CHAPTER 5
Linear Second Order Equations

IN THIS CHAPTER we study a particularly important class of second order equations. Because of their many applications in science and engineering, second order differential equation have historically been the most thoroughly studied class of differential equations. Research on the theory of second order differential equations continues to the present day. This chapter is devoted to second order equations that can be written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x).$$

Such equations are said to be *linear*. As in the case of first order linear equations, (A) is said to be *homogeneous* if $F \equiv 0$, or *nonhomogeneous* if $F \not\equiv 0$.

SECTION 5.1 is devoted to the theory of homogeneous linear equations.

SECTION 5.2 deals with homogeneous equations of the special form

$$ay'' + by' + cy = 0,$$

where a, b, and c are constant ($a \neq 0$). When you've completed this section you'll know everything there is to know about solving such equations.

SECTION 5.3 presents the theory of nonhomogeneous linear equations.

SECTIONS 5.4 AND 5.5 present the *method of undetermined coefficients*, which can be used to solve nonhomogeneous equations of the form

$$ay'' + by' + cy = F(x),$$

where a, b, and c are constants and F has a special form that is still sufficiently general to occur in many applications. In this section we make extensive use of the idea of variation of parameters introduced in Chapter 2.

SECTION 5.6 deals with *reduction of order*, a technique based on the idea of variation of parameters, which enables us to find the general solution of a nonhomogeneous linear second order equation provided that we know one nontrivial (not identically zero) solution of the associated homogeneous equation.

SECTION 5.6 deals with the method traditionally called *variation of parameters*, which enables us to find the general solution of a nonhomogeneous linear second order equation provided that we know two nontrivial solutions (with nonconstant ratio) of the associated homogeneous equation.

5.1 HOMOGENEOUS LINEAR EQUATIONS

A second order differential equation is said to be *linear* if it can be written as

$$y'' + p(x)y' + q(x)y = f(x). \tag{5.1.1}$$

We call the function f on the right a *forcing function*, since in physical applications it's often related to a force acting on some system modeled by the differential equation. We say that (5.1.1) is *homogeneous* if $f \equiv 0$ or *nonhomogeneous* if $f \not\equiv 0$. Since these definitions are like the corresponding definitions in Section 2.1 for the linear first order equation

$$y' + p(x)y = f(x), \tag{5.1.2}$$

it's natural to expect similarities between methods of solving (5.1.1) and (5.1.2). However, solving (5.1.1) is more difficult than solving (5.1.2). For example, while Theorem 2.1.1 gives a formula for the general solution of (5.1.2) in the case where $f \equiv 0$ and Theorem 2.1.2 gives a formula for the case where $f \not\equiv 0$, there are no formulas for the general solution of (5.1.1) in either case. Therefore we must be content to solve linear second order equations of special forms.

In Section 2.1 we considered the homogeneous equation $y'+p(x)y = 0$ first, and then used a nontrivial solution of this equation to find the general solution of the nonhomogeneous equation $y'+p(x)y = f(x)$. Although the progression from the homogeneous to the nonhomogeneous case isn't that simple for the linear second order equation, it's still necessary to solve the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.3}$$

in order to solve the nonhomogeneous equation (5.1.1). This section is devoted to (5.1.3).

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for (5.1.3). We omit the proof.

Theorem 5.1.1 *Suppose p and q are continuous on an open interval (a,b), let x_0 be any point in (a,b), and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem*

$$y'' + p(x)y' + q(x)y = 0, \; y(x_0) = k_0, \; y'(x_0) = k_1$$

has a unique solution on (a,b).

Since $y \equiv 0$ is obviously a solution of (5.1.3) we call it the *trivial* solution. Any other solution is *nontrivial*. Under the assumptions of Theorem 5.1.1, the only solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \; y(x_0) = 0, \; y'(x_0) = 0$$

on (a,b) is the trivial solution (Exercise 24).

The next three examples illustrate concepts that we'll develop later in this section. You shouldn't be concerned with how to *find* the given solutions of the equations in these examples. This will be explained in later sections.

Example 5.1.1 The coefficients of y' and y in

$$y'' - y = 0 \tag{5.1.4}$$

are the constant functions $p \equiv 0$ and $q \equiv -1$, which are continuous on $(-\infty,\infty)$. Therefore Theorem 5.1.1 implies that every initial value problem for (5.1.4) has a unique solution on $(-\infty,\infty)$.

(a) Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of (5.1.4) on $(-\infty, \infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants, $y = c_1 e^x + c_2 e^{-x}$ is a solution of (5.1.4) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{5.1.5}$$

SOLUTION(**a**) If $y_1 = e^x$ then $y_1' = e^x$ and $y_1'' = e^x = y_1$, so $y_1'' - y_1 = 0$. If $y_2 = e^{-x}$, then $y_2' = -e^{-x}$ and $y_2'' = e^{-x} = y_2$, so $y_2'' - y_2 = 0$.

SOLUTION(**b**) If

$$y = c_1 e^x + c_2 e^{-x} \tag{5.1.6}$$

then

$$y' = c_1 e^x - c_2 e^{-x} \tag{5.1.7}$$

and

$$y'' = c_1 e^x + c_2 e^{-x},$$

so

$$\begin{aligned} y'' - y &= (c_1 e^x + c_2 e^{-x}) - (c_1 e^x + c_2 e^{-x}) \\ &= c_1(e^x - e^x) + c_2(e^{-x} - e^{-x}) = 0 \end{aligned}$$

for all x. Therefore $y = c_1 e^x + c_2 e^{-x}$ is a solution of (5.1.4) on $(-\infty, \infty)$.

SOLUTION(**c**) We can solve (5.1.5) by choosing c_1 and c_2 in (5.1.6) so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in (5.1.6) and (5.1.7) shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3. \end{aligned}$$

Solving these equations yields $c_1 = 2$ and $c_2 = -1$. Therefore $y = 2e^x - e^{-x}$ is the unique solution of (5.1.5) on $(-\infty, \infty)$.

Example 5.1.2 Let ω be a positive constant. The coefficients of y' and y in

$$y'' + \omega^2 y = 0 \tag{5.1.8}$$

are the constant functions $p \equiv 0$ and $q \equiv \omega^2$, which are continuous on $(-\infty, \infty)$. Therefore Theorem 5.1.1 implies that every initial value problem for (5.1.8) has a unique solution on $(-\infty, \infty)$.

(a) Verify that $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$ are solutions of (5.1.8) on $(-\infty, \infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 \cos \omega x + c_2 \sin \omega x$ is a solution of (5.1.8) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' + \omega^2 y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{5.1.9}$$

SOLUTION(**a**) If $y_1 = \cos\omega x$ then $y_1' = -\omega\sin\omega x$ and $y_1'' = -\omega^2\cos\omega x = -\omega^2 y_1$, so $y_1'' + \omega^2 y_1 = 0$. If $y_2 = \sin\omega x$ then, $y_2' = \omega\cos\omega x$ and $y_2'' = -\omega^2\sin\omega x = -\omega^2 y_2$, so $y_2'' + \omega^2 y_2 = 0$.

SOLUTION(**b**) If

$$y = c_1\cos\omega x + c_2\sin\omega x \tag{5.1.10}$$

then

$$y' = \omega(-c_1\sin\omega x + c_2\cos\omega x) \tag{5.1.11}$$

and

$$y'' = -\omega^2(c_1\cos\omega x + c_2\sin\omega x),$$

so

$$\begin{aligned} y'' + \omega^2 y &= -\omega^2(c_1\cos\omega x + c_2\sin\omega x) + \omega^2(c_1\cos\omega x + c_2\sin\omega x) \\ &= c_1\omega^2(-\cos\omega x + \cos\omega x) + c_2\omega^2(-\sin\omega x + \sin\omega x) = 0 \end{aligned}$$

for all x. Therefore $y = c_1\cos\omega x + c_2\sin\omega x$ is a solution of (5.1.8) on $(-\infty, \infty)$.

SOLUTION(**c**) To solve (5.1.9), we must choosing c_1 and c_2 in (5.1.10) so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in (5.1.10) and (5.1.11) shows that $c_1 = 1$ and $c_2 = 3/\omega$. Therefore

$$y = \cos\omega x + \frac{3}{\omega}\sin\omega x$$

is the unique solution of (5.1.9) on $(-\infty, \infty)$. ■

Theorem 5.1.1 implies that if k_0 and k_1 are arbitrary real numbers then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{5.1.12}$$

has a unique solution on an interval (a, b) that contains x_0, provided that P_0, P_1, and P_2 are continuous and P_0 has no zeros on (a, b). To see this, we rewrite the differential equation in (5.1.12) as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and apply Theorem 5.1.1 with $p = P_1/P_0$ and $q = P_2/P_0$.

Example 5.1.3 The equation

$$x^2y'' + xy' - 4y = 0 \tag{5.1.13}$$

has the form of the differential equation in (5.1.12), with $P_0(x) = x^2$, $P_1(x) = x$, and $P_2(x) = -4$, which are are all continuous on $(-\infty, \infty)$. However, since $P(0) = 0$ we must consider solutions of (5.1.13) on $(-\infty, 0)$ and $(0, \infty)$. Since P_0 has no zeros on these intervals, Theorem 5.1.1 implies that the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(0, \infty)$ if $x_0 > 0$, or on $(-\infty, 0)$ if $x_0 < 0$.

(a) Verify that $y_1 = x^2$ is a solution of (5.1.13) on $(-\infty, \infty)$ and $y_2 = 1/x^2$ is a solution of (5.1.13) on $(-\infty, 0)$ and $(0, \infty)$.

(b) Verify that if c_1 and c_2 are any constants then $y = c_1x^2 + c_2/x^2$ is a solution of (5.1.13) on $(-\infty, 0)$ and $(0, \infty)$.

(c) Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(1) = 2, \quad y'(1) = 0. \tag{5.1.14}$$

(d) Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 0. \tag{5.1.15}$$

SOLUTION(**a**) If $y_1 = x^2$ then $y_1' = 2x$ and $y_1'' = 2$, so

$$x^2y_1'' + xy_1' - 4y_1 = x^2(2) + x(2x) - 4x^2 = 0$$

for x in $(-\infty, \infty)$. If $y_2 = 1/x^2$, then $y_2' = -2/x^3$ and $y_2'' = 6/x^4$, so

$$x^2y_2'' + xy_2' - 4y_2 = x^2\left(\frac{6}{x^4}\right) - x\left(\frac{2}{x^3}\right) - \frac{4}{x^2} = 0$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

SOLUTION(**b**) If

$$y = c_1x^2 + \frac{c_2}{x^2} \tag{5.1.16}$$

then

$$y' = 2c_1x - \frac{2c_2}{x^3} \tag{5.1.17}$$

and

$$y'' = 2c_1 + \frac{6c_2}{x^4},$$

so

$$\begin{aligned} x^2y'' + xy' - 4y &= x^2\left(2c_1 + \frac{6c_2}{x^4}\right) + x\left(2c_1x - \frac{2c_2}{x^3}\right) - 4\left(c_1x^2 + \frac{c_2}{x^2}\right) \\ &= c_1(2x^2 + 2x^2 - 4x^2) + c_2\left(\frac{6}{x^2} - \frac{2}{x^2} - \frac{4}{x^2}\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0 \end{aligned}$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

SOLUTION(**c**) To solve (5.1.14), we choose c_1 and c_2 in (5.1.16) so that $y(1) = 2$ and $y'(1) = 0$. Setting $x = 1$ in (5.1.16) and (5.1.17) shows that this is equivalent to

$$\begin{aligned} c_1 + \ c_2 &= 2 \\ 2c_1 - 2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1 = 1$ and $c_2 = 1$. Therefore $y = x^2 + 1/x^2$ is the unique solution of (5.1.14) on $(0, \infty)$.

SOLUTION(**d**) We can solve (5.1.15) by choosing c_1 and c_2 in (5.1.16) so that $y(-1) = 2$ and $y'(-1) = 0$. Setting $x = -1$ in (5.1.16) and (5.1.17) shows that this is equivalent to

$$\begin{aligned} c_1 + \ c_2 &= 2 \\ -2c_1 + 2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1 = 1$ and $c_2 = 1$. Therefore $y = x^2 + 1/x^2$ is the unique solution of (5.1.15) on $(-\infty, 0)$. ■

Although the *formulas* for the solutions of (5.1.14) and (5.1.15) are both $y = x^2 + 1/x^2$, you should not conclude that these two initial value problems have the same solution. Remember that a solution of an initial value problem is defined *on an interval that contains the initial point*; therefore, the solution of (5.1.14) is $y = x^2 + 1/x^2$ *on the interval* $(0, \infty)$, which contains the initial point $x_0 = 1$, while the solution of (5.1.15) is $y = x^2 + 1/x^2$ *on the interval* $(-\infty, 0)$, which contains the initial point $x_0 = -1$.

The General Solution of a Homogeneous Linear Second Order Equation

If y_1 and y_2 are defined on an interval (a, b) and c_1 and c_2 are constants, then

$$y = c_1 y_1 + c_2 y_2$$

is a *linear combination of y_1 and y_2*. For example, $y = 2\cos x + 7\sin x$ is a linear combination of $y_1 = \cos x$ and $y_2 = \sin x$, with $c_1 = 2$ and $c_2 = 7$.

The next theorem states a fact that we've already verified in Examples 5.1.1, 5.1.2, and 5.1.3.

Theorem 5.1.2 *If y_1 and y_2 are solutions of the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.18}$$

on (a, b), then any linear combination

$$y = c_1 y_1 + c_2 y_2 \tag{5.1.19}$$

of y_1 and y_2 is also a solution of (5.1.18) *on (a, b).*

Proof If

$$y = c_1 y_1 + c_2 y_2$$

then

$$y' = c_1 y_1' + c_2 y_2' \quad \text{and} \quad y'' = c_1 y_1'' + c_2 y_2''.$$

Therefore

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (c_1 y_1'' + c_2 y_2'') + p(x)(c_1 y_1' + c_2 y_2') + q(x)(c_1 y_1 + c_2 y_2) \\ &= c_1 \left(y_1'' + p(x)y_1' + q(x)y_1\right) + c_2 \left(y_2'' + p(x)y_2' + q(x)y_2\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0, \end{aligned}$$

since y_1 and y_2 are solutions of (5.1.18). ■

We say that $\{y_1, y_2\}$ is a *fundamental set of solutions of* (5.1.18) *on* (a, b) if every solution of (5.1.18) on (a, b) can be written as a linear combination of y_1 and y_2 as in (5.1.19). In this case we say that (5.1.19) is *general solution of* (5.1.18) *on* (a, b).

Linear Independence

We need a way to determine whether a given set $\{y_1, y_2\}$ of solutions of (5.1.18) is a fundamental set. The next definition will enable us to state necessary and sufficient conditions for this.

We say that two functions y_1 and y_2 defined on an interval (a, b) are *linearly independent on* (a, b) if neither is a constant multiple of the other on (a, b). (In particular, this means that neither can be the trivial solution of (5.1.18), since, for example, if $y_1 \equiv 0$ we could write $y_1 = 0y_2$.) We'll also say that the set $\{y_1, y_2\}$ *is linearly independent on* (a, b).

Theorem 5.1.3 *Suppose p and q are continuous on (a,b). Then a set $\{y_1, y_2\}$ of solutions of*

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.20}$$

on (a,b) is a fundamental set if and only if $\{y_1, y_2\}$ is linearly independent on (a,b).

We'll present the proof of Theorem 5.1.3 in steps worth regarding as theorems in their own right. However, let's first interpret Theorem 5.1.3 in terms of Examples 5.1.1, 5.1.2, and 5.1.3.

Example 5.1.4

(a) Since $e^x/e^{-x} = e^{2x}$ is nonconstant, Theorem 5.1.3 implies that $y = c_1e^x + c_2e^{-x}$ is the general solution of $y'' - y = 0$ on $(-\infty, \infty)$.

(b) Since $\cos\omega x/\sin\omega x = \cot\omega x$ is nonconstant, Theorem 5.1.3 implies that $y = c_1\cos\omega x + c_2\sin\omega x$ is the general solution of $y'' + \omega^2 y = 0$ on $(-\infty, \infty)$.

(c) Since $x^2/x^{-2} = x^4$ is nonconstant, Theorem 5.1.3 implies that $y = c_1x^2 + c_2/x^2$ is the general solution of $x^2y'' + xy' - 4y = 0$ on $(-\infty, 0)$ and $(0, \infty)$.

The Wronskian and Abel's Formula

To motivate a result that we need in order to prove Theorem 5.1.3, let's see what is required to prove that $\{y_1, y_2\}$ is a fundamental set of solutions of (5.1.20) on (a,b). Let x_0 be an arbitrary point in (a,b), and suppose y is an arbitrary solution of (5.1.20) on (a,b). Then y is the unique solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1; \tag{5.1.21}$$

that is, k_0 and k_1 are the numbers obtained by evaluating y and y' at x_0. Moreover, k_0 and k_1 can be any real numbers, since Theorem 5.1.1 implies that (5.1.21) has a solution no matter how k_0 and k_1 are chosen. Therefore $\{y_1, y_2\}$ is a fundamental set of solutions of (5.1.20) on (a,b) if and only if it's possible to write the solution of an arbitrary initial value problem (5.1.21) as $y = c_1y_1 + c_2y_2$. This is equivalent to requiring that the system

$$\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) &= k_0 \\ c_1y_1'(x_0) + c_2y_2'(x_0) &= k_1 \end{aligned} \tag{5.1.22}$$

has a solution (c_1, c_2) for every choice of (k_0, k_1). Let's try to solve (5.1.22).

Multiplying the first equation in (5.1.22) by $y_2'(x_0)$ and the second by $y_2(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_2'(x_0) + c_2y_2(x_0)y_2'(x_0) &= y_2'(x_0)k_0 \\ c_1y_1'(x_0)y_2(x_0) + c_2y_2'(x_0)y_2(x_0) &= y_2(x_0)k_1, \end{aligned}$$

and subtracting the second equation here from the first yields

$$\left(y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)\right)c_1 = y_2'(x_0)k_0 - y_2(x_0)k_1. \tag{5.1.23}$$

Multiplying the first equation in (5.1.22) by $y_1'(x_0)$ and the second by $y_1(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_1'(x_0) + c_2y_2(x_0)y_1'(x_0) &= y_1'(x_0)k_0 \\ c_1y_1'(x_0)y_1(x_0) + c_2y_2'(x_0)y_1(x_0) &= y_1(x_0)k_1, \end{aligned}$$

and subtracting the first equation here from the second yields

$$\left(y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)\right)c_2 = y_1(x_0)k_1 - y_1'(x_0)k_0. \tag{5.1.24}$$

If

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) = 0,$$

it's impossible to satisfy (5.1.23) and (5.1.24) (and therefore (5.1.22)) unless k_0 and k_1 happen to satisfy

$$\begin{aligned} y_1(x_0)k_1 - y_1'(x_0)k_0 &= 0 \\ y_2'(x_0)k_0 - y_2(x_0)k_1 &= 0. \end{aligned}$$

On the other hand, if

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0 \tag{5.1.25}$$

we can divide (5.1.23) and (5.1.24) through by the quantity on the left to obtain

$$\begin{aligned} c_1 &= \frac{y_2'(x_0)k_0 - y_2(x_0)k_1}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)} \\ c_2 &= \frac{y_1(x_0)k_1 - y_1'(x_0)k_0}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)}, \end{aligned} \tag{5.1.26}$$

no matter how k_0 and k_1 are chosen. This motivates us to consider conditions on y_1 and y_2 that imply (5.1.25).

Theorem 5.1.4 *Suppose p and q are continuous on (a,b), let y_1 and y_2 be solutions of*

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.27}$$

on (a,b), and define

$$W = y_1y_2' - y_1'y_2. \tag{5.1.28}$$

Let x_0 be any point in (a,b). Then

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p(t)\,dt}, \quad a < x < b. \tag{5.1.29}$$

Therefore either W has no zeros in (a,b) or $W \equiv 0$ on (a,b).

