B}{(x-a)^{j-1}} \qquad [j\geq 2]$$

Case 3: $D(x)$ has irreducible quadratic factors, but none is repeated.

In this case, each quadratic factor x^2+bx+c contributes a term

$$\frac{Ax+B}{x^2+bx+c}$$

to the partial fractions representation.

EXAMPLE

$$\frac{x^2-1}{(x^2+1)(x+2)}=\frac{A_1}{x+2}+\frac{A_2x+A_3}{x^2+1}$$

Multiply by $(x^2+1)(x+2)$,

$$x^2-1=A_1(x^2+1)+(A_2x+A_3)(x+2) \tag{3}$$

Let $x=-2$,

$$3=A_1(5)+0 \qquad \text{or} \qquad A_1=\frac{3}{5}$$

Comparing coefficients of x^0 (the constant terms),

$$-1=A_1+2A_3 \qquad \text{or} \qquad A_3=-\frac{1}{2}(1+A_1)=-\frac{1}{2}\left(\frac{8}{5}\right)=-\frac{4}{5}$$

Comparing coefficients of x^2,

$$1=A_1+A_2 \qquad \text{or} \qquad A_2=1-A_1=\frac{2}{5}$$

The sum for $\int f(x)\,dx$ will now include, besides terms arising from any linear factors, at least one term of the form

$$\int \frac{Ax+B}{x^2+bx+c}\,dx = \int \frac{A\left(x+\frac{b}{2}\right)+C}{\left(x+\frac{b}{2}\right)^2+\delta^2}\,dx \qquad \left[\begin{array}{l} C \equiv B - \dfrac{Ab}{2} \\ \delta^2 \equiv c - \dfrac{b^2}{4} > 0 \end{array}\right]$$

$$= \int \frac{Au+C}{u^2+\delta^2}\,du \qquad \left[\text{let } u = x + \frac{b}{2}\right]$$

$$= A\int \frac{u\,du}{u^2+\delta^2} + C\int \frac{du}{u^2+\delta^2}$$

$$= \frac{A}{2}\ln\,(u^2+\delta^2) + \frac{C}{\delta}\tan^{-1}\frac{u}{\delta}$$

(For a guarantee that δ is a *real* number, see Problem 40.7.)

Case 4: *$D(x)$ has at least one repeated irreducible quadratic factor.*

A repeated quadratic factor $(x^2+bx+c)^k$ contributes to the partial fractions representation the expression

$$\frac{A_1x+A_2}{ax^2+bx+c} + \frac{A_3x+A_4}{(ax^2+bx+c)^2} + \cdots + \frac{A_{2k-1}x+A_{2k}}{(ax^2+bx+c)^k}$$

The computations in this case may be long and tedious.

EXAMPLE

$$\frac{x^3+1}{(x^2+1)^2} = \frac{A_1x+A_2}{x^2+1} + \frac{A_3x+A_4}{(x^2+1)^2}$$

Multiply by $(x^2+1)^2$,

$$x^3+1 = (A_1x+A_2)(x^2+1) + A_3x + A_4 \qquad (4)$$

Compare coefficients of x^3,

$$1 = A_1$$

Compare coefficients of x^2,

$$0 = A_2$$

Compare coefficients of x,

$$0 = A_1 + A_3 \qquad \text{or} \qquad A_3 = -A_1 = -1$$

Compare coefficients of x^0,

$$1 = A_2 + A_4 \qquad \text{or} \qquad A_4 = 1 - A_2 = 1$$

The new contribution to $\int f(x)\,dx$ will consist of one or more terms of the form

$$\int \frac{Ax+B}{(x^2+bx+c)^j}\,dx = A\int \frac{u\,du}{(u^2+\delta^2)^j} + C\int \frac{du}{(u^2+\delta^2)^j} \qquad [\text{as in Case 3}]$$

$$= \frac{E}{(u^2+\delta^2)^{j-1}} + F\int \cos^{2(j-1)}\theta\,d\theta \qquad [\text{let } u = \delta\tan\theta]$$

and we know how to evaluate the trigonometric integral [see Problem 38.12(*a*) or example (*b*) of Section 39.1].