Proof Differentiating (5.1.28) yields

$$W' = y_1'y_2' + y_1y_2'' - y_1'y_2' - y_1''y_2 = y_1y_2'' - y_1''y_2. \tag{5.1.30}$$

Since y_1 and y_2 both satisfy (5.1.27),

$$y_1'' = -py_1' - qy_1 \quad \text{and} \quad y_2'' = -py_2' - qy_2.$$

Substituting these into (5.1.30) yields

$$\begin{aligned} W' &= -y_1\left(py_2' + qy_2\right) + y_2\left(py_1' + qy_1\right) \\ &= -p(y_1y_2' - y_2y_1') - q(y_1y_2 - y_2y_1) \\ &= -p(y_1y_2' - y_2y_1') = -pW. \end{aligned}$$

Therefore $W' + p(x)W = 0$; that is, W is the solution of the initial value problem

$$y' + p(x)y = 0, \quad y(x_0) = W(x_0).$$

We leave it to you to verify by separation of variables that this implies (5.1.29). If $W(x_0) \neq 0$, (5.1.29) implies that W has no zeros in (a, b), since an exponential is never zero. On the other hand, if $W(x_0) = 0$, (5.1.29) implies that $W(x) = 0$ for all x in (a, b). ■

The function W defined in (5.1.28) is the *Wronskian of* $\{y_1, y_2\}$. Formula (5.1.29) is *Abel's formula*.

The Wronskian of $\{y_1, y_2\}$ is usually written as the determinant

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

The expressions in (5.1.26) for c_1 and c_2 can be written in terms of determinants as

$$c_1 = \frac{1}{W(x_0)} \begin{vmatrix} k_0 & y_2(x_0) \\ k_1 & y_2'(x_0) \end{vmatrix} \quad \text{and} \quad c_2 = \frac{1}{W(x_0)} \begin{vmatrix} y_1(x_0) & k_0 \\ y_1'(x_0) & k_1 \end{vmatrix}.$$

If you've taken linear algebra you may recognize this as *Cramer's rule*.

Example 5.1.5 Verify Abel's formula for the following differential equations and the corresponding solutions, from Examples 5.1.1, 5.1.2, and 5.1.3:

(a) $y'' - y = 0; \quad y_1 = e^x, \; y_2 = e^{-x}$

(b) $y'' + \omega^2 y = 0; \quad y_1 = \cos \omega x, \; y_2 = \sin \omega x$

(c) $x^2 y'' + xy' - 4y = 0; \quad y_1 = x^2, \; y_2 = 1/x^2$

SOLUTION**(a)** Since $p \equiv 0$, we can verify Abel's formula by showing that W is constant, which is true, since

$$W(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^x e^{-x} = -2$$

for all x.

SOLUTION**(b)** Again, since $p \equiv 0$, we can verify Abel's formula by showing that W is constant, which is true, since

$$\begin{aligned} W(x) &= \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix} \\ &= \cos \omega x (\omega \cos \omega x) - (-\omega \sin \omega x) \sin \omega x \\ &= \omega(\cos^2 \omega x + \sin^2 \omega x) = \omega \end{aligned}$$

for all x.

SOLUTION**(c)** Computing the Wronskian of $y_1 = x^2$ and $y_2 = 1/x^2$ directly yields

$$W = \begin{vmatrix} x^2 & 1/x^2 \\ 2x & -2/x^3 \end{vmatrix} = x^2 \left(-\frac{2}{x^3}\right) - 2x \left(\frac{1}{x^2}\right) = -\frac{4}{x}. \tag{5.1.31}$$

To verify Abel's formula we rewrite the differential equation as

$$y'' + \frac{1}{x} y' - \frac{4}{x^2} y = 0$$

to see that $p(x) = 1/x$. If x_0 and x are either both in $(-\infty, 0)$ or both in $(0, \infty)$ then

$$\int_{x_0}^{x} p(t)\,dt = \int_{x_0}^{x} \frac{dt}{t} = \ln\left(\frac{x}{x_0}\right),$$

so Abel's formula becomes

$$\begin{aligned} W(x) &= W(x_0)e^{-\ln(x/x_0)} = W(x_0)\frac{x_0}{x} \\ &= -\left(\frac{4}{x_0}\right)\left(\frac{x_0}{x}\right) \quad \text{from (5.1.31)} \\ &= -\frac{4}{x}, \end{aligned}$$

which is consistent with (5.1.31). ■

The next theorem will enable us to complete the proof of Theorem 5.1.3.

Theorem 5.1.5 *Suppose p and q are continuous on an open interval (a,b), let y_1 and y_2 be solutions of*

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.32}$$

on (a,b), and let $W = y_1y_2' - y_1'y_2$. Then y_1 and y_2 are linearly independent on (a,b) if and only if W has no zeros on (a,b).

Proof We first show that if $W(x_0) = 0$ for some x_0 in (a,b), then y_1 and y_2 are linearly dependent on (a,b). Let I be a subinterval of (a,b) on which y_1 has no zeros. (If there's no such subinterval, $y_1 \equiv 0$ on (a,b), so y_1 and y_2 are linearly independent, and we're finished with this part of the proof.) Then y_2/y_1 is defined on I, and

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W}{y_1^2}. \tag{5.1.33}$$

However, if $W(x_0) = 0$, Theorem 5.1.4 implies that $W \equiv 0$ on (a,b). Therefore (5.1.33) implies that $(y_2/y_1)' \equiv 0$, so $y_2/y_1 = c$ (constant) on I. This shows that $y_2(x) = cy_1(x)$ for all x in I. However, we want to show that $y_2 = cy_1(x)$ for all x in (a,b). Let $Y = y_2 - cy_1$. Then Y is a solution of (5.1.32) on (a,b) such that $Y \equiv 0$ on I, and therefore $Y' \equiv 0$ on I. Consequently, if x_0 is chosen arbitrarily in I then Y is a solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0,$$

which implies that $Y \equiv 0$ on (a,b), by the paragraph following Theorem 5.1.1. (See also Exercise 24). Hence, $y_2 - cy_1 \equiv 0$ on (a,b), which implies that y_1 and y_2 are not linearly independent on (a,b).

Now suppose W has no zeros on (a,b). Then y_1 can't be identically zero on (a,b) (why not?), and therefore there is a subinterval I of (a,b) on which y_1 has no zeros. Since (5.1.33) implies that y_2/y_1 is nonconstant on I, y_2 isn't a constant multiple of y_1 on (a,b). A similar argument shows that y_1 isn't a constant multiple of y_2 on (a,b), since

$$\left(\frac{y_1}{y_2}\right)' = \frac{y_1'y_2 - y_1y_2'}{y_2^2} = -\frac{W}{y_2^2}$$

on any subinterval of (a,b) where y_2 has no zeros. ■

We can now complete the proof of Theorem 5.1.3. From Theorem 5.1.5, two solutions y_1 and y_2 of (5.1.32) are linearly independent on (a,b) if and only if W has no zeros on (a,b). From Theorem 5.1.4 and the motivating comments preceding it, $\{y_1, y_2\}$ is a fundamental set of solutions of (5.1.32) if and only if W has no zeros on (a,b). Therefore $\{y_1, y_2\}$ is a fundamental set for (5.1.32) on (a,b) if and only if $\{y_1, y_2\}$ is linearly independent on (a,b). ■

The next theorem summarizes the relationships among the concepts discussed in this section.

Theorem 5.1.6 *Suppose p and q are continuous on an open interval (a,b) and let y_1 and y_2 be solutions of*

$$y''+p(x)y'+q(x)y=0 \tag{5.1.34}$$

on (a,b). Then the following statements are equivalent; that is, they are either all true or all false.

(a) *The general solution of (5.1.34) on (a,b) is $y=c_1y_1+c_2y_2$.*

(b) *$\{y_1,y_2\}$ is a fundamental set of solutions of (5.1.34) on (a,b).*

(c) *$\{y_1,y_2\}$ is linearly independent on (a,b).*

(d) *The Wronskian of $\{y_1,y_2\}$ is nonzero at some point in (a,b).*

(e) *The Wronskian of $\{y_1,y_2\}$ is nonzero at all points in (a,b).*

We can apply this theorem to an equation written as

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0$$

on an interval (a,b) where P_0, P_1, and P_2 are continuous and P_0 has no zeros.

Theorem 5.1.7 *Suppose c is in (a,b) and α and β are real numbers, not both zero. Under the assumptions of Theorem 5.1.7, suppose y_1 and y_2 are solutions of (5.1.34) such that*

$$\alpha y_1(c)+\beta y_1'(c)=0 \quad \textit{and} \quad \alpha y_2(c)+\beta y_2'(c)=0. \tag{5.1.35}$$

Then $\{y_1,y_2\}$ isn't linearly independent on (a,b).

Proof Since α and β are not both zero, (5.1.35) implies that

$$\left|\begin{array}{cc} y_1(c) & y_1'(c) \\ y_2(c) & y_2'(c) \end{array}\right|=0, \quad \text{so} \quad \left|\begin{array}{cc} y_1(c) & y_2(c) \\ y_1'(c) & y_2'(c) \end{array}\right|=0$$

and Theorem 5.1.6 implies the stated conclusion.

5.1 Exercises

1. **(a)** Verify that $y_1=e^{2x}$ and $y_2=e^{5x}$ are solutions of

$$y''-7y'+10y=0 \tag{A}$$

on $(-\infty,\infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants then $y=c_1e^{2x}+c_2e^{5x}$ is a solution of (A) on $(-\infty,\infty)$.

(c) Solve the initial value problem

$$y''-7y'+10y=0, \quad y(0)=-1, \quad y'(0)=1.$$

(d) Solve the initial value problem

$$y''-7y'+10y=0, \quad y(0)=k_0, \quad y'(0)=k_1.$$

2. **(a)** Verify that $y_1=e^x\cos x$ and $y_2=e^x\sin x$ are solutions of

$$y''-2y'+2y=0 \tag{A}$$

on $(-\infty,\infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants then $y = c_1e^x \cos x + c_2e^x \sin x$ is a solution of (A) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = 3, \quad y'(0) = -2.$$

(d) Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

3. **(a)** Verify that $y_1 = e^x$ and $y_2 = xe^x$ are solutions of

$$y'' - 2y' + y = 0 \qquad \text{(A)}$$

on $(-\infty, \infty)$.

(b) Verify that if c_1 and c_2 are arbitrary constants then $y = e^x(c_1 + c_2x)$ is a solution of (A) on $(-\infty, \infty)$.

(c) Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 7, \quad y'(0) = 4.$$

(d) Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

4. **(a)** Verify that $y_1 = 1/(x - 1)$ and $y_2 = 1/(x + 1)$ are solutions of

$$(x^2 - 1)y'' + 4xy' + 2y = 0 \qquad \text{(A)}$$

on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. What is the general solution of (A) on each of these intervals?

(b) Solve the initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = 0, \quad y(0) = -5, \quad y'(0) = 1.$$

What is the interval of validity of the solution?

(c) C/G Graph the solution of the initial value problem.

(d) Verify Abel's formula for y_1 and y_2, with $x_0 = 0$.

5. Compute the Wronskians of the given sets of functions.

(a) $\{1, e^x\}$

(b) $\{e^x, e^x \sin x\}$

(c) $\{x + 1, x^2 + 2\}$

(d) $\{x^{1/2}, x^{-1/3}\}$

(e) $\left\{\frac{\sin x}{x}, \frac{\cos x}{x}\right\}$

(f) $\{x \ln |x|, x^2 \ln |x|\}$

(g) $\{e^x \cos \sqrt{x}, e^x \sin \sqrt{x}\}$

6. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$y'' + 3(x^2 + 1)y' - 2y = 0,$$

given that $W(\pi) = 0$.

7. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0,$$

given that $W(0) = 1$. (This is *Legendre's equation*.)

8. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

given that $W(1) = 1$. (This is *Bessel's equation*.)

9. (This exercise shows that if you know one nontrivial solution of $y'' + p(x)y' + q(x)y = 0$, you can use Abel's formula to find another.)

Suppose p and q are continuous and y_1 is a solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

that has no zeros on (a,b). Let $P(x) = \int p(x)\,dx$ be any antiderivative of p on (a,b).

(a) Show that if K is an arbitrary nonzero constant and y_2 satisfies

$$y_1y_2' - y_1'y_2 = Ke^{-P(x)} \tag{B}$$

on (a,b), then y_2 also satisfies (A) on (a,b), and $\{y_1, y_2\}$ is a fundamental set of solutions on (A) on (a,b).

(b) Conclude from **(a)** that if $y_2 = uy_1$ where $u' = K\dfrac{e^{-P(x)}}{y_1^2(x)}$, then $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on (a,b).

In Exercises 10–23 use the method suggested by Exercise 9 to find a second solution y_2 that isn't a constant multiple of the solution y_1. Choose K conveniently to simplify y_2.

10. $y'' - 2y' - 3y = 0; \quad y_1 = e^{3x}$

11. $y'' - 6y' + 9y = 0; \quad y_1 = e^{3x}$

12. $y'' - 2ay' + a^2y = 0$ (a = constant); $\quad y_1 = e^{ax}$

13. $x^2y'' + xy' - y = 0; \quad y_1 = x$

14. $x^2y'' - xy' + y = 0; \quad y_1 = x$

15. $x^2y'' - (2a-1)xy' + a^2y = 0$ (a = nonzero constant); $x > 0; \quad y_1 = x^a$

16. $4x^2y'' - 4xy' + (3 - 16x^2)y = 0; \quad y_1 = x^{1/2}e^{2x}$

17. $(x-1)y'' - xy' + y = 0; \quad y_1 = e^x$

18. $x^2y'' - 2xy' + (x^2+2)y = 0; \quad y_1 = x\cos x$

19. $4x^2(\sin x)y'' - 4x(x\cos x + \sin x)y' + (2x\cos x + 3\sin x)y = 0; \quad y_1 = x^{1/2}$

20. $(3x-1)y'' - (3x+2)y' - (6x-8)y = 0; \quad y_1 = e^{2x}$

21. $(x^2-4)y'' + 4xy' + 2y = 0; \quad y_1 = \dfrac{1}{x-2}$

22. $(2x+1)xy'' - 2(2x^2-1)y' - 4(x+1)y = 0; \quad y_1 = \dfrac{1}{x}$

23. $(x^2-2x)y'' + (2-x^2)y' + (2x-2)y = 0; \quad y_1 = e^x$

24. Suppose p and q are continuous on an open interval (a,b) and let x_0 be in (a,b). Use Theorem 5.1.1 to show that the only solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0$$

on (a,b) is the trivial solution $y \equiv 0$.

25. Suppose P_0, P_1, and P_2 are continuous on (a,b) and let x_0 be in (a,b). Show that if either of the following statements is true then $P_0(x) = 0$ for some x in (a,b).

(a) The initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has more than one solution on (a,b).

(b) The initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0$$

has a nontrivial solution on (a,b).

26. Suppose p and q are continuous on (a,b) and y_1 and y_2 are solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

on (a,b). Let

$$z_1 = \alpha y_1 + \beta y_2 \quad \text{and} \quad z_2 = \gamma y_1 + \delta y_2,$$

where α, β, γ, and δ are constants. Show that if $\{z_1, z_2\}$ is a fundamental set of solutions of (A) on (a,b) then so is $\{y_1, y_2\}$.

27. Suppose p and q are continuous on (a,b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

on (a,b). Let

$$z_1 = \alpha y_1 + \beta y_2 \quad \text{and} \quad z_2 = \gamma y_1 + \delta y_2,$$

where α, β, γ, and δ are constants. Show that $\{z_1, z_2\}$ is a fundamental set of solutions of (A) on (a,b) if and only if $\alpha\gamma - \beta\delta \neq 0$.

28. Suppose y_1 is differentiable on an interval (a,b) and $y_2 = ky_1$, where k is a constant. Show that the Wronskian of $\{y_1, y_2\}$ is identically zero on (a,b).

29. Let

$$y_1 = x^3 \quad \text{and} \quad y_2 = \begin{cases} x^3, & x \geq 0, \\ -x^3, & x < 0. \end{cases}$$

(a) Show that the Wronskian of $\{y_1, y_2\}$ is defined and identically zero on $(-\infty, \infty)$.

(b) Suppose $a < 0 < b$. Show that $\{y_1, y_2\}$ is linearly independent on (a,b).

(c) Use Exercise 25**(b)** to show that these results don't contradict Theorem 5.1.5, because neither y_1 nor y_2 can be a solution of an equation

$$y'' + p(x)y' + q(x)y = 0$$

on (a,b) if p and q are continuous on (a,b).

30. Suppose p and q are continuous on (a,b) and $\{y_1, y_2\}$ is a set of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on (a,b) such that either $y_1(x_0) = y_2(x_0) = 0$ or $y_1'(x_0) = y_2'(x_0) = 0$ for some x_0 in (a,b). Show that $\{y_1, y_2\}$ is linearly dependent on (a,b).

31. Suppose p and q are continuous on (a,b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on (a,b). Show that if $y_1(x_1) = y_1(x_2) = 0$, where $a < x_1 < x_2 < b$, then $y_2(x) = 0$ for some x in (x_1, x_2). HINT: *Show that if y_2 has no zeros in (x_1, x_2), then y_1/y_2 is either strictly increasing or strictly decreasing on (x_1, x_2), and deduce a contradiction.*

32. Suppose p and q are continuous on (a,b) and every solution of

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(A)}$$

on (a,b) can be written as a linear combination of the twice differentiable functions $\{y_1, y_2\}$. Use Theorem 5.1.1 to show that y_1 and y_2 are themselves solutions of (A) on (a,b).

33. Suppose p_1, p_2, q_1, and q_2 are continuous on (a,b) and the equations

$$y'' + p_1(x)y' + q_1(x)y = 0 \quad \text{and} \quad y'' + p_2(x)y' + q_2(x)y = 0$$

have the same solutions on (a,b). Show that $p_1 = p_2$ and $q_1 = q_2$ on (a,b). HINT: *Use Abel's formula.*

34. (For this exercise you have to know about 3×3 determinants.) Show that if y_1 and y_2 are twice continuously differentiable on (a,b) and the Wronskian W of $\{y_1, y_2\}$ has no zeros in (a,b) then the equation

$$\frac{1}{W}\begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix} = 0$$

can be written as

$$y'' + p(x)y' + q(x)y = 0, \qquad \text{(A)}$$

where p and q are continuous on (a,b) and $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on (a,b). HINT: *Expand the determinant by cofactors of its first column.*

35. Use the method suggested by Exercise 34 to find a linear homogeneous equation for which the given functions form a fundamental set of solutions on some interval.

(a) $e^x \cos 2x, \quad e^x \sin 2x$ **(b)** $x, \quad e^{2x}$

(c) $x, \quad x \ln x$ **(d)** $\cos(\ln x), \quad \sin(\ln x)$

(e) $\cosh x, \quad \sinh x$ **(f)** $x^2 - 1, \quad x^2 + 1$

36. Suppose p and q are continuous on (a,b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{(A)}$$

on (a,b). Show that if y is a solution of (A) on (a,b), there's exactly one way to choose c_1 and c_2 so that $y = c_1y_1 + c_2y_2$ on (a,b).

37. Suppose p and q are continuous on (a,b) and x_0 is in (a,b). Let y_1 and y_2 be the solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \; y_2'(x_0) = 1.$$

(Theorem 5.1.1 implies that each of these initial value problems has a unique solution on (a,b).)

(a) Show that $\{y_1, y_2\}$ is linearly independent on (a,b).

(b) Show that an arbitrary solution y of (A) on (a,b) can be written as $y = y(x_0)y_1 + y'(x_0)y_2$.

(c) Express the solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2.

38. Find solutions y_1 and y_2 of the equation $y'' = 0$ that satisfy the initial conditions

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Then use Exercise 37 **(c)** to write the solution of the initial value problem

$$y'' = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

as a linear combination of y_1 and y_2.

39. Let x_0 be an arbitrary real number. Given (Example 5.1.1) that e^x and e^{-x} are solutions of $y'' - y = 0$, find solutions y_1 and y_2 of $y'' - y = 0$ such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \; y_2'(x_0) = 1.$$

Then use Exercise 37 **(c)** to write the solution of the initial value problem

$$y'' - y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2.

40. Let x_0 be an arbitrary real number. Given (Example 5.1.2) that $\cos\omega x$ and $\sin\omega x$ are solutions of $y'' + \omega^2 y = 0$, find solutions of $y'' + \omega^2 y = 0$ such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \; y_2'(x_0) = 1.$$

Then use Exercise 37 **(c)** to write the solution of the initial value problem

$$y'' + \omega^2 y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2. Use the identities

$$\begin{aligned} \cos(A+B) &= \cos A\cos B - \sin A\sin B \\ \sin(A+B) &= \sin A\cos B + \cos A\sin B \end{aligned}$$

to simplify your expressions for y_1, y_2, and y.

41. Recall from Exercise 4 that $1/(x-1)$ and $1/(x+1)$ are solutions of

$$(x^2-1)y''+4xy'+2y=0 \tag{A}$$

on $(-1,1)$. Find solutions of (A) such that

$$y_1(0)=1, \quad y_1'(0)=0 \quad \text{and} \quad y_2(0)=0,\ y_2'(0)=1.$$

Then use Exercise 37 **(c)** to write the solution of initial value problem

$$(x^2-1)y''+4xy'+2y=0, \quad y(0)=k_0, \quad y'(0)=k_1$$

as a linear combination of y_1 and y_2.

42. **(a)** Verify that $y_1=x^2$ and $y_2=x^3$ satisfy

$$x^2y''-4xy'+6y=0 \tag{A}$$

on $(-\infty,\infty)$ and that $\{y_1,y_2\}$ is a fundamental set of solutions of (A) on $(-\infty,0)$ and $(0,\infty)$.

(b) Let a_1, a_2, b_1, and b_2 be constants. Show that

$$y=\begin{cases} a_1x^2+a_2x^3, & x\ge 0,\\ b_1x^2+b_2x^3, & x<0\end{cases}$$

is a solution of (A) on $(-\infty,\infty)$ if and only if $a_1=b_1$. From this, justify the statement that y is a solution of (A) on $(-\infty,\infty)$ if and only if

$$y=\begin{cases} c_1x^2+c_2x^3, & x\ge 0,\\ c_1x^2+c_3x^3, & x<0,\end{cases}$$

where c_1, c_2, and c_3 are arbitrary constants.

(c) For what values of k_0 and k_1 does the initial value problem

$$x^2y''-4xy'+6y=0, \quad y(0)=k_0, \quad y'(0)=k_1$$

have a solution? What are the solutions?

(d) Show that if $x_0\ne 0$ and k_0,k_1 are arbitrary constants, the initial value problem

$$x^2y''-4xy'+6y=0, \quad y(x_0)=k_0, \quad y'(x_0)=k_1 \tag{B}$$

has infinitely many solutions on $(-\infty,\infty)$. On what interval does (B) have a unique solution?

43. **(a)** Verify that $y_1=x$ and $y_2=x^2$ satisfy

$$x^2y''-2xy'+2y=0 \tag{A}$$

on $(-\infty,\infty)$ and that $\{y_1,y_2\}$ is a fundamental set of solutions of (A) on $(-\infty,0)$ and $(0,\infty)$.

(b) Let a_1, a_2, b_1, and b_2 be constants. Show that

$$y=\begin{cases} a_1x+a_2x^2, & x\ge 0,\\ b_1x+b_2x^2, & x<0\end{cases}$$

is a solution of (A) on $(-\infty,\infty)$ if and only if $a_1=b_1$ and $a_2=b_2$. From this, justify the statement that the general solution of (A) on $(-\infty,\infty)$ is $y=c_1x+c_2x^2$, where c_1 and c_2 are arbitrary constants.

(c) For what values of k_0 and k_1 does the initial value problem

$$x^2y'' - 2xy' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

(d) Show that if $x_0 \neq 0$ and k_0, k_1 are arbitrary constants then the initial value problem

$$x^2y'' - 2xy' + 2y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(-\infty, \infty)$.

44. **(a)** Verify that $y_1 = x^3$ and $y_2 = x^4$ satisfy

$$x^2y'' - 6xy' + 12y = 0 \tag{A}$$

on $(-\infty, \infty)$, and that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on $(-\infty, 0)$ and $(0, \infty)$.

(b) Show that y is a solution of (A) on $(-\infty, \infty)$ if and only if

$$y = \begin{cases} a_1x^3 + a_2x^4, & x \geq 0, \\ b_1x^3 + b_2x^4, & x < 0, \end{cases}$$

where a_1, a_2, b_1, and b_2 are arbitrary constants.

(c) For what values of k_0 and k_1 does the initial value problem

$$x^2y'' - 6xy' + 12y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

(d) Show that if $x_0 \neq 0$ and k_0, k_1 are arbitrary constants then the initial value problem

$$x^2y'' - 6xy' + 12y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{B}$$

has infinitely many solutions on $(-\infty, \infty)$. On what interval does (B) have a unique solution?

5.2 CONSTANT COEFFICIENT HOMOGENEOUS EQUATIONS

If a, b, and c are real constants and $a \neq 0$, then

$$ay'' + by' + cy = F(x)$$

is said to be a *constant coefficient equation*. In this section we consider the homogeneous constant coefficient equation

$$ay'' + by' + cy = 0. \tag{5.2.1}$$

As we'll see, all solutions of (5.2.1) are defined on $(-\infty, \infty)$. This being the case, we'll omit references to the interval on which solutions are defined, or on which a given set of solutions is a fundamental set, etc., since the interval will always be $(-\infty, \infty)$.

The key to solving (5.2.1) is that if $y = e^{rx}$ where r is a constant then the left side of (5.2.1) is a multiple of e^{rx}; thus, if $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2e^{rx}$, so

$$ay'' + by' + cy = ar^2e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx}. \tag{5.2.2}$$

The quadratic polynomial

$$p(r) = ar^2 + br + c$$

is the *characteristic polynomial* of (5.2.1), and $p(r) = 0$ is the *characteristic equation*. From (5.2.2) we can see that $y = e^{rx}$ is a solution of (5.2.1) if and only if $p(r) = 0$.

The roots of the characteristic equation are given by the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{5.2.3}$$

We consider three cases:

CASE 1. $b^2 - 4ac > 0$, so the characteristic equation has two distinct real roots.

CASE 2. $b^2 - 4ac = 0$, so the characteristic equation has a repeated real root.

CASE 3. $b^2 - 4ac < 0$, so the characteristic equation has complex roots.

In each case we'll start with an example.

Case 1: Distinct Real Roots

Example 5.2.1

(a) Find the general solution of

$$y'' + 6y' + 5y = 0. \tag{5.2.4}$$

(b) Solve the initial value problem

$$y'' + 6y' + 5y = 0, \quad y(0) = 3, \ y'(0) = -1. \tag{5.2.5}$$

SOLUTION(**a**) The characteristic polynomial of (5.2.4) is

$$p(r) = r^2 + 6r + 5 = (r + 1)(r + 5).$$

Since $p(-1) = p(-5) = 0$, $y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are solutions of (5.2.4). Since $y_2/y_1 = e^{-4x}$ is nonconstant, 5.1.6 implies that the general solution of (5.2.4) is

$$y = c_1e^{-x} + c_2e^{-5x}. \tag{5.2.6}$$

SOLUTION(**b**) We must determine c_1 and c_2 in (5.2.6) so that y satisfies the initial conditions in (5.2.5). Differentiating (5.2.6) yields

$$y' = -c_1e^{-x} - 5c_2e^{-5x}. \tag{5.2.7}$$

Imposing the initial conditions $y(0) = 3,\ y'(0) = -1$ in (5.2.6) and (5.2.7) yields

$$\begin{aligned} c_1 + \ c_2 &= \ \ 3 \\ -c_1 - 5c_2 &= -1. \end{aligned}$$

The solution of this system is $c_1 = 7/2, c_2 = -1/2$. Therefore the solution of (5.2.5) is

$$y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}.$$

Figure 5.2.1 is a graph of this solution.

If the characteristic equation has arbitrary distinct real roots r_1 and r_2, then $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are solutions of $ay'' + by' + cy = 0$. Since $y_2/y_1 = e^{(r_2-r_1)x}$ is nonconstant, Theorem 5.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of $ay'' + by' + cy = 0$.

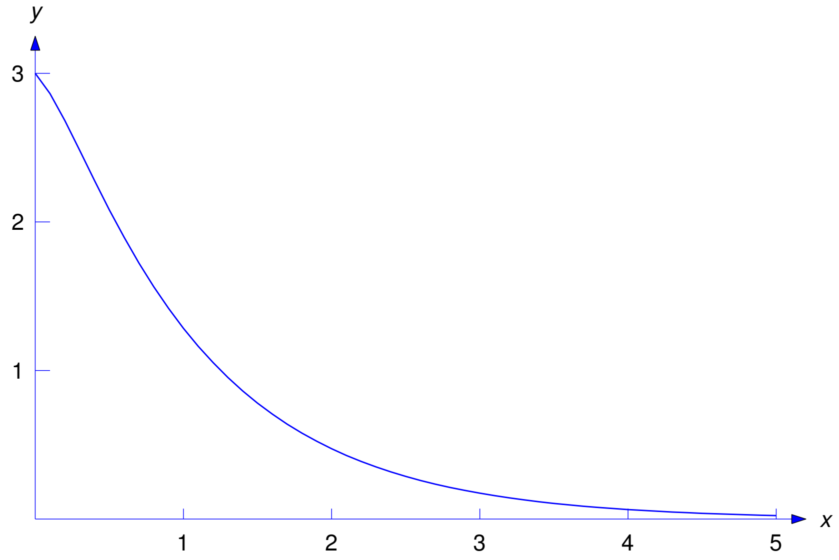

Figure 5.2.1 $y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}$

Case 2: A Repeated Real Root

Example 5.2.2

(a) Find the general solution of

$$y'' + 6y' + 9y = 0. \tag{5.2.8}$$

(b) Solve the initial value problem

$$y'' + 6y' + 9y = 0, \quad y(0) = 3, \; y'(0) = -1. \tag{5.2.9}$$

SOLUTION(**a**) The characteristic polynomial of (5.2.8) is

$$p(r) = r^2 + 6r + 9 = (r + 3)^2,$$

so the characteristic equation has the repeated real root $r_1 = -3$. Therefore $y_1 = e^{-3x}$ is a solution of (5.2.8). Since the characteristic equation has no other roots, (5.2.8) has no other solutions of the form e^{rx}. We look for solutions of the form $y = uy_1 = ue^{-3x}$, where u is a function that we'll now determine. (This should remind you of the method of variation of parameters used in Section 2.1 to solve the nonhomogeneous equation $y' + p(x)y = f(x)$, given a solution y_1 of the complementary equation $y' + p(x)y = 0$. It's also a special case of a method called *reduction of order* that we'll study in Section 5.6. For other ways to obtain a second solution of (5.2.8) that's not a multiple of e^{-3x}, see Exercises 5.1.9, 5.1.12, and 33.

If $y = ue^{-3x}$, then

$$y' = u'e^{-3x} - 3ue^{-3x} \quad \text{and} \quad y'' = u''e^{-3x} - 6u'e^{-3x} + 9ue^{-3x},$$

so

$$\begin{aligned} y'' + 6y' + 9y &= e^{-3x}\left[(u'' - 6u' + 9u) + 6(u' - 3u) + 9u\right] \\ &= e^{-3x}\left[u'' - (6-6)u' + (9 - 18 + 9)u\right] = u''e^{-3x}. \end{aligned}$$

Therefore $y = ue^{-3x}$ is a solution of (5.2.8) if and only if $u'' = 0$, which is equivalent to $u = c_1 + c_2x$, where c_1 and c_2 are constants. Therefore any function of the form

$$y = e^{-3x}(c_1 + c_2x) \tag{5.2.10}$$

is a solution of (5.2.8). Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-3x}$ that we already knew. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = xe^{-3x}$. Since $y_2/y_1 = x$ is nonconstant, 5.1.6 implies that $\{y_1, y_2\}$ is fundamental set of solutions of (5.2.8), and (5.2.10) is the general solution.

SOLUTION(**b**) Differentiating (5.2.10) yields

$$y' = -3e^{-3x}(c_1 + c_2x) + c_2e^{-3x}. \tag{5.2.11}$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ in (5.2.10) and (5.2.11) yields $c_1 = 3$ and $-3c_1 + c_2 = -1$, so $c_2 = 8$. Therefore the solution of (5.2.9) is

$$y = e^{-3x}(3 + 8x).$$

Figure 5.2.2 is a graph of this solution.

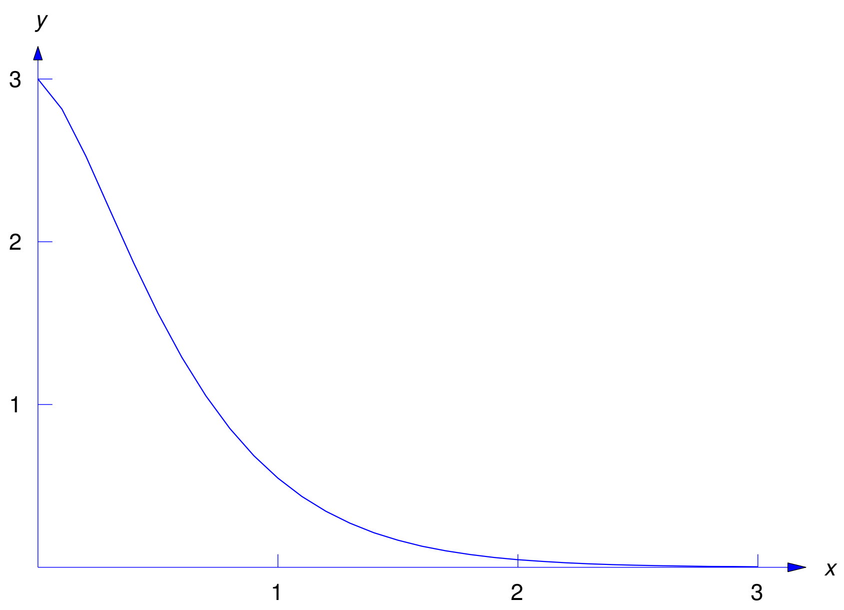

Figure 5.2.2 $y = e^{-3x}(3 + 8x)$

If the characteristic equation of $ay''+by'+cy=0$ has an arbitrary repeated root r_1, the characteristic polynomial must be

$$p(r)=a(r-r_1)^2=a(r^2-2r_1r+r_1^2).$$

Therefore

$$ar^2+br+c=ar^2-(2ar_1)r+ar_1^2,$$

which implies that $b=-2ar_1$ and $c=ar_1^2$. Therefore $ay''+by'+cy=0$ can be written as $a(y''-2r_1y'+r_1^2y)=0$. Since $a\neq 0$ this equation has the same solutions as

$$y''-2r_1y'+r_1^2y=0. \tag{5.2.12}$$

Since $p(r_1)=0$, t $y_1=e^{r_1x}$ is a solution of $ay''+by'+cy=0$, and therefore of (5.2.12). Proceeding as in Example 5.2.2, we look for other solutions of (5.2.12) of the form $y=ue^{r_1x}$; then

$$y'=u'e^{r_1x}+rue^{r_1x} \quad \text{and} \quad y''=u''e^{r_1x}+2r_1u'e^{r_1x}+r_1^2ue^{r_1x},$$

so

$$\begin{aligned} y''-2r_1y'+r_1^2y &= e^{rx}\left[(u''+2r_1u'+r_1^2u)-2r_1(u'+r_1u)+r_1^2u\right] \\ &= e^{r_1x}\left[u''+(2r_1-2r_1)u'+(r_1^2-2r_1^2+r_1^2)u\right]=u''e^{r_1x}. \end{aligned}$$

Therefore $y=ue^{r_1x}$ is a solution of (5.2.12) if and only if $u''=0$, which is equivalent to $u=c_1+c_2x$, where c_1 and c_2 are constants. Hence, any function of the form

$$y=e^{r_1x}(c_1+c_2x) \tag{5.2.13}$$

is a solution of (5.2.12). Letting $c_1=1$ and $c_2=0$ here yields the solution $y_1=e^{r_1x}$ that we already knew. Letting $c_1=0$ and $c_2=1$ yields the second solution $y_2=xe^{r_1x}$. Since $y_2/y_1=x$ is nonconstant, 5.1.6 implies that $\{y_1,y_2\}$ is a fundamental set of solutions of (5.2.12), and (5.2.13) is the general solution.

Case 3: Complex Conjugate Roots

Example 5.2.3

(a) Find the general solution of

$$y''+4y'+13y=0. \tag{5.2.14}$$

(b) Solve the initial value problem

$$y''+4y'+13y=0, \quad y(0)=2,\ y'(0)=-3. \tag{5.2.15}$$

SOLUTION(**a**) The characteristic polynomial of (5.2.14) is

$$p(r)=r^2+4r+13=r^2+4r+4+9=(r+2)^2+9.$$

The roots of the characteristic equation are $r_1=-2+3i$ and $r_2=-2-3i$. By analogy with Case 1, it's reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions of (5.2.14). This is true (see Exercise 34); however, there are difficulties here, since you are probably not familiar with exponential functions with complex arguments, and even if you are, it's inconvenient to work with them, since they are complex–valued. We'll take a simpler approach, which we motivate as follows: the exponential notation suggests that

$$e^{(-2+3i)x}=e^{-2x}e^{3ix} \quad \text{and} \quad e^{(-2-3i)x}=e^{-2x}e^{-3ix},$$

so even though we haven't defined e^{3ix} and e^{-3ix}, it's reasonable to expect that every linear combination of $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ can be written as $y = ue^{-2x}$, where u depends upon x. To determine u, we note that if $y = ue^{-2x}$ then

$$y' = u'e^{-2x} - 2ue^{-2x} \quad \text{and} \quad y'' = u''e^{-2x} - 4u'e^{-2x} + 4ue^{-2x},$$

so

$$\begin{aligned} y'' + 4y' + 13y &= e^{-2x}\left[(u'' - 4u' + 4u) + 4(u' - 2u) + 13u\right] \\ &= e^{-2x}\left[u'' - (4-4)u' + (4-8+13)u\right] = e^{-2x}(u'' + 9u). \end{aligned}$$

Therefore $y = ue^{-2x}$ is a solution of (5.2.14) if and only if

$$u'' + 9u = 0.$$

From Example 5.1.2, the general solution of this equation is

$$u = c_1 \cos 3x + c_2 \sin 3x.$$

Therefore any function of the form

$$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) \tag{5.2.16}$$

is a solution of (5.2.14). Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-2x}\cos 3x$. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = e^{-2x}\sin 3x$. Since $y_2/y_1 = \tan 3x$ is nonconstant, 5.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (5.2.14), and (5.2.16) is the general solution.

SOLUTION(**b**) Imposing the condition $y(0) = 2$ in (5.2.16) shows that $c_1 = 2$. Differentiating (5.2.16) yields

$$y' = -2e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + 3e^{-2x}(-c_1 \sin 3x + c_2 \cos 3x),$$

and imposing the initial condition $y'(0) = -3$ here yields $-3 = -2c_1 + 3c_2 = -4 + 3c_2$, so $c_2 = 1/3$. Therefore the solution of (5.2.15) is

$$y = e^{-2x}(2\cos 3x + \frac{1}{3}\sin 3x).$$

Figure 5.2.3 is a graph of this function. ■

Now suppose the characteristic equation of $ay'' + by' + cy = 0$ has arbitrary complex roots; thus, $b^2 - 4ac < 0$ and, from (5.2.3), the roots are

$$r_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a}, \quad r_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a},$$

which we rewrite as

$$r_1 = \lambda + i\omega, \quad r_2 = \lambda - i\omega, \tag{5.2.17}$$

with

$$\lambda = -\frac{b}{2a}, \quad \omega = \frac{\sqrt{4ac - b^2}}{2a}.$$

Don't memorize these formulas. Just remember that r_1 and r_2 are of the form (5.2.17), where λ is an arbitrary real number and ω is positive; λ and ω are the *real* and *imaginary parts*, respectively, of r_1. Similarly, λ and $-\omega$ are the real and imaginary parts of r_2. We say that r_1 and r_2 are *complex conjugates*,

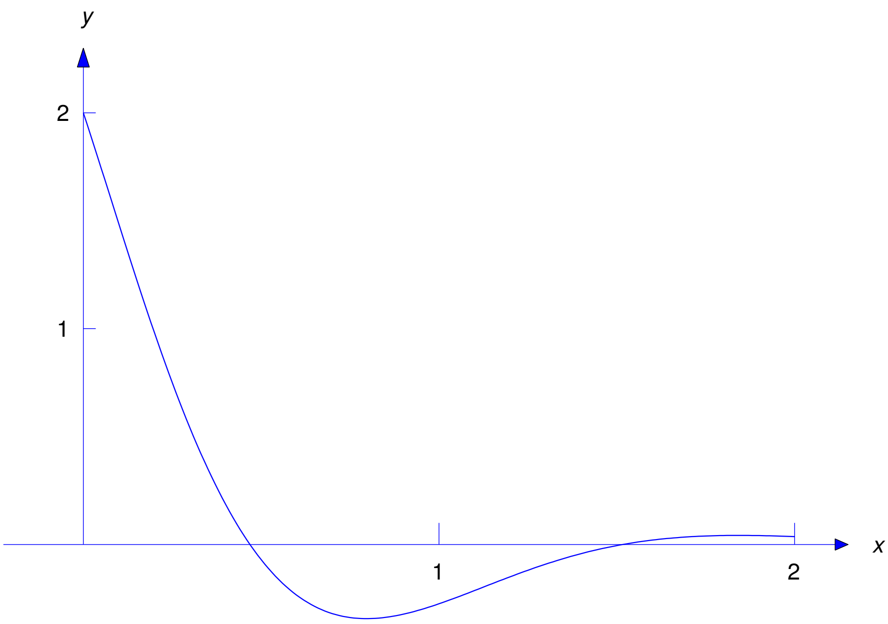

Figure 5.2.3 $y = e^{-2x}(2\cos 3x + \frac{1}{3}\sin 3x)$

which means that they have the same real part and their imaginary parts have the same absolute values, but opposite signs.