Solved Problems

40.1 Evaluate $\int \frac{2x^3 + x^2 - 6x + 7}{x^2 + x - 6}\,dx$.

The numerator has greater degree than the denominator. Therefore, divide the numerator by the denominator,

$$\begin{array}{r|l} & 2x - 1 \\ \hline x^2 + x - 6 & 2x^3 + x^2 - 6x + 7 \\ & 2x^3 + 2x^2 - 12x \\ \hline & -x^2 + 6x + 7 \\ & -x^2 - x + 6 \\ \hline & 7x + 1 \end{array}$$

Thus, $$\frac{2x^3 + x^2 - 6x + 7}{x^2 + x - 6} = 2x - 1 + \frac{7x + 1}{x^2 + x - 6}$$

Next, factor the denominator, $x^2 + x - 6 = (x + 3)(x - 2)$. The partial fractions decomposition has the form (Case 1)

$$\frac{7x + 1}{(x + 3)(x - 2)} = \frac{A_1}{x + 3} + \frac{A_2}{x - 2}$$

Multiply by the denominator $(x + 3)(x - 2)$,

$$7x + 1 = A_1(x - 2) + A_2(x + 3)$$

Let $x = 2$, $\quad 15 = 0 + 5A_2 \quad$ or $\quad A_2 = 3$

Let $x = -3$, $\quad -20 = -5A_1 + 0 \quad$ or $\quad A_1 = 4$

Thus, $$\frac{7x + 1}{(x + 3)(x - 2)} = \frac{4}{x + 3} + \frac{3}{x - 2}$$

and $$\int \frac{2x^3 + x^2 - 6x + 7}{x^2 + x - 6}\,dx = \int (2x - 1)\,dx + \int \frac{4}{x + 3}\,dx + \int \frac{3}{x - 2}\,dx$$
$$= x^2 - x + 4 \ln|x + 3| + 3 \ln|x - 2| + C$$

40.2 Find $\int \frac{x^2\,dx}{x^3 - 3x^2 - 9x + 27}$.

Testing the factors of 27, we find that 3 is a root of $D(x)$. Dividing $D(x)$ by $x - 3$ yields

$$x^3 - 3x^2 - 9x + 27 = (x - 3)(x^2 - 9) = (x - 3)(x - 3)(x + 3) = (x - 3)^2(x + 3)$$

and so the partial fractions representation is (Case 2)

$$\frac{x^2}{(x - 3)^2(x + 3)} = \frac{A_1}{x - 3} + \frac{A_2}{(x - 3)^2} + \frac{A_3}{x + 3}$$

Multiply by $(x - 3)^2(x + 3)$,

$$x^2 = A_1(x - 3)(x + 3) + A_2(x + 3) + A_3(x - 3)^2$$

Let $x = 3$, $\quad 9 = 0 + 6A_2 + 0 \quad$ or $\quad A_2 = \frac{3}{2}$

Let $x = -3$, $$9 = 0 + 0 + A_3(-6)^2 \quad \text{or} \quad A_3 = \frac{1}{4}$$

Compare coefficients of x^2,

$$1 = A_1 + A_3 \quad \text{or} \quad A_1 = 1 - A_3 = \frac{3}{4}$$

Thus, $$\frac{x^2}{x^3 - 3x^2 - 9x + 27} = \frac{3}{4}\frac{1}{x-3} + \frac{3}{2}\frac{1}{(x-3)^2} + \frac{1}{4}\frac{1}{x+3}$$

and $$\int \frac{x^2\,dx}{x^3 - 3x^2 - 9x + 27} = \frac{3}{4}\ln|x-3| - \frac{3}{2}\frac{1}{x-3} + \frac{1}{4}\ln|x+3| + C$$

40.3 Find $\int \frac{x+1}{x(x^2+2)}\,dx$.

This is Case 3,

$$\frac{x+1}{x(x^2+2)} = \frac{A_1}{x} + \frac{A_2 x + A_3}{x^2+2}$$

Multiply by $x(x^2+2)$,

$$x + 1 = A_1(x^2+2) + x(A_2 x + A_3)$$

Let $x = 0$, $$1 = 2A_1 + 0 \quad \text{or} \quad A_1 = \frac{1}{2}$$

Compare coefficients of x^2,

$$0 = A_1 + A_2 \quad \text{or} \quad A_2 = -A_1 = -\frac{1}{2}$$

Compare coefficients of x,

$$1 = A_3$$

Thus, $$\frac{x+1}{x(x^2+2)} = \frac{1}{2}\left(\frac{1}{x}\right) + \frac{(-\frac{1}{2})x + 1}{x^2+2}$$

and $$\int \frac{x+1}{x(x^2+2)}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{x\,dx}{x^2+2} + \int \frac{dx}{x^2+2}$$

Because the quadratic factor $x^2 + 2$ is a complete square, we can perform the integrations on the right without a change of variable,

$$\int \frac{x+1}{x(x^2+2)}\,dx = \frac{1}{2}\ln|x| - \frac{1}{4}\ln(x^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

40.4 Evaluate $\int \frac{1}{1 - \sin x + \cos x}\,dx$.

Observe that the integrand is a *rational function* of $\sin x$ *and* $\cos x$. Any rational function of the six trigonometric functions reduces to a function of this type, and the method we shall use to solve this particular problem will work for any such function.