As in Example 5.2.3, it's reasonable to to expect that the solutions of $ay'' + by' + cy = 0$ are linear combinations of $e^{(\lambda+i\omega)x}$ and $e^{(\lambda-i\omega)x}$. Again, the exponential notation suggests that

$$e^{(\lambda+i\omega)x} = e^{\lambda x}e^{i\omega x} \quad \text{and} \quad e^{(\lambda-i\omega)x} = e^{\lambda x}e^{-i\omega x},$$

so even though we haven't defined $e^{i\omega x}$ and $e^{-i\omega x}$, it's reasonable to expect that every linear combination of $e^{(\lambda+i\omega)x}$ and $e^{(\lambda-i\omega)x}$ can be written as $y = ue^{\lambda x}$, where u depends upon x. To determine u we first observe that since $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ are the roots of the characteristic equation, p must be of the form

$$\begin{aligned} p(r) &= a(r - r_1)(r - r_2) \\ &= a(r - \lambda - i\omega)(r - \lambda + i\omega) \\ &= a\left[(r - \lambda)^2 + \omega^2\right] \\ &= a(r^2 - 2\lambda r + \lambda^2 + \omega^2). \end{aligned}$$

Therefore $ay'' + by' + cy = 0$ can be written as

$$a\left[y'' - 2\lambda y' + (\lambda^2 + \omega^2)y\right] = 0.$$

Since $a \neq 0$ this equation has the same solutions as

$$y'' - 2\lambda y' + (\lambda^2 + \omega^2)y = 0. \tag{5.2.18}$$

To determine u we note that if $y = ue^{\lambda x}$ then

$$y' = u'e^{\lambda x} + \lambda ue^{\lambda x} \quad \text{and} \quad y'' = u''e^{\lambda x} + 2\lambda u'e^{\lambda x} + \lambda^2 ue^{\lambda x}.$$

Substituting these expressions into (5.2.18) and dropping the common factor $e^{\lambda x}$ yields

$$(u'' + 2\lambda u' + \lambda^2 u) - 2\lambda(u' + \lambda u) + (\lambda^2 + \omega^2)u = 0,$$

which simplifies to

$$u'' + \omega^2 u = 0.$$

From Example 5.1.2, the general solution of this equation is

$$u = c_1 \cos \omega x + c_2 \sin \omega x.$$

Therefore any function of the form

$$y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x) \tag{5.2.19}$$

is a solution of (5.2.18). Letting $c_1 = 1$ and $c_2 = 0$ here yields the solution $y_1 = e^{\lambda x} \cos \omega x$. Letting $c_1 = 0$ and $c_2 = 1$ yields a second solution $y_2 = e^{\lambda x} \sin \omega x$. Since $y_2/y_1 = \tan \omega x$ is nonconstant, so Theorem 5.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (5.2.18), and (5.2.19) is the general solution.

Summary

The next theorem summarizes the results of this section.

Theorem 5.2.1 *Let $p(r) = ar^2 + br + c$ be the characteristic polynomial of*

$$ay'' + by' + cy = 0. \tag{5.2.20}$$

Then:

(a) *If $p(r) = 0$ has distinct real roots r_1 and r_2, then the general solution of (5.2.20) is*

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

(b) *If $p(r) = 0$ has a repeated root r_1, then the general solution of (5.2.20) is*

$$y = e^{r_1 x}(c_1 + c_2 x).$$

(c) *If $p(r) = 0$ has complex conjugate roots $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ (where $\omega > 0$), then the general solution of (5.2.20) is*

$$y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x).$$

5.2 Exercises

In Exercises 1–12 find the general solution.

1. $y'' + 5y' - 6y = 0$

2. $y'' - 4y' + 5y = 0$

3. $y'' + 8y' + 7y = 0$

4. $y'' - 4y' + 4y = 0$

5. $y'' + 2y' + 10y = 0$

6. $y'' + 6y' + 10y = 0$

7. $y'' - 8y' + 16y = 0$

8. $y'' + y' = 0$

9. $y'' - 2y' + 3y = 0$

10. $y'' + 6y' + 13y = 0$

11. $4y'' + 4y' + 10y = 0$

12. $10y'' - 3y' - y = 0$

In Exercises 13–17 solve the initial value problem.

13. $y'' + 14y' + 50y = 0, \quad y(0) = 2, \quad y'(0) = -17$

14. $6y'' - y' - y = 0, \quad y(0) = 10, \quad y'(0) = 0$

15. $6y'' + y' - y = 0, \quad y(0) = -1, \quad y'(0) = 3$

16. $4y'' - 4y' - 3y = 0, \quad y(0) = \frac{13}{12}, \quad y'(0) = \frac{23}{24}$

17. $4y'' - 12y' + 9y = 0, \quad y(0) = 3, \quad y'(0) = \frac{5}{2}$

In Exercises 18–21 solve the initial value problem and graph the solution.

18. C/G $y'' + 7y' + 12y = 0, \quad y(0) = -1, \quad y'(0) = 0$

19. C/G $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$

20. C/G $36y'' - 12y' + y = 0, \quad y(0) = 3, \quad y'(0) = \frac{5}{2}$

21. C/G $y'' + 4y' + 10y = 0, \quad y(0) = 3, \quad y'(0) = -2$

22. **(a)** Suppose y is a solution of the constant coefficient homogeneous equation

$$ay'' + by' + cy = 0. \tag{A}$$

Let $z(x) = y(x - x_0)$, where x_0 is an arbitrary real number. Show that

$$az'' + bz' + cz = 0.$$

(b) Let $z_1(x) = y_1(x - x_0)$ and $z_2(x) = y_2(x - x_0)$, where $\{y_1, y_2\}$ is a fundamental set of solutions of (A). Show that $\{z_1, z_2\}$ is also a fundamental set of solutions of (A).

(c) The statement of Theorem 5.2.1 is convenient for solving an initial value problem

$$ay'' + by' + cy = 0, \quad y(0) = k_0, \quad y'(0) = k_1,$$

where the initial conditions are imposed at $x_0 = 0$. However, if the initial value problem is

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1, \tag{B}$$

where $x_0 \neq 0$, then determining the constants in

$$y = c_1e^{r_1x} + c_2e^{r_2x}, \quad y = e^{r_1x}(c_1 + c_2x), \text{ or } y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x)$$

(whichever is applicable) is more complicated. Use **(b)** to restate Theorem 5.2.1 in a form more convenient for solving (B).

In Exercises 23–28 use a method suggested by Exercise 22 to solve the initial value problem.

23. $y'' + 3y' + 2y = 0, \quad y(1) = -1, \quad y'(1) = 4$

24. $y'' - 6y' - 7y = 0, \quad y(2) = -\frac{1}{3}, \quad y'(2) = -5$

25. $y'' - 14y' + 49y = 0, \quad y(1) = 2, \quad y'(1) = 11$

26. $9y'' + 6y' + y = 0, \quad y(2) = 2, \quad y'(2) = -\frac{14}{3}$

27. $9y'' + 4y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$

28. $y'' + 3y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -1$

29. Prove: If the characteristic equation of

$$ay'' + by' + cy = 0 \tag{A}$$

has a repeated negative root or two roots with negative real parts, then every solution of (A) approaches zero as $x \to \infty$.

30. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has distinct real roots r_1 and r_2. Use a method suggested by Exercise 22 to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

31. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has a repeated real root r_1. Use a method suggested by Exercise 22 to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

32. Suppose the characteristic polynomial of $ay''+by'+cy = 0$ has complex conjugate roots $\lambda \pm i\omega$. Use a method suggested by Exercise 22 to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

33. Suppose the characteristic equation of

$$ay'' + by' + cy = 0 \tag{A}$$

has a repeated real root r_1. Temporarily, think of e^{rx} as a function of two real variables x and r.

(a) Show that

$$a\frac{\partial^2}{\partial^2 x}(e^{rx}) + b\frac{\partial}{\partial x}(e^{rx}) + ce^{rx} = a(r - r_1)^2 e^{rx}. \tag{B}$$

(b) Differentiate (B) with respect to r to obtain

$$a\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial^2 x}(e^{rx})\right) + b\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}(e^{rx})\right) + c(xe^{rx}) = [2 + (r - r_1)x]a(r - r_1)e^{rx}. \tag{C}$$

(c) Reverse the orders of the partial differentiations in the first two terms on the left side of (C) to obtain

$$a\frac{\partial^2}{\partial x^2}(xe^{rx}) + b\frac{\partial}{\partial x}(xe^{rx}) + c(xe^{rx}) = [2 + (r - r_1)x]a(r - r_1)e^{rx}. \tag{D}$$

(d) Set $r = r_1$ in (B) and (D) to see that $y_1 = e^{r_1 x}$ and $y_2 = xe^{r_1 x}$ are solutions of (A)

34. In calculus you learned that e^u, $\cos u$, and $\sin u$ can be represented by the infinite series

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots + \frac{u^n}{n!} + \cdots \tag{A}$$

$$\cos u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!} = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots + (-1)^n \frac{u^{2n}}{(2n)!} + \cdots, \tag{B}$$

and

$$\sin u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!} = u - \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots + (-1)^n \frac{u^{2n+1}}{(2n+1)!} + \cdots \tag{C}$$

for all real values of u. Even though you have previously considered (A) only for real values of u, we can set $u = i\theta$, where θ is real, to obtain

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}. \tag{D}$$

Given the proper background in the theory of infinite series with complex terms, it can be shown that the series in (D) converges for all real θ.

(a) Recalling that $i^2 = -1$, write enough terms of the sequence $\{i^n\}$ to convince yourself that the sequence is repetitive:

$$1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, 1, i, -1, -i, \cdots.$$

Use this to group the terms in (D) as

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}. \end{aligned}$$

By comparing this result with (B) and (C), conclude that

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{E}$$

This is *Euler's identity*.

(b) Starting from

$$e^{i\theta_1} e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2),$$

collect the real part (the terms not multiplied by i) and the imaginary part (the terms multiplied by i) on the right, and use the trigonometric identities

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ \sin(\theta_1 + \theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \end{aligned}$$

to verify that

$$e^{i(\theta_1+\theta_2)} = e^{i\theta_1} e^{i\theta_2},$$

as you would expect from the use of the exponential notation $e^{i\theta}$.

(c) If α and β are real numbers, define

$$e^{\alpha+i\beta} = e^{\alpha}e^{i\beta} = e^{\alpha}(\cos\beta + i\sin\beta). \tag{F}$$

Show that if $z_1 = \alpha_1 + i\beta_1$ and $z_2 = \alpha_2 + i\beta_2$ then

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

(d) Let a, b, and c be real numbers, with $a \neq 0$. Let $z = u + iv$ where u and v are real-valued functions of x. Then we say that z is a solution of

$$ay'' + by' + cy = 0 \tag{G}$$

if u and v are both solutions of (G). Use Theorem 5.2.1**(c)** to verify that if the characteristic equation of (G) has complex conjugate roots $\lambda \pm i\omega$ then $z_1 = e^{(\lambda+i\omega)x}$ and $z_2 = e^{(\lambda-i\omega)x}$ are both solutions of (G).

5.3 NONHOMOGENEOUS LINEAR EQUATIONS

We'll now consider the nonhomogeneous linear second order equation

$$y'' + p(x)y' + q(x)y = f(x), \tag{5.3.1}$$

where the forcing function f isn't identically zero. The next theorem, an extension of Theorem 5.1.1, gives sufficient conditions for existence and uniqueness of solutions of initial value problems for (5.3.1). We omit the proof, which is beyond the scope of this book.

Theorem 5.3.1 *Suppose p, , q and f are continuous on an open interval (a,b), let x_0 be any point in (a,b), and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem*

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on (a,b).

To find the general solution of (5.3.1) on an interval (a,b) where p, q, and f are continuous, it's necessary to find the general solution of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{5.3.2}$$

on (a,b). We call (5.3.2) the *complementary equation* for (5.3.1).

The next theorem shows how to find the general solution of (5.3.1) if we know one solution y_p of (5.3.1) and a fundamental set of solutions of (5.3.2). We call y_p a *particular solution* of (5.3.1); it can be any solution that we can find, one way or another.

Theorem 5.3.2 *Suppose p, q, and f are continuous on (a,b). Let y_p be a particular solution of*

$$y'' + p(x)y' + q(x)y = f(x) \tag{5.3.3}$$

on (a,b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation

$$y'' + p(x)y' + q(x)y = 0 \tag{5.3.4}$$

on (a,b). Then y is a solution of (5.3.3) on (a,b) if and only if

$$y = y_p + c_1y_1 + c_2y_2, \tag{5.3.5}$$

where c_1 and c_2 are constants.

Proof We first show that y in (5.3.5) is a solution of (5.3.3) for any choice of the constants c_1 and c_2. Differentiating (5.3.5) twice yields

$$y' = y_p' + c_1y_1' + c_2y_2' \quad \text{and} \quad y'' = y_p'' + c_1y_1'' + c_2y_2'',$$

so

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (y_p'' + c_1y_1'' + c_2y_2'') + p(x)(y_p' + c_1y_1' + c_2y_2') \\ &\quad + q(x)(y_p + c_1y_1 + c_2y_2) \\ &= (y_p'' + p(x)y_p' + q(x)y_p) + c_1(y_1'' + p(x)y_1' + q(x)y_1) \\ &\quad + c_2(y_2'' + p(x)y_2' + q(x)y_2) \\ &= f + c_1 \cdot 0 + c_2 \cdot 0 = f, \end{aligned}$$

since y_p satisfies (5.3.3) and y_1 and y_2 satisfy (5.3.4).

Now we'll show that every solution of (5.3.3) has the form (5.3.5) for some choice of the constants c_1 and c_2. Suppose y is a solution of (5.3.3). We'll show that $y - y_p$ is a solution of (5.3.4), and therefore of the form $y - y_p = c_1y_1 + c_2y_2$, which implies (5.3.5). To see this, we compute

$$\begin{aligned} (y - y_p)'' + p(x)(y - y_p)' + q(x)(y - y_p) &= (y'' - y_p'') + p(x)(y' - y_p') \\ &\quad + q(x)(y - y_p) \\ &= (y'' + p(x)y' + q(x)y) \\ &\quad - (y_p'' + p(x)y_p' + q(x)y_p) \\ &= f(x) - f(x) = 0, \end{aligned}$$

since y and y_p both satisfy (5.3.3). ■

We say that (5.3.5) is the *general solution of* (5.3.3) *on* (a, b).

If P_0, P_1, and F are continuous and P_0 has no zeros on (a, b), then Theorem 5.3.2 implies that the general solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{5.3.6}$$

on (a, b) is $y = y_p + c_1y_1 + c_2y_2$, where y_p is a particular solution of (5.3.6) on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

on (a, b). To see this, we rewrite (5.3.6) as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = \frac{F(x)}{P_0(x)}$$

and apply Theorem 5.3.2 with $p = P_1/P_0$, $q = P_2/P_0$, and $f = F/P_0$.

To avoid awkward wording in examples and exercises, we won't specify the interval (a, b) when we ask for the general solution of a specific linear second order equation, or for a fundamental set of solutions of a homogeneous linear second order equation. Let's agree that this always means that we want the general solution (or a fundamental set of solutions, as the case may be) on every open interval on which p, q, and f are continuous if the equation is of the form (5.3.3), or on which P_0, P_1, P_2, and F are continuous and P_0 has no zeros, if the equation is of the form (5.3.6). We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_0, P_1, P_2, and F are all continuous on an open interval (a, b), but P_0 *does* have a zero in (a, b), then (5.3.6) may fail to have a general solution on (a, b) in the sense just defined. Exercises 42–44 illustrate this point for a homogeneous equation.

In this section we to limit ourselves to applications of Theorem 5.3.2 where we can guess at the form of the particular solution.

Example 5.3.1

(a) Find the general solution of

$$y'' + y = 1. \tag{5.3.7}$$

(b) Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7. \tag{5.3.8}$$

SOLUTION(**a**) We can apply Theorem 5.3.2 with $(a, b) = (-\infty, \infty)$, since the functions $p \equiv 0$, $q \equiv 1$, and $f \equiv 1$ in (5.3.7) are continuous on $(-\infty, \infty)$. By inspection we see that $y_p \equiv 1$ is a particular solution of (5.3.7). Since $y_1 = \cos x$ and $y_2 = \sin x$ form a fundamental set of solutions of the complementary equation $y'' + y = 0$, the general solution of (5.3.7) is

$$y = 1 + c_1 \cos x + c_2 \sin x. \tag{5.3.9}$$

SOLUTION(**b**) Imposing the initial condition $y(0) = 2$ in (5.3.9) yields $2 = 1 + c_1$, so $c_1 = 1$. Differentiating (5.3.9) yields

$$y' = -c_1 \sin x + c_2 \cos x.$$

Imposing the initial condition $y'(0) = 7$ here yields $c_2 = 7$, so the solution of (5.3.8) is

$$y = 1 + \cos x + 7 \sin x.$$

Figure 5.3.1 is a graph of this function.

Example 5.3.2

(a) Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2. \tag{5.3.10}$$

(b) Solve the initial value problem

$$y'' - 2y' + y = -3 - x + x^2, \quad y(0) = -2, \quad y'(0) = 1. \tag{5.3.11}$$

SOLUTION(**a**) The characteristic polynomial of the complementary equation

$$y'' - 2y' + y = 0$$

is $r^2 - 2r + 1 = (r - 1)^2$, so $y_1 = e^x$ and $y_2 = xe^x$ form a fundamental set of solutions of the complementary equation. To guess a form for a particular solution of (5.3.10), we note that substituting a second degree polynomial $y_p = A + Bx + Cx^2$ into the left side of (5.3.10) will produce another second degree polynomial with coefficients that depend upon A, B, and C. The trick is to choose A, B, and C so the polynomials on the two sides of (5.3.10) have the same coefficients; thus, if

$$y_p = A + Bx + Cx^2 \quad \text{then} \quad y_p' = B + 2Cx \quad \text{and} \quad y_p'' = 2C,$$

so

$$\begin{aligned} y_p'' - 2y_p' + y_p &= 2C - 2(B + 2Cx) + (A + Bx + Cx^2) \\ &= (2C - 2B + A) + (-4C + B)x + Cx^2 = -3 - x + x^2. \end{aligned}$$

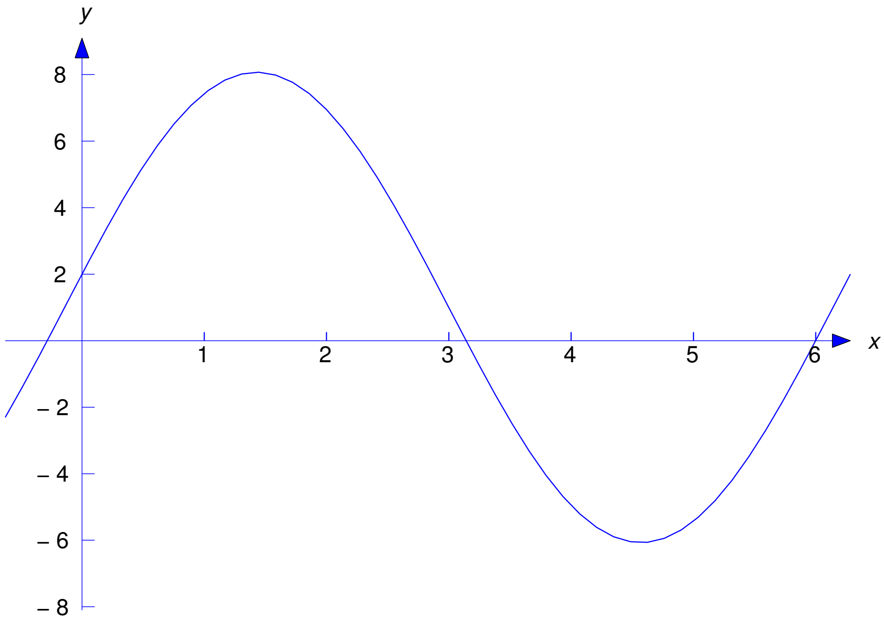

Figure 5.3.1 $y = 1 + \cos x + 7 \sin x$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} C &= 1 \\ B - 4C &= -1 \\ A - 2B + 2C &= -3, \end{aligned}$$

so $C = 1$, $B = -1 + 4C = 3$, and $A = -3 - 2C + 2B = 1$. Therefore $y_p = 1 + 3x + x^2$ is a particular solution of (5.3.10) and Theorem 5.3.2 implies that

$$y = 1 + 3x + x^2 + e^x(c_1 + c_2x) \tag{5.3.12}$$

is the general solution of (5.3.10).

SOLUTION(**b**) Imposing the initial condition $y(0) = -2$ in (5.3.12) yields $-2 = 1 + c_1$, so $c_1 = -3$. Differentiating (5.3.12) yields

$$y' = 3 + 2x + e^x(c_1 + c_2x) + c_2e^x,$$

and imposing the initial condition $y'(0) = 1$ here yields $1 = 3 + c_1 + c_2$, so $c_2 = 1$. Therefore the solution of (5.3.11) is

$$y = 1 + 3x + x^2 - e^x(3 - x).$$

Figure 5.3.2 is a graph of this solution.

Example 5.3.3 Find the general solution of

$$x^2y'' + xy' - 4y = 2x^4 \tag{5.3.13}$$

on $(-\infty, 0)$ and $(0, \infty)$.

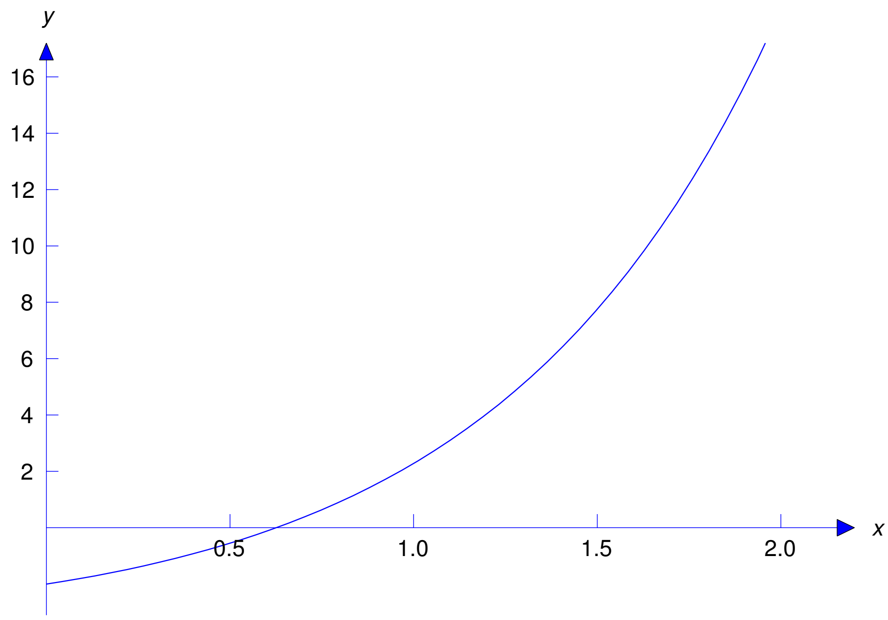

Figure 5.3.2 $y = 1 + 3x + x^2 - e^x(3 - x)$

Solution In Example 5.1.3, we verified that $y_1 = x^2$ and $y_2 = 1/x^2$ form a fundamental set of solutions of the complementary equation

$$x^2 y'' + xy' - 4y = 0$$

on $(-\infty, 0)$ and $(0, \infty)$. To find a particular solution of (5.3.13), we note that if $y_p = Ax^4$, where A is a constant then both sides of (5.3.13) will be constant multiples of x^4 and we may be able to choose A so the two sides are equal. This is true in this example, since if $y_p = Ax^4$ then

$$x^2 y_p'' + xy_p' - 4y_p = x^2(12Ax^2) + x(4Ax^3) - 4Ax^4 = 12Ax^4 = 2x^4$$

if $A = 1/6$; therefore, $y_p = x^4/6$ is a particular solution of (5.3.13) on $(-\infty, \infty)$. Theorem 5.3.2 implies that the general solution of (5.3.13) on $(-\infty, 0)$ and $(0, \infty)$ is

$$y = \frac{x^4}{6} + c_1 x^2 + \frac{c_2}{x^2}.$$

The Principle of Superposition

The next theorem enables us to break a nonhomogeneous equation into simpler parts, find a particular solution for each part, and then combine their solutions to obtain a particular solution of the original problem.

Theorem 5.3.3 *[The Principle of Superposition] Suppose y_{p_1} is a particular solution of*

$$y'' + p(x)y' + q(x)y = f_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x)$$

on (a, b). Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

on (a, b).

Proof If $y_p = y_{p_1} + y_{p_2}$ then

$$\begin{aligned} y_p'' + p(x)y_p' + q(x)y_p &= (y_{p_1} + y_{p_2})'' + p(x)(y_{p_1} + y_{p_2})' + q(x)(y_{p_1} + y_{p_2}) \\ &= \left(y_{p_1}'' + p(x)y_{p_1}' + q(x)y_{p_1}\right) + \left(y_{p_2}'' + p(x)y_{p_2}' + q(x)y_{p_2}\right) \\ &= f_1(x) + f_2(x). \ \blacksquare \end{aligned}$$

It's easy to generalize Theorem 5.3.3 to the equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{5.3.14}$$

where

$$f = f_1 + f_2 + \cdots + f_k;$$

thus, if y_{p_i} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_i(x)$$

on (a, b) for $i = 1, 2, \dots, k$, then $y_{p_1} + y_{p_2} + \cdots + y_{p_k}$ is a particular solution of (5.3.14) on (a, b). Moreover, by a proof similar to the proof of Theorem 5.3.3 we can formulate the principle of superposition in terms of a linear equation written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$$

(Exercise 39); that is, if y_{p_1} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_2(x)$$

on (a, b), then $y_{p_1} + y_{p_2}$ is a solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x) + F_2(x)$$

on (a, b).

Example 5.3.4 The function $y_{p_1} = x^4/15$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 \tag{5.3.15}$$

on $(-\infty, \infty)$ and $y_{p_2} = x^2/3$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 4x^2 \tag{5.3.16}$$

on $(-\infty, \infty)$. Use the principle of superposition to find a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 + 4x^2 \tag{5.3.17}$$

on $(-\infty, \infty)$.

Solution The right side $F(x) = 2x^4 + 4x^2$ in (5.3.17) is the sum of the right sides

$$F_1(x) = 2x^4 \quad \text{and} \quad F_2(x) = 4x^2.$$

in (5.3.15) and (5.3.16). Therefore the principle of superposition implies that

$$y_p = y_{p_1} + y_{p_2} = \frac{x^4}{15} + \frac{x^2}{3}$$

is a particular solution of (5.3.17).

5.3 Exercises

In Exercises 1–6 find a particular solution by the method used in Example 5.3.2. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

1. $y'' + 5y' - 6y = 22 + 18x - 18x^2$
2. $y'' - 4y' + 5y = 1 + 5x$
3. $y'' + 8y' + 7y = -8 - x + 24x^2 + 7x^3$
4. $y'' - 4y' + 4y = 2 + 8x - 4x^2$
5. C/G $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3, \quad y(0) = 2, \quad y'(0) = 9$
6. C/G $y'' + 6y' + 10y = 22 + 20x, \quad y(0) = 2, \; y'(0) = -2$
7. Show that the method used in Example 5.3.2 won't yield a particular solution of
$$y'' + y' = 1 + 2x + x^2; \tag{A}$$
that is, (A) does'nt have a particular solution of the form $y_p = A + Bx + Cx^2$, where A, B, and C are constants.

In Exercises 8–13 find a particular solution by the method used in Example 5.3.3.

8. $x^2y'' + 7xy' + 8y = \dfrac{6}{x}$
9. $x^2y'' - 7xy' + 7y = 13x^{1/2}$
10. $x^2y'' - xy' + y = 2x^3$
11. $x^2y'' + 5xy' + 4y = \dfrac{1}{x^3}$
12. $x^2y'' + xy' + y = 10x^{1/3}$
13. $x^2y'' - 3xy' + 13y = 2x^4$
14. Show that the method suggested for finding a particular solution in Exercises 8-13 won't yield a particular solution of
$$x^2y'' + 3xy' - 3y = \frac{1}{x^3}; \tag{A}$$
that is, (A) doesn't have a particular solution of the form $y_p = A/x^3$.
15. Prove: If a, b, c, α, and M are constants and $M \neq 0$ then
$$ax^2y'' + bxy' + cy = Mx^\alpha$$
has a particular solution $y_p = Ax^\alpha$ (A = constant) if and only if $a\alpha(\alpha - 1) + b\alpha + c \neq 0$.

If a, b, c, and α are constants, then

$$a(e^{\alpha x})'' + b(e^{\alpha x})' + ce^{\alpha x} = (a\alpha^2 + b\alpha + c)e^{\alpha x}.$$

Use this in Exercises 16–21 to find a particular solution . Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

16. $y'' + 5y' - 6y = 6e^{3x}$ **17.** $y'' - 4y' + 5y = e^{2x}$

18. C/G $y'' + 8y' + 7y = 10e^{-2x}, \quad y(0) = -2, \; y'(0) = 10$

19. C/G $y'' - 4y' + 4y = e^{x}, \quad y(0) = 2, \quad y'(0) = 0$

20. $y'' + 2y' + 10y = e^{x/2}$ **21.** $y'' + 6y' + 10y = e^{-3x}$

22. Show that the method suggested for finding a particular solution in Exercises 16-21 won't yield a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}; \tag{A}$$

that is, (A) doesn't have a particular solution of the form $y_p = Ae^{4x}$.

23. Prove: If α and M are constants and $M \neq 0$ then constant coefficient equation

$$ay'' + by' + cy = Me^{\alpha x}$$

has a particular solution $y_p = Ae^{\alpha x}$ (A = constant) if and only if $e^{\alpha x}$ isn't a solution of the complementary equation.

If ω is a constant, differentiating a linear combination of $\cos\omega x$ *and* $\sin\omega x$ *with respect to x yields another linear combination of* $\cos\omega x$ *and* $\sin\omega x$. *In Exercises 24–29 use this to find a particular solution of the equation. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.*

24. $y'' - 8y' + 16y = 23\cos x - 7\sin x$

25. $y'' + y' = -8\cos 2x + 6\sin 2x$

26. $y'' - 2y' + 3y = -6\cos 3x + 6\sin 3x$

27. $y'' + 6y' + 13y = 18\cos x + 6\sin x$

28. C/G $y'' + 7y' + 12y = -2\cos 2x + 36\sin 2x, \quad y(0) = -3, \quad y'(0) = 3$

29. C/G $y'' - 6y' + 9y = 18\cos 3x + 18\sin 3x, \quad y(0) = 2, \quad y'(0) = 2$

30. Find the general solution of

$$y'' + \omega_0^2 y = M\cos\omega x + N\sin\omega x,$$

where M and N are constants and ω and ω_0 are distinct positive numbers.

31. Show that the method suggested for finding a particular solution in Exercises 24-29 won't yield a particular solution of

$$y'' + y = \cos x + \sin x; \tag{A}$$

that is, (A) does not have a particular solution of the form $y_p = A\cos x + B\sin x$.

32. Prove: If M, N are constants (not both zero) and $\omega > 0$, the constant coefficient equation

$$ay'' + by' + cy = M\cos\omega x + N\sin\omega x \tag{A}$$

has a particular solution that's a linear combination of $\cos\omega x$ and $\sin\omega x$ if and only if the left side of (A) is not of the form $a(y'' + \omega^2 y)$, so that $\cos\omega x$ and $\sin\omega x$ are solutions of the complementary equation.

In Exercises 33–38 refer to the cited exercises and use the principal of superposition to find a particular solution. Then find the general solution.

33. $y'' + 5y' - 6y = 22 + 18x - 18x^2 + 6e^{3x}$ (See Exercises 1 and 16.)

34. $y'' - 4y' + 5y = 1 + 5x + e^{2x}$ (See Exercises 2 and 17.)

35. $y'' + 8y' + 7y = -8 - x + 24x^2 + 7x^3 + 10e^{-2x}$ (See Exercises 3 and 18.)

36. $y'' - 4y' + 4y = 2 + 8x - 4x^2 + e^x$ (See Exercises 4 and 19.)

37. $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3 + e^{x/2}$ (See Exercises 5 and 20.)

38. $y'' + 6y' + 10y = 22 + 20x + e^{-3x}$ (See Exercises 6 and 21.)

39. Prove: If y_{p_1} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_2(x)$$

on (a, b), then $y_p = y_{p_1} + y_{p_2}$ is a solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x) + F_2(x)$$

on (a, b).

40. Suppose p, q, and f are continuous on (a, b). Let y_1, y_2, and y_p be twice differentiable on (a, b), such that $y = c_1y_1 + c_2y_2 + y_p$ is a solution of

$$y'' + p(x)y' + q(x)y = f$$

on (a, b) for every choice of the constants c_1, c_2. Show that y_1 and y_2 are solutions of the complementary equation on (a, b).

5.4 THE METHOD OF UNDETERMINED COEFFICIENTS I

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x}G(x), \tag{5.4.1}$$

where α is a constant and G is a polynomial.

From Theorem 5.3.2, the general solution of (5.4.1) is $y = y_p + c_1y_1 + c_2y_2$, where y_p is a particular solution of (5.4.1) and $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation

$$ay'' + by' + cy = 0.$$

In Section 5.2 we showed how to find $\{y_1, y_2\}$. In this section we'll show how to find y_p. The procedure that we'll use is called *the method of undetermined coefficients*.

Our first example is similar to Exercises 16–21.

Example 5.4.1 Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{5.4.2}$$

Then find the general solution.

Solution Substituting $y_p = Ae^{2x}$ for y in (5.4.2) will produce a constant multiple of Ae^{2x} on the left side of (5.4.2), so it may be possible to choose A so that y_p is a solution of (5.4.2). Let's try it; if $y_p = Ae^{2x}$ then

$$y_p'' - 7y_p' + 12y_p = 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 2Ae^{2x} = 4e^{2x}$$

if $A = 2$. Therefore $y_p = 2e^{2x}$ is a particular solution of (5.4.2). To find the general solution, we note that the characteristic polynomial of the complementary equation

$$y'' - 7y' + 12y = 0 \tag{5.4.3}$$

is $p(r) = r^2 - 7r + 12 = (r-3)(r-4)$, so $\{e^{3x}, e^{4x}\}$ is a fundamental set of solutions of (5.4.3). Therefore the general solution of (5.4.2) is

$$y = 2e^{2x} + c_1e^{3x} + c_2e^{4x}.$$

Example 5.4.2 Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{5.4.4}$$

Then find the general solution.

Solution Fresh from our success in finding a particular solution of (5.4.2) — where we chose $y_p = Ae^{2x}$ because the right side of (5.4.2) is a constant multiple of e^{2x} — it may seem reasonable to try $y_p = Ae^{4x}$ as a particular solution of (5.4.4). However, this won't work, since we saw in Example 5.4.1 that e^{4x} is a solution of the complementary equation (5.4.3), so substituting $y_p = Ae^{4x}$ into the left side of (5.4.4) produces zero on the left, no matter how we choose$A$. To discover a suitable form for y_p, we use the same approach that we used in Section 5.2 to find a second solution of

$$ay'' + by' + cy = 0$$

in the case where the characteristic equation has a repeated real root: we look for solutions of (5.4.4) in the form $y = ue^{4x}$, where u is a function to be determined. Substituting

$$y = ue^{4x}, \quad y' = u'e^{4x} + 4ue^{4x}, \quad \text{and} \quad y'' = u''e^{4x} + 8u'e^{4x} + 16ue^{4x} \tag{5.4.5}$$

into (5.4.4) and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 7(u' + 4u) + 12u = 5,$$

or

$$u'' + u' = 5.$$

By inspection we see that $u_p = 5x$ is a particular solution of this equation, so $y_p = 5xe^{4x}$ is a particular solution of (5.4.4). Therefore

$$y = 5xe^{4x} + c_1e^{3x} + c_2e^{4x}$$

is the general solution.

Example 5.4.3 Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \tag{5.4.6}$$

Solution Since the characteristic polynomial of the complementary equation

$$y'' - 8y' + 16y = 0 \tag{5.4.7}$$

is $p(r) = r^2 - 8r + 16 = (r-4)^2$, both $y_1 = e^{4x}$ and $y_2 = xe^{4x}$ are solutions of (5.4.7). Therefore (5.4.6) does not have a solution of the form $y_p = Ae^{4x}$ or $y_p = Axe^{4x}$. As in Example 5.4.2, we look for solutions of (5.4.6) in the form $y = ue^{4x}$, where u is a function to be determined. Substituting from (5.4.5) into (5.4.6) and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 8(u' + 4u) + 16u = 2,$$

or

$$u'' = 2.$$

Integrating twice and taking the constants of integration to be zero shows that $u_p = x^2$ is a particular solution of this equation, so $y_p = x^2e^{4x}$ is a particular solution of (5.4.4). Therefore

$$y = e^{4x}(x^2 + c_1 + c_2x)$$

is the general solution. ■

The preceding examples illustrate the following facts concerning the form of a particular solution y_p of a constant coefficent equation

$$ay'' + by' + cy = ke^{\alpha x},$$

where k is a nonzero constant:

(a) If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay'' + by' + cy = 0, \tag{5.4.8}$$

then $y_p = Ae^{\alpha x}$, where A is a constant. (See Example 5.4.1).

(b) If $e^{\alpha x}$ is a solution of (5.4.8) but $xe^{\alpha x}$ is not, then $y_p = Axe^{\alpha x}$, where A is a constant. (See Example 5.4.2.)

(c) If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of (5.4.8), then $y_p = Ax^2e^{\alpha x}$, where A is a constant. (See Example 5.4.3.)

See Exercise 30 for the proofs of these facts.