Make the change of variable $z = \tan(x/2)$; that is, $x = 2\tan^{-1} z$. Then,

$$dx = \frac{2}{1+z^2}\,dz$$

and, by Theorem 26.8,

$$\begin{aligned}\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\,\frac{\tan(x/2)}{\sec^2(x/2)}\\ &= 2\,\frac{\tan(x/2)}{1+\tan^2(x/2)} = \frac{2z}{1+z^2}\\ \cos x &= 1 - 2\sin^2\frac{x}{2} = 1 - 2\,\frac{\tan^2(x/2)}{\sec^2(x/2)}\\ &= 1 - 2\,\frac{\tan^2(x/2)}{1+\tan^2(x/2)} = 1 - \frac{2z^2}{1+z^2} = \frac{1-z^2}{1+z^2}\end{aligned}$$

When these substitutions are made, the resulting integrand will be a rational function of z (because compositions and products of rational functions are rational functions). The method of partial fractions can then be applied,

$$\begin{aligned}\int (1 - \sin x + \cos x)^{-1}\,dx &= \int \left(1 - \frac{2z}{1+z^2} + \frac{1-z^2}{1+z^2}\right)^{-1} \frac{2}{1+z^2}\,dz\\ &= \int \left(\frac{(1+z^2) - 2z + (1-z^2)}{1+z^2}\right)^{-1} \frac{2}{1+z^2}\,dz\\ &= \int \left(\frac{2-2z}{1+z^2}\right)^{-1} \frac{2}{1+z^2}\,dz = \int \frac{1+z^2}{2-2z}\,\frac{2}{1+z^2}\,dz\\ &= \int \frac{1}{1-z}\,dz = -\ln|1-z| + C\\ &= -\ln\left|1 - \tan\frac{x}{2}\right| + C\end{aligned}$$

40.5 Find $\displaystyle\int \frac{x\,dx}{(x+1)(x^2+2x+2)^2}$.

This is Case 4 for $D(x)$, and so

$$\frac{x}{(x+1)(x^2+2x+2)^2} = \frac{A_1}{x+1} + \frac{A_2x + A_3}{x^2+2x+2} + \frac{A_4x + A_5}{(x^2+2x+2)^2}$$

Multiply by $(x+1)(x^2+2x+2)^2$,

$$x = A_1(x^2+2x+2)^2 + (A_2x + A_3)(x+1)(x^2+2x+2) + (A_4x + A_5)(x+1)$$

or, partially expanding the right-hand side,

$$x = A_1(x^4 + 4x^3 + 8x^2 + 8x + 4) + (A_2x + A_3)(x^3 + 3x^2 + 4x + 2) + (A_4x + A_5)(x+1) \tag{1}$$

In (1), let $x = -1$, $\qquad -1 = A_1(1) = A_1$

Compare coefficients of x^4,

$$0 = A_1 + A_2 \quad\text{or}\quad A_2 = -A_1 = 1$$

Compare coefficients of x^3,

$$0 = 4A_1 + 3A_2 + A_3 \quad\text{or}\quad A_3 = -4A_1 - 3A_2 = 1$$

Compare coefficients of x^2,

$$0 = 8A_1 + 4A_2 + 3A_3 + A_4 \quad\text{or}\quad A_4 = -8A_1 - 4A_2 - 3A_3 = 1$$

Compare coefficients of x^0,

$$0 = 4A_1 + 2A_3 + A_5 \quad\text{or}\quad A_5 = -4A_1 - 2A_3 = 2$$

Therefore,

$$\int \frac{x\,dx}{(x+1)(x^2+2x+2)^2} = -1\int \frac{dx}{x+1} + \int \frac{(x+1)\,dx}{x^2+2x+2} + \int \frac{(x+2)\,dx}{(x^2+2x+2)^2}$$

$$= -\ln|x+1| + \int \frac{u\,du}{u^2+1} + \int \frac{u\,du}{(u^2+1)^2} + \int \frac{du}{(u^2+1)^2}$$

[by quick formulas II and I]

$$= -\ln|x+1| + \frac{1}{2}\ln(u^2+1) - \frac{1}{2}\left(\frac{1}{u^2+1}\right) + \int \cos^2\theta\,d\theta$$