In all three cases you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p'' + by_p' + cy_p = ke^{\alpha x},$$

and solve for the constant A, as we did in Example 5.4.1. (See Exercises 31–33.) However, if the equation is

$$ay'' + by' + cy = ke^{\alpha x}G(x),$$

where G is a polynomial of degree greater than zero, we recommend that you use the substitution $y = ue^{\alpha x}$ as we did in Examples 5.4.2 and 5.4.3. The equation for u will turn out to be

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x), \tag{5.4.9}$$

where $p(r) = ar^2 + br + c$ is the characteristic polynomial of the complementary equation and $p'(r) = 2ar + b$ (Exercise 30); however, you shouldn't memorize this since it's easy to derive the equation for u in any particular case. Note, however, that if $e^{\alpha x}$ is a solution of the complementary equation then $p(\alpha) = 0$, so (5.4.9) reduces to

$$au'' + p'(\alpha)u' = G(x),$$

while if both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the complementary equation then $p(r) = a(r - \alpha)^2$ and $p'(r) = 2a(r - \alpha)$, so $p(\alpha) = p'(\alpha) = 0$ and (5.4.9) reduces to

$$au'' = G(x).$$

Example 5.4.4 Find a particular solution of

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2). \tag{5.4.10}$$

Solution Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and } y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into (5.4.10) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 3(u' + 3u) + 2u = -1 + 2x + x^2,$$

or

$$u'' + 3u' + 2u = -1 + 2x + x^2. \tag{5.4.11}$$

As in Example 2, in order to guess a form for a particular solution of (5.4.11), we note that substituting a second degree polynomial $u_p = A + Bx + Cx^2$ for u in the left side of (5.4.11) produces another second degree polynomial with coefficients that depend upon A, B, and C; thus,

$$\text{if} \quad u_p = A + Bx + Cx^2 \quad \text{then} \quad u_p' = B + 2Cx \quad \text{and} \quad u_p'' = 2C.$$

If u_p is to satisfy (5.4.11), we must have

$$\begin{aligned} u_p'' + 3u_p' + 2u_p &= 2C + 3(B + 2Cx) + 2(A + Bx + Cx^2) \\ &= (2C + 3B + 2A) + (6C + 2B)x + 2Cx^2 = -1 + 2x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} 2C &= 1 \\ 2B + 6C &= 2 \\ 2A + 3B + 2C &= -1. \end{aligned}$$

Solving these equations for C, B, and A (in that order) yields $C = 1/2, B = -1/2, A = -1/4$. Therefore

$$u_p = -\frac{1}{4}(1 + 2x - 2x^2)$$

is a particular solution of (5.4.11), and

$$y_p = u_pe^{3x} = -\frac{e^{3x}}{4}(1 + 2x - 2x^2)$$

is a particular solution of (5.4.10).

Example 5.4.5 Find a particular solution of

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2). \tag{5.4.12}$$

Solution Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and } y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into (5.4.12) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 4(u' + 3u) + 3u = 6 + 8x + 12x^2,$$

or

$$u'' + 2u' = 6 + 8x + 12x^2. \tag{5.4.13}$$

There's no u term in this equation, since e^{3x} is a solution of the complementary equation for (5.4.12). (See Exercise 30.) Therefore (5.4.13) does not have a particular solution of the form $u_p = A + Bx + Cx^2$ that we used successfully in Example 5.4.4, since with this choice of u_p,

$$u_p'' + 2u_p' = 2C + (B + 2Cx)$$

can't contain the last term ($12x^2$) on the right side of (5.4.13). Instead, let's try $u_p = Ax + Bx^2 + Cx^3$ on the grounds that

$$u_p' = A + 2Bx + 3Cx^2 \quad \text{and} \quad u_p'' = 2B + 6Cx$$

together contain all the powers of x that appear on the right side of (5.4.13).

Substituting these expressions in place of u' and u'' in (5.4.13) yields

$$(2B + 6Cx) + 2(A + 2Bx + 3Cx^2) = (2B + 2A) + (6C + 4B)x + 6Cx^2 = 6 + 8x + 12x^2.$$

Comparing coefficients of like powers of x on the two sides of the last equality shows that u_p satisfies (5.4.13) if

$$\begin{aligned} 6C &= 12 \\ 4B + 6C &= 8 \\ 2A + 2B \quad &= 6. \end{aligned}$$

Solving these equations successively yields $C = 2$, $B = -1$, and $A = 4$. Therefore

$$u_p = x(4 - x + 2x^2)$$

is a particular solution of (5.4.13), and

$$y_p = u_p e^{3x} = xe^{3x}(4 - x + 2x^2)$$

is a particular solution of (5.4.12).

Example 5.4.6 Find a particular solution of

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2). \tag{5.4.14}$$

Solution Substituting

$$y = ue^{-x/2}, \quad y' = u'e^{-x/2} - \frac{1}{2}ue^{-x/2}, \quad \text{and} \quad y'' = u''e^{-x/2} - u'e^{-x/2} + \frac{1}{4}ue^{-x/2}$$

into (5.4.14) and canceling $e^{-x/2}$ yields

$$4\left(u''-u'+\frac{u}{4}\right)+4\left(u'-\frac{u}{2}\right)+u=4u''=-8+48x+144x^2,$$

or

$$u''=-2+12x+36x^2, \tag{5.4.15}$$

which does not contain u or u' because $e^{-x/2}$ and $xe^{-x/2}$ are both solutions of the complementary equation. (See Exercise 30.) To obtain a particular solution of (5.4.15) we integrate twice, taking the constants of integration to be zero; thus,

$$u_p'=-2x+6x^2+12x^3 \quad \text{and} \quad u_p=-x^2+2x^3+3x^4=x^2(-1+2x+3x^2).$$

Therefore

$$y_p=u_pe^{-x/2}=x^2e^{-x/2}(-1+2x+3x^2)$$

is a particular solution of (5.4.14).

Summary

The preceding examples illustrate the following facts concerning particular solutions of a constant coefficent equation of the form

$$ay''+by'+cy=e^{\alpha x}G(x),$$

where G is a polynomial (see Exercise 30):

(a) If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay''+by'+cy=0, \tag{5.4.16}$$

then $y_p=e^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 5.4.4).

(b) If $e^{\alpha x}$ is a solution of (5.4.16) but $xe^{\alpha x}$ is not, then $y_p=xe^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 5.4.5.)

(c) If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of (5.4.16), then $y_p=x^2e^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 5.4.6.)

In all three cases, you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p''+by_p'+cy_p=e^{\alpha x}G(x),$$

and solve for the coefficients of the polynomial Q. However, if you try this you will see that the computations are more tedious than those that you encounter by making the substitution $y=ue^{\alpha x}$ and finding a particular solution of the resulting equation for u. (See Exercises 34-36.) In Case **(a)** the equation for u will be of the form

$$au''+p'(\alpha)u'+p(\alpha)u=G(x),$$

with a particular solution of the form $u_p=Q(x)$, a polynomial of the same degree as G, whose coefficients can be found by the method used in Example 5.4.4. In Case **(b)** the equation for u will be of the form

$$au''+p'(\alpha)u'=G(x)$$

(no u term on the left), with a particular solution of the form $u_p=xQ(x)$, where Q is a polynomial of the same degree as G whose coefficents can be found by the method used in Example 5.4.5. In Case **(c)** the equation for u will be of the form

$$au''=G(x)$$

with a particular solution of the form $u_p = x^2Q(x)$ that can be obtained by integrating $G(x)/a$ twice and taking the constants of integration to be zero, as in Example 5.4.6.

Using the Principle of Superposition

The next example shows how to combine the method of undetermined coefficients and Theorem 5.3.3, the principle of superposition.

Example 5.4.7 Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}. \tag{5.4.17}$$

Solution In Example 5.4.1 we found that $y_{p_1} = 2e^{2x}$ is a particular solution of

$$y'' - 7y' + 12y = 4e^{2x},$$

and in Example 5.4.2 we found that $y_{p_2} = 5xe^{4x}$ is a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}.$$

Therefore the principle of superposition implies that $y_p = 2e^{2x} + 5xe^{4x}$ is a particular solution of (5.4.17).

5.4 Exercises

In Exercises 1–14 find a particular solution.

1. $y'' - 3y' + 2y = e^{3x}(1 + x)$

2. $y'' - 6y' + 5y = e^{-3x}(35 - 8x)$

3. $y'' - 2y' - 3y = e^{x}(-8 + 3x)$

4. $y'' + 2y' + y = e^{2x}(-7 - 15x + 9x^2)$

5. $y'' + 4y = e^{-x}(7 - 4x + 5x^2)$

6. $y'' - y' - 2y = e^{x}(9 + 2x - 4x^2)$

7. $y'' - 4y' - 5y = -6xe^{-x}$

8. $y'' - 3y' + 2y = e^{x}(3 - 4x)$

9. $y'' + y' - 12y = e^{3x}(-6 + 7x)$

10. $2y'' - 3y' - 2y = e^{2x}(-6 + 10x)$

11. $y'' + 2y' + y = e^{-x}(2 + 3x)$

12. $y'' - 2y' + y = e^{x}(1 - 6x)$

13. $y'' - 4y' + 4y = e^{2x}(1 - 3x + 6x^2)$

14. $9y'' + 6y' + y = e^{-x/3}(2 - 4x + 4x^2)$

In Exercises 15–19 find the general solution.

15. $y'' - 3y' + 2y = e^{3x}(1 + x)$

16. $y'' - 6y' + 8y = e^{x}(11 - 6x)$

17. $y'' + 6y' + 9y = e^{2x}(3 - 5x)$

18. $y'' + 2y' - 3y = -16xe^{x}$

19. $y'' - 2y' + y = e^{x}(2 - 12x)$

In Exercises 20–23 solve the initial value problem and plot the solution.

20. C/G $y''-4y'-5y=9e^{2x}(1+x),\quad y(0)=0,\quad y'(0)=-10$

21. C/G $y''+3y'-4y=e^{2x}(7+6x),\quad y(0)=2,\quad y'(0)=8$

22. C/G $y''+4y'+3y=-e^{-x}(2+8x),\quad y(0)=1,\quad y'(0)=2$

23. C/G $y''-3y'-10y=7e^{-2x},\quad y(0)=1,\quad y'(0)=-17$

In Exercises 24–29 use the principle of superposition to find a particular solution.

24. $y''+y'+y=xe^{x}+e^{-x}(1+2x)$

25. $y''-7y'+12y=-e^{x}(17-42x)-e^{3x}$

26. $y''-8y'+16y=6xe^{4x}+2+16x+16x^2$

27. $y''-3y'+2y=-e^{2x}(3+4x)-e^{x}$

28. $y''-2y'+2y=e^{x}(1+x)+e^{-x}(2-8x+5x^2)$

29. $y''+y=e^{-x}(2-4x+2x^2)+e^{3x}(8-12x-10x^2)$

30. **(a)** Prove that y is a solution of the constant coefficient equation

$$ay''+by'+cy=e^{\alpha x}G(x) \tag{A}$$

if and only if $y=ue^{\alpha x}$, where u satisfies

$$au''+p'(\alpha)u'+p(\alpha)u=G(x) \tag{B}$$

and $p(r)=ar^2+br+c$ is the characteristic polynomial of the complementary equation

$$ay''+by'+cy=0.$$

For the rest of this exercise, let G be a polynomial. Give the requested proofs for the case where

$$G(x)=g_0+g_1x+g_2x^2+g_3x^3.$$

(b) Prove that if $e^{\alpha x}$ isn't a solution of the complementary equation then (B) has a particular solution of the form $u_p=A(x)$, where A is a polynomial of the same degree as G, as in Example 5.4.4. Conclude that (A) has a particular solution of the form $y_p=e^{\alpha x}A(x)$.

(c) Show that if $e^{\alpha x}$ is a solution of the complementary equation and $xe^{\alpha x}$ isn't , then (B) has a particular solution of the form $u_p=xA(x)$, where A is a polynomial of the same degree as G, as in Example 5.4.5. Conclude that (A) has a particular solution of the form $y_p=xe^{\alpha x}A(x)$.

(d) Show that if $e^{\alpha x}$ and $xe^{\alpha x}$ are both solutions of the complementary equation then (B) has a particular solution of the form $u_p=x^2A(x)$, where A is a polynomial of the same degree as G, and $x^2A(x)$ can be obtained by integrating G/a twice, taking the constants of integration to be zero, as in Example 5.4.6. Conclude that (A) has a particular solution of the form $y_p=x^2e^{\alpha x}A(x)$.

Exercises 31–36 treat the equations considered in Examples 5.4.1–5.4.6. Substitute the suggested form of y_p into the equation and equate the resulting coefficients of like functions on the two sides of the resulting equation to derive a set of simultaneous equations for the coefficients in y_p. Then solve for the coefficients to obtain y_p. Compare the work you've done with the work required to obtain the same results in Examples 5.4.1–5.4.6.

31. Compare with Example 5.4.1:

$$y'' - 7y' + 12y = 4e^{2x}; \quad y_p = Ae^{2x}$$

32. Compare with Example 5.4.2:

$$y'' - 7y' + 12y = 5e^{4x}; \quad y_p = Axe^{4x}$$

33. Compare with Example 5.4.3.

$$y'' - 8y' + 16y = 2e^{4x}; \quad y_p = Ax^2e^{4x}$$

34. Compare with Example 5.4.4:

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2), \quad y_p = e^{3x}(A + Bx + Cx^2)$$

35. Compare with Example 5.4.5:

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2), \quad y_p = e^{3x}(Ax + Bx^2 + Cx^3)$$

36. Compare with Example 5.4.6:

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2), \quad y_p = e^{-x/2}(Ax^2 + Bx^3 + Cx^4)$$

37. Write $y = ue^{\alpha x}$ to find the general solution.

(a) $y'' + 2y' + y = \dfrac{e^{-x}}{\sqrt{x}}$ **(b)** $y'' + 6y' + 9y = e^{-3x}\ln x$

(c) $y'' - 4y' + 4y = \dfrac{e^{2x}}{1+x}$ **(d)** $4y'' + 4y' + y = 4e^{-x/2}\left(\dfrac{1}{x} + x\right)$

38. Suppose $\alpha \neq 0$ and k is a positive integer. In most calculus books integrals like $\int x^k e^{\alpha x}\,dx$ are evaluated by integrating by parts k times. This exercise presents another method. Let

$$y = \int e^{\alpha x} P(x)\,dx$$

with

$$P(x) = p_0 + p_1 x + \cdots + p_k x^k, \quad \text{(where } p_k \neq 0\text{)}.$$

(a) Show that $y = e^{\alpha x}u$, where

$$u' + \alpha u = P(x). \tag{A}$$

(b) Show that (A) has a particular solution of the form

$$u_p = A_0 + A_1 x + \cdots + A_k x^k,$$

where $A_k, A_{k-1}, \dots, A_0$ can be computed successively by equating coefficients of $x^k, x^{k-1}, \dots, 1$ on both sides of the equation

$$u_p' + \alpha u_p = P(x).$$

(c) Conclude that

$$\int e^{\alpha x} P(x)\,dx = \left(A_0 + A_1 x + \cdots + A_k x^k\right) e^{\alpha x} + c,$$

where c is a constant of integration.

39. Use the method of Exercise 38 to evaluate the integral.

(a) $\int e^x(4+x)\,dx$ **(b)** $\int e^{-x}(-1+x^2)\,dx$

(c) $\int x^3e^{-2x}\,dx$ **(d)** $\int e^x(1+x)^2\,dx$

(e) $\int e^{3x}(-14+30x+27x^2)\,dx$ **(f)** $\int e^{-x}(1+6x^2-14x^3+3x^4)\,dx$

40. Use the method suggested in Exercise 38 to evaluate $\int x^ke^{\alpha x}\,dx$, where k is an arbitrary positive integer and $\alpha \ne 0$.

5.5 THE METHOD OF UNDETERMINED COEFFICIENTS II

In this section we consider the constant coefficient equation

$$ay''+by'+cy=e^{\lambda x}\left(P(x)\cos\omega x+Q(x)\sin\omega x\right) \tag{5.5.1}$$

where λ and ω are real numbers, $\omega \ne 0$, and P and Q are polynomials. We want to find a particular solution of (5.5.1). As in Section 5.4, the procedure that we will use is called *the method of undetermined coefficients*.

Forcing Functions Without Exponential Factors

We begin with the case where $\lambda = 0$ in (5.5.1); thus, we we want to find a particular solution of

$$ay''+by'+cy=P(x)\cos\omega x+Q(x)\sin\omega x, \tag{5.5.2}$$

where P and Q are polynomials.

Differentiating $x^r\cos\omega x$ and $x^r\sin\omega x$ yields

$$\frac{d}{dx}x^r\cos\omega x = -\omega x^r\sin\omega x+rx^{r-1}\cos\omega x$$

and

$$\frac{d}{dx}x^r\sin\omega x = \omega x^r\cos\omega x+rx^{r-1}\sin\omega x.$$

This implies that if

$$y_p=A(x)\cos\omega x+B(x)\sin\omega x$$

where A and B are polynomials, then

$$ay_p''+by_p'+cy_p=F(x)\cos\omega x+G(x)\sin\omega x,$$

where F and G are polynomials with coefficients that can be expressed in terms of the coefficients of A and B. This suggests that we try to choose A and B so that $F=P$ and $G=Q$, respectively. Then y_p will be a particular solution of (5.5.2). The next theorem tells us how to choose the proper form for y_p. For the proof see Exercise 37.

Theorem 5.5.1 *Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation*

$$ay''+by'+cy=P(x)\cos\omega x+Q(x)\sin\omega x$$

has a particular solution

$$y_p=A(x)\cos\omega x+B(x)\sin\omega x, \tag{5.5.3}$$

where

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k \quad \textit{and} \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k,$$

provided that $\cos \omega x$ *and* $\sin \omega x$ *are not solutions of the complementary equation. The solutions of*

$$a(y'' + \omega^2 y) = P(x) \cos \omega x + Q(x) \sin \omega x$$

(for which $\cos \omega x$ *and* $\sin \omega x$ *are solutions of the complementary equation) are of the form* (5.5.3), *where*

$$A(x) = A_0 x + A_1 x^2 + \cdots + A_k x^{k+1} \quad \textit{and} \quad B(x) = B_0 x + B_1 x^2 + \cdots + B_k x^{k+1}.$$

For an analog of this theorem that's applicable to (5.5.1), see Exercise 38.

Example 5.5.1 Find a particular solution of

$$y'' - 2y' + y = 5 \cos 2x + 10 \sin 2x. \tag{5.5.4}$$

Solution In (5.5.4) the coefficients of $\cos 2x$ and $\sin 2x$ are both zero degree polynomials (constants). Therefore Theorem 5.5.1 implies that (5.5.4) has a particular solution

$$y_p = A \cos 2x + B \sin 2x.$$

Since

$$y_p' = -2A \sin 2x + 2B \cos 2x \quad \text{and} \quad y_p'' = -4(A \cos 2x + B \sin 2x),$$

replacing y by y_p in (5.5.4) yields

$$\begin{aligned} y_p'' - 2y_p' + y_p &= -4(A \cos 2x + B \sin 2x) - 4(-A \sin 2x + B \cos 2x) \\ &\quad + (A \cos 2x + B \sin 2x) \\ &= (-3A - 4B) \cos 2x + (4A - 3B) \sin 2x. \end{aligned}$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$ here with the corresponding coefficients on the right side of (5.5.4) shows that y_p is a solution of (5.5.4) if

$$\begin{aligned} -3A - 4B &= 5 \\ 4A - 3B &= 10. \end{aligned}$$

Solving these equations yields $A = 1$, $B = -2$. Therefore

$$y_p = \cos 2x - 2 \sin 2x$$

is a particular solution of (5.5.4).

Example 5.5.2 Find a particular solution of

$$y'' + 4y = 8 \cos 2x + 12 \sin 2x. \tag{5.5.5}$$

Solution The procedure used in Example 5.5.1 doesn't work here; substituting $y_p = A \cos 2x + B \sin 2x$ for y in (5.5.5) yields

$$y_p'' + 4y_p = -4(A \cos 2x + B \sin 2x) + 4(A \cos 2x + B \sin 2x) = 0$$

for any choice of A and B, since $\cos 2x$ and $\sin 2x$ are both solutions of the complementary equation for (5.5.5). We're dealing with the second case mentioned in Theorem 5.5.1, and should therefore try a particular solution of the form

$$y_p = x(A\cos 2x + B\sin 2x). \tag{5.5.6}$$

Then

$$y_p' = A\cos 2x + B\sin 2x + 2x(-A\sin 2x + B\cos 2x)$$

and

$$\begin{aligned} y_p'' &= -4A\sin 2x + 4B\cos 2x - 4x(A\cos 2x + B\sin 2x) \\ &= -4A\sin 2x + 4B\cos 2x - 4y_p \text{ (see (5.5.6))}, \end{aligned}$$

so

$$y_p'' + 4y_p = -4A\sin 2x + 4B\cos 2x.$$

Therefore y_p is a solution of (5.5.5) if

$$-4A\sin 2x + 4B\cos 2x = 8\cos 2x + 12\sin 2x,$$

which holds if $A = -3$ and $B = 2$. Therefore

$$y_p = -x(3\cos 2x - 2\sin 2x)$$

is a particular solution of (5.5.5).

Example 5.5.3 Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x. \tag{5.5.7}$$

Solution The coefficients of $\cos x$ and $\sin x$ in (5.5.7) are polynomials of degree one and zero, respectively. Therefore Theorem 5.5.1 tells us to look for a particular solution of (5.5.7) of the form

$$y_p = (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x. \tag{5.5.8}$$

Then

$$y_p' = (A_1 + B_0 + B_1x)\cos x + (B_1 - A_0 - A_1x)\sin x \tag{5.5.9}$$

and

$$y_p'' = (2B_1 - A_0 - A_1x)\cos x - (2A_1 + B_0 + B_1x)\sin x, \tag{5.5.10}$$

so

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= [A_0 + 3A_1 + 3B_0 + 2B_1 + (A_1 + 3B_1)x]\cos x \\ &\quad + [B_0 + 3B_1 - 3A_0 - 2A_1 + (B_1 - 3A_1)x]\sin x. \end{aligned} \tag{5.5.11}$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in (5.5.7) shows that y_p is a solution of (5.5.7) if

$$\begin{aligned} A_1 + 3B_1 &= 20 \\ -3A_1 + B_1 &= 0 \\ A_0 + 3B_0 + 3A_1 + 2B_1 &= 16 \\ -3A_0 + B_0 - 2A_1 + 3B_1 &= 10. \end{aligned}$$

Solving the first two equations yields $A_1 = 2$, $B_1 = 6$. Substituting these into the last two equations yields

$$\begin{aligned} A_0 + 3B_0 &= 16 - 3A_1 - 2B_1 = -2 \\ -3A_0 + B_0 &= 10 + 2A_1 - 3B_1 = -4. \end{aligned}$$

Solving these equations yields $A_0 = 1$, $B_0 = -1$. Substituting $A_0 = 1$, $A_1 = 2$, $B_0 = -1$, $B_1 = 6$ into (5.5.8) shows that

$$y_p = (1 + 2x)\cos x - (1 - 6x)\sin x$$

is a particular solution of (5.5.7).

A Useful Observation

In (5.5.9), (5.5.10), and (5.5.11) the polynomials multiplying $\sin x$ can be obtained by replacing A_0, A_1, B_0, and B_1 by B_0, B_1, $-A_0$, and $-A_1$, respectively, in the polynomials mutiplying $\cos x$. An analogous result applies in general, as follows (Exercise 36).

Theorem 5.5.2 *If*

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x,$$

where $A(x)$ and $B(x)$ are polynomials with coefficients $A_0 \dots, A_k$ and $B_0, \dots, B_k$, then the polynomials multiplying $\sin\omega x$ in

$$y_p', \quad y_p'', \quad ay_p'' + by_p' + cy_p \quad \text{and} \quad y_p'' + \omega^2 y_p$$

can be obtained by replacing $A_0, \dots, A_k$ by $B_0, \dots, B_k$ and $B_0, \dots, B_k$ by $-A_0, \dots, -A_k$ in the corresponding polynomials multiplying $\cos\omega x$.

We won't use this theorem in our examples, but we recommend that you use it to check your manipulations when you work the exercises.

Example 5.5.4 Find a particular solution of

$$y'' + y = (8 - 4x)\cos x - (8 + 8x)\sin x. \tag{5.5.12}$$

Solution According to Theorem 5.5.1, we should look for a particular solution of the form

$$y_p = (A_0x + A_1x^2)\cos x + (B_0x + B_1x^2)\sin x, \tag{5.5.13}$$

since $\cos x$ and $\sin x$ are solutions of the complementary equation. However, let's try

$$y_p = (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x \tag{5.5.14}$$

first, so you can see why it doesn't work. From (5.5.10),

$$y_p'' = (2B_1 - A_0 - A_1x)\cos x - (2A_1 + B_0 + B_1x)\sin x,$$

which together with (5.5.14) implies that

$$y_p'' + y_p = 2B_1\cos x - 2A_1\sin x.$$

Since the right side of this equation does not contain $x\cos x$ or $x\sin x$, (5.5.14) can't satisfy (5.5.12) no matter how we choose A_0, A_1, B_0, and B_1.

Now let y_p be as in (5.5.13). Then

$$\begin{aligned} y_p' &= \left[A_0 + (2A_1 + B_0)x + B_1x^2\right]\cos x \\ &\quad + \left[B_0 + (2B_1 - A_0)x - A_1x^2\right]\sin x \end{aligned}$$

and

$$\begin{aligned} y_p'' &= \left[2A_1 + 2B_0 - (A_0 - 4B_1)x - A_1x^2\right]\cos x \\ &\quad + \left[2B_1 - 2A_0 - (B_0 + 4A_1)x - B_1x^2\right]\sin x, \end{aligned}$$

so

$$y_p'' + y_p = (2A_1 + 2B_0 + 4B_1x)\cos x + (2B_1 - 2A_0 - 4A_1x)\sin x.$$

Comparing the coefficients of $\cos x$ and $\sin x$ here with the corresponding coefficients in (5.5.12) shows that y_p is a solution of (5.5.12) if

$$\begin{aligned} 4B_1 &= -4 \\ -4A_1 &= -8 \\ 2B_0 + 2A_1 &= 8 \\ -2A_0 + 2B_1 &= -8. \end{aligned}$$

The solution of this system is $A_1 = 2$, $B_1 = -1$, $A_0 = 3$, $B_0 = 2$. Therefore

$$y_p = x\left[(3 + 2x)\cos x + (2 - x)\sin x\right]$$

is a particular solution of (5.5.12).

Forcing Functions with Exponential Factors

To find a particular solution of

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{5.5.15}$$

when $\lambda \neq 0$, we recall from Section 5.4 that substituting $y = ue^{\lambda x}$ into (5.5.15) will produce a constant coefficient equation for u with the forcing function $P(x)\cos\omega x + Q(x)\sin\omega x$. We can find a particular solution u_p of this equation by the procedure that we used in Examples 5.5.1–5.5.4. Then $y_p = u_p e^{\lambda x}$ is a particular solution of (5.5.15).

Example 5.5.5 Find a particular solution of

$$y'' - 3y' + 2y = e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right]. \tag{5.5.16}$$

Solution Let $y = ue^{-2x}$. Then

$$\begin{aligned} y'' - 3y' + 2y &= e^{-2x}\left[(u'' - 4u' + 4u) - 3(u' - 2u) + 2u\right] \\ &= e^{-2x}(u'' - 7u' + 12u) \\ &= e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right] \end{aligned}$$

if

$$u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x. \tag{5.5.17}$$

Since $\cos 3x$ and $\sin 3x$ aren't solutions of the complementary equation

$$u'' - 7u' + 12u = 0,$$

Theorem 5.5.1 tells us to look for a particular solution of (5.5.17) of the form

$$u_p = (A_0 + A_1x)\cos 3x + (B_0 + B_1x)\sin 3x. \tag{5.5.18}$$

Then

$$\begin{aligned} u_p' &= (A_1 + 3B_0 + 3B_1x)\cos 3x + (B_1 - 3A_0 - 3A_1x)\sin 3x \\ \text{and} \qquad u_p'' &= (-9A_0 + 6B_1 - 9A_1x)\cos 3x - (9B_0 + 6A_1 + 9B_1x)\sin 3x, \end{aligned}$$

so

$$\begin{aligned} u_p'' - 7u_p' + 12u_p &= [3A_0 - 21B_0 - 7A_1 + 6B_1 + (3A_1 - 21B_1)x] \cos 3x \\ &+ [21A_0 + 3B_0 - 6A_1 - 7B_1 + (21A_1 + 3B_1)x] \sin 3x. \end{aligned}$$

Comparing the coefficients of $x \cos 3x$, $x \sin 3x$, $\cos 3x$, and $\sin 3x$ here with the corresponding coefficients on the right side of (5.5.17) shows that u_p is a solution of (5.5.17) if

$$\begin{aligned} 3A_1 - 21B_1 &= 0 \\ 21A_1 + 3B_1 &= 150 \\ 3A_0 - 21B_0 - 7A_1 + 6B_1 &= 2 \\ 21A_0 + 3B_0 - 6A_1 - 7B_1 &= -34. \end{aligned} \tag{5.5.19}$$

Solving the first two equations yields $A_1 = 7$, $B_1 = 1$. Substituting these values into the last two equations of (5.5.19) yields

$$\begin{aligned} 3A_0 - 21B_0 &= 2 + 7A_1 - 6B_1 = 45 \\ 21A_0 + 3B_0 &= -34 + 6A_1 + 7B_1 = 15. \end{aligned}$$

Solving this system yields $A_0 = 1$, $B_0 = -2$. Substituting $A_0 = 1$, $A_1 = 7$, $B_0 = -2$, and $B_1 = 1$ into (5.5.18) shows that

$$u_p = (1 + 7x) \cos 3x - (2 - x) \sin 3x$$

is a particular solution of (5.5.17). Therefore

$$y_p = e^{-2x} [(1 + 7x) \cos 3x - (2 - x) \sin 3x]$$

is a particular solution of (5.5.16).

Example 5.5.6 Find a particular solution of

$$y'' + 2y' + 5y = e^{-x} [(6 - 16x) \cos 2x - (8 + 8x) \sin 2x]. \tag{5.5.20}$$

Solution Let $y = ue^{-x}$. Then

$$\begin{aligned} y'' + 2y' + 5y &= e^{-x} \left[(u'' - 2u' + u) + 2(u' - u) + 5u\right] \\ &= e^{-x}(u'' + 4u) \\ &= e^{-x} [(6 - 16x) \cos 2x - (8 + 8x) \sin 2x] \end{aligned}$$

if

$$u'' + 4u = (6 - 16x) \cos 2x - (8 + 8x) \sin 2x. \tag{5.5.21}$$

Since $\cos 2x$ and $\sin 2x$ are solutions of the complementary equation

$$u'' + 4u = 0,$$

Theorem 5.5.1 tells us to look for a particular solution of (5.5.21) of the form

$$u_p = (A_0x + A_1x^2) \cos 2x + (B_0x + B_1x^2) \sin 2x.$$

Then

$$u_p' = \left[A_0+(2A_1+2B_0)x+2B_1x^2\right]\cos 2x + \left[B_0+(2B_1-2A_0)x-2A_1x^2\right]\sin 2x$$

and

$$u_p'' = \left[2A_1+4B_0-(4A_0-8B_1)x-4A_1x^2\right]\cos 2x + \left[2B_1-4A_0-(4B_0+8A_1)x-4B_1x^2\right]\sin 2x,$$

so

$$u_p''+4u_p=(2A_1+4B_0+8B_1x)\cos 2x+(2B_1-4A_0-8A_1x)\sin 2x.$$

Equating the coefficients of $x\cos 2x$, $x\sin 2x$, $\cos 2x$, and $\sin 2x$ here with the corresponding coefficients on the right side of (5.5.21) shows that u_p is a solution of (5.5.21) if

$$\begin{aligned} 8B_1 &= -16\\ -8A_1 &= -\ 8\\ 4B_0+2A_1 &= \ \ 6\\ -4A_0+2B_1 &= -8. \end{aligned} \tag{5.5.22}$$

The solution of this system is $A_1=1$, $B_1=-2$, $B_0=1$, $A_0=1$. Therefore

$$u_p=x[(1+x)\cos 2x+(1-2x)\sin 2x]$$

is a particular solution of (5.5.21), and

$$y_p=xe^{-x}\left[(1+x)\cos 2x+(1-2x)\sin 2x\right]$$

is a particular solution of (5.5.20). ■

You can also find a particular solution of (5.5.20) by substituting

$$y_p=xe^{-x}\left[(A_0+A_1x)\cos 2x+(B_0+B_1x)\sin 2x\right]$$

for y in (5.5.20) and equating the coefficients of $xe^{-x}\cos 2x$, $xe^{-x}\sin 2x$, $e^{-x}\cos 2x$, and $e^{-x}\sin 2x$ in the resulting expression for

$$y_p''+2y_p'+5y_p$$

with the corresponding coefficients on the right side of (5.5.20). (See Exercise 38). This leads to the same system (5.5.22) of equations for A_0, A_1, B_0, and B_1 that we obtained in Example 5.5.6. However, if you try this approach you'll see that deriving (5.5.22) this way is much more tedious than the way we did it in Example 5.5.6.

5.5 Exercises

In Exercises 1–17 find a particular solution.

1. $y''+3y'+2y=7\cos x-\sin x$

2. $y''+3y'+y=(2-6x)\cos x-9\sin x$

3. $y''+2y'+y=e^x(6\cos x+17\sin x)$

4. $y''+3y'-2y=-e^{2x}(5\cos 2x+9\sin 2x)$

5. $y''-y'+y=e^x(2+x)\sin x$

6. $y''+3y'-2y=e^{-2x}\left[(4+20x)\cos 3x+(26-32x)\sin 3x\right]$

7. $y'' + 4y = -12\cos 2x - 4\sin 2x$

8. $y'' + y = (-4 + 8x)\cos x + (8 - 4x)\sin x$

9. $4y'' + y = -4\cos x/2 - 8x\sin x/2$

10. $y'' + 2y' + 2y = e^{-x}(8\cos x - 6\sin x)$

11. $y'' - 2y' + 5y = e^{x}\left[(6 + 8x)\cos 2x + (6 - 8x)\sin 2x\right]$

12. $y'' + 2y' + y = 8x^2\cos x - 4x\sin x$

13. $y'' + 3y' + 2y = (12 + 20x + 10x^2)\cos x + 8x\sin x$

14. $y'' + 3y' + 2y = (1 - x - 4x^2)\cos 2x - (1 + 7x + 2x^2)\sin 2x$

15. $y'' - 5y' + 6y = -e^{x}\left[(4 + 6x - x^2)\cos x - (2 - 4x + 3x^2)\sin x\right]$

16. $y'' - 2y' + y = -e^{x}\left[(3 + 4x - x^2)\cos x + (3 - 4x - x^2)\sin x\right]$

17. $y'' - 2y' + 2y = e^{x}\left[(2 - 2x - 6x^2)\cos x + (2 - 10x + 6x^2)\sin x\right]$

In Exercises 1–17 find a particular solution and graph it.

18. C/G $y'' + 2y' + y = e^{-x}\left[(5 - 2x)\cos x - (3 + 3x)\sin x\right]$

19. C/G $y'' + 9y = -6\cos 3x - 12\sin 3x$

20. C/G $y'' + 3y' + 2y = (1 - x - 4x^2)\cos 2x - (1 + 7x + 2x^2)\sin 2x$

21. C/G $y'' + 4y' + 3y = e^{-x}\left[(2 + x + x^2)\cos x + (5 + 4x + 2x^2)\sin x\right]$

In Exercises 22–26 solve the initial value problem.

22. $y'' - 7y' + 6y = -e^{x}(17\cos x - 7\sin x), \quad y(0) = 4,\ y'(0) = 2$

23. $y'' - 2y' + 2y = -e^{x}(6\cos x + 4\sin x), \quad y(0) = 1,\ y'(0) = 4$

24. $y'' + 6y' + 10y = -40e^{x}\sin x, \quad y(0) = 2, \quad y'(0) = -3$

25. $y'' - 6y' + 10y = -e^{3x}(6\cos x + 4\sin x), \quad y(0) = 2, \quad y'(0) = 7$

26. $y'' - 3y' + 2y = e^{3x}\left[21\cos x - (11 + 10x)\sin x\right], \ y(0) = 0, \quad y'(0) = 6$

In Exercises 27–32 use the principle of superposition to find a particular solution. Where indicated, solve the initial value problem.

27. $y'' - 2y' - 3y = 4e^{3x} + e^{x}(\cos x - 2\sin x)$

28. $y'' + y = 4\cos x - 2\sin x + xe^{x} + e^{-x}$

29. $y'' - 3y' + 2y = xe^{x} + 2e^{2x} + \sin x$

30. $y'' - 2y' + 2y = 4xe^{x}\cos x + xe^{-x} + 1 + x^2$

31. $y'' - 4y' + 4y = e^{2x}(1 + x) + e^{2x}(\cos x - \sin x) + 3e^{3x} + 1 + x$

32. $y'' - 4y' + 4y = 6e^{2x} + 25\sin x, \quad y(0) = 5,\ y'(0) = 3$

In Exercises 33–35 solve the initial value problem and graph the solution.

33. C/G $y'' + 4y = -e^{-2x}\left[(4 - 7x)\cos x + (2 - 4x)\sin x\right], \ y(0) = 3, \quad y'(0) = 1$

34. C/G $y'' + 4y' + 4y = 2\cos 2x + 3\sin 2x + e^{-x}, \quad y(0) = -1,\ y'(0) = 2$

35. C/G $y'' + 4y = e^{x}(11 + 15x) + 8\cos 2x - 12\sin 2x, \quad y(0) = 3,\ y'(0) = 5$

36. **(a)** Verify that if

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x$$

where A and B are twice differentiable, then

$$\begin{aligned} y_p' &= (A' + \omega B)\cos\omega x + (B' - \omega A)\sin\omega x \text{ and} \\ y_p'' &= (A'' + 2\omega B' - \omega^2 A)\cos\omega x + (B'' - 2\omega A' - \omega^2 B)\sin\omega x. \end{aligned}$$

(b) Use the results of **(a)** to verify that

$$\begin{aligned} ay_p'' + by_p' + cy_p &= \left[(c - a\omega^2)A + b\omega B + 2a\omega B' + bA' + aA''\right]\cos\omega x + \\ &\quad \left[-b\omega A + (c - a\omega^2)B - 2a\omega A' + bB' + aB''\right]\sin\omega x. \end{aligned}$$

(c) Use the results of **(a)** to verify that

$$y_p'' + \omega^2 y_p = (A'' + 2\omega B')\cos\omega x + (B'' - 2\omega A')\sin\omega x.$$

(d) Prove Theorem 5.5.2.

37. Let a, b, c, and ω be constants, with $a \ne 0$ and $\omega > 0$, and let

$$P(x) = p_0 + p_1 x + \cdots + p_k x^k \quad \text{and} \quad Q(x) = q_0 + q_1 x + \cdots + q_k x^k,$$

where at least one of the coefficients p_k, q_k is nonzero, so k is the larger of the degrees of P and Q.

(a) Show that if $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation

$$ay'' + by' + cy = 0,$$

then there are polynomials

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k \quad \text{and} \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k \qquad \text{(A)}$$

such that

$$\begin{aligned} (c - a\omega^2)A + b\omega B + 2a\omega B' + bA' + aA'' &= P \\ -b\omega A + (c - a\omega^2)B - 2a\omega A' + bB' + aB'' &= Q, \end{aligned}$$

where (A_k, B_k), (A_{k-1}, B_{k-1}), …,(A_0, B_0) can be computed successively by solving the systems

$$\begin{aligned} (c - a\omega^2)A_k + b\omega B_k &= p_k \\ -b\omega A_k + (c - a\omega^2)B_k &= q_k, \end{aligned}$$

and, if $1 \le r \le k$,

$$\begin{aligned} (c - a\omega^2)A_{k-r} + b\omega B_{k-r} &= p_{k-r} + \cdots \\ -b\omega A_{k-r} + (c - a\omega^2)B_{k-r} &= q_{k-r} + \cdots, \end{aligned}$$

where the terms indicated by "$\cdots$" depend upon the previously computed coefficients with subscripts greater than $k - r$. Conclude from this and Exercise 36**(b)** that

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x \qquad \text{(B)}$$

is a particular solution of

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x.$$

(b) Conclude from Exercise 36**(c)** that the equation

$$a(y'' + \omega^2 y) = P(x)\cos\omega x + Q(x)\sin\omega x \tag{C}$$

does not have a solution of the form (B) with A and B as in (A). Then show that there are polynomials

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \text{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}$$

such that

$$\begin{aligned} a(A'' + 2\omega B') &= P \\ a(B'' - 2\omega A') &= Q, \end{aligned}$$

where the pairs (A_k, B_k), (A_{k-1}, B_{k-1}), $\dots$, (A_0, B_0) can be computed successively as follows:

$$\begin{aligned} A_k &= -\frac{q_k}{2a\omega(k+1)} \\ B_k &= \frac{p_k}{2a\omega(k+1)}, \end{aligned}$$

and, if $k \ge 1$,

$$\begin{aligned} A_{k-j} &= -\frac{1}{2\omega}\left[\frac{q_{k-j}}{a(k-j+1)} - (k-j+2)B_{k-j+1}\right] \\ B_{k-j} &= \frac{1}{2\omega}\left[\frac{p_{k-j}}{a(k-j+1)} - (k-j+2)A_{k-j+1}\right] \end{aligned}$$

for $1 \le j \le k$. Conclude that (B) with this choice of the polynomials A and B is a particular solution of (C).