[Case 4: let $u = \tan\theta$]

$$= -\ln|x+1| + \frac{1}{2}\ln(x^2+2x+2) - \frac{1}{2}\left(\frac{1}{x^2+2x+2}\right) + \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C$$

Now $\theta = \tan^{-1} u = \tan^{-1}(x+1)$

and (see Problem 40.4)

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2(x+1)}{x^2+2x+2}$$

so that we have, finally,

$$\int \frac{x\,dx}{(x+1)(x^2+2x+2)^2} = -\ln|x+1| + \frac{1}{2}\ln(x^2+2x+2) - \frac{1}{2}\left(\frac{1}{x^2+2x+2}\right)$$

$$+ \frac{1}{2}\tan^{-1}(x+1) + \frac{1}{2}\left(\frac{x+1}{x^2+2x+2}\right) + C$$

$$= \frac{1}{2}\left(\ln(x^2+2x+2) + \frac{x}{x^2+2x+2} + \tan^{-1}(x+1)\right) - \ln|x+1| + C$$

Supplementary Problems

40.6 Find the following antiderivatives:

(a) $\int \frac{dx}{x^2-9}$ (b) $\int \frac{x\,dx}{(x+2)(x+3)}$ (c) $\int \frac{x^4-4x^2+x+1}{x^2-4}\,dx$

(d) $\int \frac{2x^2+1}{(x-1)(x-2)(x-3)}\,dx$ (e) $\int \frac{x^2-4}{x^3-3x^2-x+3}\,dx$ (f) $\int \frac{x^3+1}{x(x+3)(x+2)(x-1)}\,dx$

(g) $\int \frac{x\,dx}{x^4-13x^2+36}$ (h) $\int \frac{x-5}{x^2(x+1)}\,dx$ (i) $\int \frac{2x\,dx}{(x-2)^2(x+2)}$

(j) $\int \frac{x+4}{x^3+6x^2+9x}\,dx$ (k) $\int \frac{x^4\,dx}{x^3-2x^2-7x-4}$ (l) $\int \frac{dx}{x(x^2+5)}$

(m) $\int \frac{x^2\,dx}{(x-1)(x^2+4x+5)}$ (n) $\int \frac{dx}{(x^2+1)(x^2+4)}$ (o) $\int \frac{x^4+1}{x^3+9x}\,dx$

(p) $\int \frac{dx}{x(x^2+1)^2}$ (q) $\int \frac{x^2\,dx}{(x-1)(x^2+4)^2}$ (r) $\int \frac{x^3+1}{x(x^2+x+1)^2}\,dx$

(s) $\int \frac{x-1}{x^3+2x^2-x-2}\,dx$ (t) $\int \frac{x^2+2}{x(x^2+5x+6)}\,dx$ (u) $\int \frac{dx}{1+e^x}$

40.7 Show that $p(x) = x^2 + bx + c$ is irreducible if and only if $c - (b^2/4) > 0$. [*Hint:* A quadratic polynomial is irreducible if and only if it has no linear factor; that is (by Theorem 7.2), if and only if it has no real root.]

40.8 (*a*) Find the area of the region in the first quadrant under the curve $y = 1/(x^3 + 27)$ and to the left of the line $x = 3$.

(*b*) Find the volume of the solid generated by revolving the region of part (*a*) around the y-axis.

40.9 Find $\displaystyle\int \frac{dx}{1 - \sin x}$. [*Hint:* See Problem 40.4.]

40.10 Find $\displaystyle\int \frac{\cos x\, dx}{\sin x - 1}$. (*a*) Use the method of Problem 40.4. (*b*) Use the quick formula II. (*c*) Verify that your answers are equivalent.

40.11 Evaluate the following integrals involving fractional powers:

(*a*) $\displaystyle\int \frac{dx}{\sqrt[3]{x} - x}$ (*b*) $\displaystyle\int \frac{dx}{1 + \sqrt[4]{x - 1}}$ (*c*) $\displaystyle\int \frac{dx}{x\sqrt{1 + 3x}}$ (*d*) $\displaystyle\int \frac{dx}{\sqrt[3]{x} + \sqrt{x}}$

(*e*) $\displaystyle\int \frac{dx}{\sqrt{\sqrt{x} + 1}}$ (*f*) $\displaystyle\int \sqrt{1 + e^x}\, dx$ (*g*) $\displaystyle\int \frac{x^{2/3}\, dx}{x + 1}$

[*Hints:* Part (*a*) let $x = z^3$; part (*b*) let $x - 1 = z^4$; part (*c*) let $1 + 3x = z^2$; part (*d*) let $x = z^6$.]