38. Show that Theorem 5.5.1 implies the next theorem: *Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation*

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)$$

has a particular solution

$$y_p = e^{\lambda x}\left(A(x)\cos\omega x + B(x)\sin\omega x\right), \tag{A}$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k \quad \textit{and} \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

provided that $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are not solutions of the complementary equation. The equation

$$a\left[y'' - 2\lambda y' + (\lambda^2 + \omega^2)y\right] = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)$$

(for which $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are solutions of the complementary equation) has a particular solution of the form (A)*, where*

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \textit{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}.$$

39. This exercise presents a method for evaluating the integral

$$y = \int e^{\lambda x} \left(P(x) \cos \omega x + Q(x) \sin \omega x \right) \, dx$$

where $\omega \neq 0$ and

$$P(x) = p_0 + p_1 x + \cdots + p_k x^k, \quad Q(x) = q_0 + q_1 x + \cdots + q_k x^k.$$

(a) Show that $y = e^{\lambda x} u$, where

$$u' + \lambda u = P(x) \cos \omega x + Q(x) \sin \omega x. \tag{A}$$

(b) Show that (A) has a particular solution of the form

$$u_p = A(x) \cos \omega x + B(x) \sin \omega x,$$

where

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k, \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k,$$

and the pairs of coefficients (A_k, B_k), (A_{k-1}, B_{k-1}), …,(A_0, B_0) can be computed successively as the solutions of pairs of equations obtained by equating the coefficients of $x^r \cos \omega x$ and $x^r \sin \omega x$ for $r = k, k-1, \ldots, 0$.

(c) Conclude that

$$\int e^{\lambda x} \left(P(x) \cos \omega x + Q(x) \sin \omega x \right) \, dx = e^{\lambda x} \left(A(x) \cos \omega x + B(x) \sin \omega x \right) + c,$$

where c is a constant of integration.

40. Use the method of Exercise 39 to evaluate the integral.

(a) $\int x^2 \cos x \, dx$ **(b)** $\int x^2 e^x \cos x \, dx$

(c) $\int x e^{-x} \sin 2x \, dx$ **(d)** $\int x^2 e^{-x} \sin x \, dx$

(e) $\int x^3 e^x \sin x \, dx$ **(f)** $\int e^x \left[x \cos x - (1 + 3x) \sin x \right] \, dx$

(g) $\int e^{-x} \left[(1 + x^2) \cos x + (1 - x^2) \sin x \right] \, dx$

5.6 REDUCTION OF ORDER

In this section we give a method for finding the general solution of

$$P_0(x) y'' + P_1(x) y' + P_2(x) y = F(x) \tag{5.6.1}$$

if we know a nontrivial solution y_1 of the complementary equation

$$P_0(x) y'' + P_1(x) y' + P_2(x) y = 0. \tag{5.6.2}$$

The method is called *reduction of order* because it reduces the task of solving (5.6.1) to solving a first order equation. Unlike the method of undetermined coefficients, it does not require P_0, P_1, and P_2 to be constants, or F to be of any special form.

By now you shoudn't be surprised that we look for solutions of (5.6.1) in the form

$$y = uy_1 \tag{5.6.3}$$

where u is to be determined so that y satisfies (5.6.1). Substituting (5.6.3) and

$$\begin{aligned} y' &= u'y_1 + uy_1' \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

into (5.6.1) yields

$$P_0(x)(u''y_1 + 2u'y_1' + uy_1'') + P_1(x)(u'y_1 + uy_1') + P_2(x)uy_1 = F(x).$$

Collecting the coefficients of u, u', and u'' yields

$$(P_0y_1)u'' + (2P_0y_1' + P_1y_1)u' + (P_0y_1'' + P_1y_1' + P_2y_1)u = F. \tag{5.6.4}$$

However, the coefficient of u is zero, since y_1 satisfies (5.6.2). Therefore (5.6.4) reduces to

$$Q_0(x)u'' + Q_1(x)u' = F, \tag{5.6.5}$$

with

$$Q_0 = P_0y_1 \quad \text{and} \quad Q_1 = 2P_0y_1' + P_1y_1.$$

(It isn't worthwhile to memorize the formulas for Q_0 and Q_1!) Since (5.6.5) is a linear first order equation in u', we can solve it for u' by variation of parameters as in Section 1.2, integrate the solution to obtain u, and then obtain y from (5.6.3).

Example 5.6.1

(a) Find the general solution of

$$xy'' - (2x+1)y' + (x+1)y = x^2, \tag{5.6.6}$$

given that $y_1 = e^x$ is a solution of the complementary equation

$$xy'' - (2x+1)y' + (x+1)y = 0. \tag{5.6.7}$$

(b) As a byproduct of **(a)**, find a fundamental set of solutions of (5.6.7).

SOLUTION**(a)** If $y = ue^x$, then $y' = u'e^x + ue^x$ and $y'' = u''e^x + 2u'e^x + ue^x$, so

$$\begin{aligned} xy'' - (2x+1)y' + (x+1)y &= x(u''e^x + 2u'e^x + ue^x) \\ &\quad -(2x+1)(u'e^x + ue^x) + (x+1)ue^x \\ &= (xu'' - u')e^x. \end{aligned}$$

Therefore $y = ue^x$ is a solution of (5.6.6) if and only if

$$(xu'' - u')e^x = x^2,$$

which is a first order equation in u'. We rewrite it as

$$u'' - \frac{u'}{x} = xe^{-x}. \tag{5.6.8}$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that (5.6.8) becomes

$$z' - \frac{z}{x} = xe^{-x}. \tag{5.6.9}$$

We leave it to you to show (by separation of variables) that $z_1 = x$ is a solution of the complementary equation

$$z' - \frac{z}{x} = 0$$

for (5.6.9). By applying variation of parameters as in Section 1.2, we can now see that every solution of (5.6.9) is of the form

$$z = vx \quad \text{where} \quad v'x = xe^{-x}, \quad \text{so} \quad v' = e^{-x} \quad \text{and} \quad v = -e^{-x} + C_1.$$

Since $u' = z = vx$, u is a solution of (5.6.8) if and only if

$$u' = vx = -xe^{-x} + C_1 x.$$

Integrating this yields

$$u = (x+1)e^{-x} + \frac{C_1}{2}x^2 + C_2.$$

Therefore the general solution of (5.6.6) is

$$y = ue^x = x + 1 + \frac{C_1}{2}x^2e^x + C_2e^x. \tag{5.6.10}$$

SOLUTION(**b**) By letting $C_1 = C_2 = 0$ in (5.6.10), we see that $y_{p_1} = x + 1$ is a solution of (5.6.6). By letting $C_1 = 2$ and $C_2 = 0$, we see that $y_{p_2} = x + 1 + x^2e^x$ is also a solution of (5.6.6). Since the difference of two solutions of (5.6.6) is a solution of (5.6.7), $y_2 = y_{p_1} - y_{p_2} = x^2e^x$ is a solution of (5.6.7). Since y_2/y_1 is nonconstant and we already know that $y_1 = e^x$ is a solution of (5.6.6), Theorem 5.1.6 implies that $\{e^x, x^2e^x\}$ is a fundamental set of solutions of (5.6.7). ■

Although (5.6.10) is a correct form for the general solution of (5.6.6), it's silly to leave the arbitrary coefficient of x^2e^x as $C_1/2$ where C_1 is an arbitrary constant. Moreover, it's sensible to make the subscripts of the coefficients of $y_1 = e^x$ and $y_2 = x^2e^x$ consistent with the subscripts of the functions themselves. Therefore we rewrite (5.6.10) as

$$y = x + 1 + c_1e^x + c_2x^2e^x$$

by simply renaming the arbitrary constants. We'll also do this in the next two examples, and in the answers to the exercises.

Example 5.6.2

(a) Find the general solution of

$$x^2y'' + xy' - y = x^2 + 1,$$

given that $y_1 = x$ is a solution of the complementary equation

$$x^2y'' + xy' - y = 0. \tag{5.6.11}$$

As a byproduct of this result, find a fundamental set of solutions of (5.6.11).

(b) Solve the initial value problem

$$x^2y'' + xy' - y = x^2 + 1, \quad y(1) = 2, \; y'(1) = -3. \tag{5.6.12}$$

SOLUTION(**a**) If $y = ux$, then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y'' + xy' - y &= x^2(u''x + 2u') + x(u'x + u) - ux \\ &= x^3u'' + 3x^2u'. \end{aligned}$$

Therefore $y = ux$ is a solution of (5.6.12) if and only if

$$x^3u'' + 3x^2u' = x^2 + 1,$$

which is a first order equation in u'. We rewrite it as

$$u'' + \frac{3}{x}u' = \frac{1}{x} + \frac{1}{x^3}. \tag{5.6.13}$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that (5.6.13) becomes

$$z' + \frac{3}{x}z = \frac{1}{x} + \frac{1}{x^3}. \tag{5.6.14}$$

We leave it to you to show by separation of variables that $z_1 = 1/x^3$ is a solution of the complementary equation

$$z' + \frac{3}{x}z = 0$$

for (5.6.14). By variation of parameters, every solution of (5.6.14) is of the form

$$z = \frac{v}{x^3} \quad \text{where} \quad \frac{v'}{x^3} = \frac{1}{x} + \frac{1}{x^3}, \quad \text{so} \quad v' = x^2 + 1 \quad \text{and} \quad v = \frac{x^3}{3} + x + C_1.$$

Since $u' = z = v/x^3$, u is a solution of (5.6.14) if and only if

$$u' = \frac{v}{x^3} = \frac{1}{3} + \frac{1}{x^2} + \frac{C_1}{x^3}.$$

Integrating this yields

$$u = \frac{x}{3} - \frac{1}{x} - \frac{C_1}{2x^2} + C_2.$$

Therefore the general solution of (5.6.12) is

$$y = ux = \frac{x^2}{3} - 1 - \frac{C_1}{2x} + C_2x. \tag{5.6.15}$$

Reasoning as in the solution of Example 5.6.1(**a**), we conclude that $y_1 = x$ and $y_2 = 1/x$ form a fundamental set of solutions for (5.6.11).

As we explained above, we rename the constants in (5.6.15) and rewrite it as

$$y = \frac{x^2}{3} - 1 + c_1x + \frac{c_2}{x}. \tag{5.6.16}$$

SOLUTION(**b**) Differentiating (5.6.16) yields

$$y' = \frac{2x}{3} + c_1 - \frac{c_2}{x^2}. \tag{5.6.17}$$

Setting $x = 1$ in (5.6.16) and (5.6.17) and imposing the initial conditions $y(1) = 2$ and $y'(1) = -3$ yields

$$\begin{aligned} c_1 + c_2 &= \frac{8}{3} \\ c_1 - c_2 &= -\frac{11}{3}. \end{aligned}$$

Solving these equations yields $c_1 = -1/2$, $c_2 = 19/6$. Therefore the solution of (5.6.12) is

$$y = \frac{x^2}{3} - 1 - \frac{x}{2} + \frac{19}{6x}.$$

Using reduction of order to find the general solution of a homogeneous linear second order equation leads to a homogeneous linear first order equation in u' that can be solved by separation of variables. The next example illustrates this.

Example 5.6.3 Find the general solution and a fundamental set of solutions of

$$x^2y'' - 3xy' + 3y = 0, \tag{5.6.18}$$

given that $y_1 = x$ is a solution.

Solution If $y = ux$ then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(u''x + 2u') - 3x(u'x + u) + 3ux \\ &= x^3u'' - x^2u'. \end{aligned}$$

Therefore $y = ux$ is a solution of (5.6.18) if and only if

$$x^3u'' - x^2u' = 0.$$

Separating the variables u' and x yields

$$\frac{u''}{u'} = \frac{1}{x},$$

so

$$\ln|u'| = \ln|x| + k, \quad \text{or, equivalently,} \quad u' = C_1x.$$

Therefore

$$u = \frac{C_1}{2}x^2 + C_2,$$

so the general solution of (5.6.18) is

$$y = ux = \frac{C_1}{2}x^3 + C_2x,$$

which we rewrite as

$$y = c_1x + c_2x^3.$$

Therefore $\{x, x^3\}$ is a fundamental set of solutions of (5.6.18).

5.6 Exercises

In Exercises 1–17 find the general solution, given that y_1 satisfies the complementary equation. As a byproduct, find a fundamental set of solutions of the complementary equation.

1. $(2x+1)y''-2y'-(2x+3)y=(2x+1)^2; \quad y_1=e^{-x}$

2. $x^2y''+xy'-y=\dfrac{4}{x^2}; \quad y_1=x$

3. $x^2y''-xy'+y=x; \quad y_1=x$

4. $y''-3y'+2y=\dfrac{1}{1+e^{-x}}; \quad y_1=e^{2x}$

5. $y''-2y'+y=7x^{3/2}e^x; \quad y_1=e^x$

6. $4x^2y''+(4x-8x^2)y'+(4x^2-4x-1)y=4x^{1/2}e^x(1+4x); \quad y_1=x^{1/2}e^x$

7. $y''-2y'+2y=e^x\sec x; \quad y_1=e^x\cos x$

8. $y''+4xy'+(4x^2+2)y=8e^{-x(x+2)}; \quad y_1=e^{-x^2}$

9. $x^2y''+xy'-4y=-6x-4; \quad y_1=x^2$

10. $x^2y''+2x(x-1)y'+(x^2-2x+2)y=x^3e^{2x}; \quad y_1=xe^{-x}$

11. $x^2y''-x(2x-1)y'+(x^2-x-1)y=x^2e^x; \quad y_1=xe^x$

12. $(1-2x)y''+2y'+(2x-3)y=(1-4x+4x^2)e^x; \quad y_1=e^x$

13. $x^2y''-3xy'+4y=4x^4; \quad y_1=x^2$

14. $2xy''+(4x+1)y'+(2x+1)y=3x^{1/2}e^{-x}; \quad y_1=e^{-x}$

15. $xy''-(2x+1)y'+(x+1)y=-e^x; \quad y_1=e^x$

16. $4x^2y''-4x(x+1)y'+(2x+3)y=4x^{5/2}e^{2x}; \quad y_1=x^{1/2}$

17. $x^2y''-5xy'+8y=4x^2; \quad y_1=x^2$

In Exercises 18–30 find a fundamental set of solutions, given that y_1 is a solution.

18. $xy''+(2-2x)y'+(x-2)y=0; \quad y_1=e^x$

19. $x^2y''-4xy'+6y=0; \quad y_1=x^2$

20. $x^2(\ln|x|)^2y''-(2x\ln|x|)y'+(2+\ln|x|)y=0; \quad y_1=\ln|x|$

21. $4xy''+2y'+y=0; \quad y_1=\sin\sqrt{x}$

22. $xy''-(2x+2)y'+(x+2)y=0; \quad y_1=e^x$

23. $x^2y''-(2a-1)xy'+a^2y=0; \quad y_1=x^a$

24. $x^2y''-2xy'+(x^2+2)y=0; \quad y_1=x\sin x$

25. $xy''-(4x+1)y'+(4x+2)y=0; \quad y_1=e^{2x}$

26. $4x^2(\sin x)y''-4x(x\cos x+\sin x)y'+(2x\cos x+3\sin x)y=0; \quad y_1=x^{1/2}$

27. $4x^2y''-4xy'+(3-16x^2)y=0; \quad y_1=x^{1/2}e^{2x}$

28. $(2x+1)xy''-2(2x^2-1)y'-4(x+1)y=0; \quad y_1=1/x$

29. $(x^2-2x)y''+(2-x^2)y'+(2x-2)y=0; \quad y_1=e^x$

30. $xy'' - (4x+1)y' + (4x+2)y = 0; \quad y_1 = e^{2x}$

In Exercises 31–33 solve the initial value problem, given that y_1 satisfies the complementary equation.

31. $x^2y'' - 3xy' + 4y = 4x^4, \quad y(-1) = 7, \quad y'(-1) = -8; \quad y_1 = x^2$

32. $(3x-1)y'' - (3x+2)y' - (6x-8)y = 0, \quad y(0) = 2,\ y'(0) = 3; \quad y_1 = e^{2x}$

33. $(x+1)^2y'' - 2(x+1)y' - (x^2+2x-1)y = (x+1)^3e^x, \quad y(0) = 1, \quad y'(0) = -1;$
$y_1 = (x+1)e^x$

In Exercises 34 and 35 solve the initial value problem and graph the solution, given that y_1 satisfies the complementary equation.

34. C/G $x^2y'' + 2xy' - 2y = x^2, \quad y(1) = \frac{5}{4},\ y'(1) = \frac{3}{2}; \quad y_1 = x$

35. C/G $(x^2-4)y'' + 4xy' + 2y = x+2, \quad y(0) = -\frac{1}{3}, \quad y'(0) = -1; \quad y_1 = \frac{1}{x-2}$

36. Suppose p_1 and p_2 are continuous on (a,b). Let y_1 be a solution of

$$y'' + p_1(x)y' + p_2(x)y = 0 \tag{A}$$

that has no zeros on (a,b), and let x_0 be in (a,b). Use reduction of order to show that y_1 and

$$y_2(x) = y_1(x)\int_{x_0}^{x} \frac{1}{y_1^2(t)} \exp\left(-\int_{x_0}^{t} p_1(s)\,ds\right) dt$$

form a fundamental set of solutions of (A) on (a,b). (NOTE: This exercise is related to Exercise 9.)

37. The nonlinear first order equation

$$y' + y^2 + p(x)y + q(x) = 0 \tag{A}$$

is a *Riccati equation*. (See Exercise 2.4.55.) Assume that p and q are continuous.

(a) Show that y is a solution of (A) if and only if $y = z'/z$, where

$$z'' + p(x)z' + q(x)z = 0. \tag{B}$$

(b) Show that the general solution of (A) is

$$y = \frac{c_1z_1' + c_2z_2'}{c_1z_1 + c_2z_2}, \tag{C}$$

where $\{z_1, z_2\}$ is a fundamental set of solutions of (B) and c_1 and c_2 are arbitrary constants.

(c) Does the formula (C) imply that the first order equation (A) has a two–parameter family of solutions? Explain your answer.

38. Use a method suggested by Exercise 37 to find all solutions. of the equation.

(a) $y' + y^2 + k^2 = 0$
(b) $y' + y^2 - 3y + 2 = 0$
(c) $y' + y^2 + 5y - 6 = 0$
(d) $y' + y^2 + 8y + 7 = 0$
(e) $y' + y^2 + 14y + 50 = 0$
(f) $6y' + 6y^2 - y - 1 = 0$
(g) $36y' + 36y^2 - 12y + 1 = 0$

39. Use a method suggested by Exercise 37 and reduction of order to find all solutions of the equation, given that y_1 is a solution.

(a) $x^2(y' + y^2) - x(x + 2)y + x + 2 = 0; \quad y_1 = 1/x$

(b) $y' + y^2 + 4xy + 4x^2 + 2 = 0; \quad y_1 = -2x$

(c) $(2x + 1)(y' + y^2) - 2y - (2x + 3) = 0; \quad y_1 = -1$

(d) $(3x - 1)(y' + y^2) - (3x + 2)y - 6x + 8 = 0; \quad y_1 = 2$

(e) $x^2(y' + y^2) + xy + x^2 - \dfrac{1}{4} = 0; \quad y_1 = -\tan x - \dfrac{1}{2x}$

(f) $x^2(y' + y^2) - 7xy + 7 = 0; \quad y_1 = 1/x$

40. The nonlinear first order equation

$$y' + r(x)y^2 + p(x)y + q(x) = 0 \tag{A}$$

is the *generalized Riccati equation*. (See Exercise 2.4.55.) Assume that p and q are continuous and r is differentiable.

(a) Show that y is a solution of (A) if and only if $y = z'/rz$, where

$$z'' + \left[p(x) - \frac{r'(x)}{r(x)}\right]z' + r(x)q(x)z = 0. \tag{B}$$

(b) Show that the general solution of (A) is

$$y = \frac{c_1z_1' + c_2z_2'}{r(c_1z_1 + c_2z_2)},$$

where $\{z_1, z_2\}$ is a fundamental set of solutions of (B) and c_1 and c_2 are arbitrary constants.

5.7 VARIATION OF PARAMETERS

In this section we give a method called *variation of parameters* for finding a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{5.7.1}$$

if we know a fundamental set $\{y_1, y_2\}$ of solutions of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{5.7.2}$$

Having found a particular solution y_p by this method, we can write the general solution of (5.7.1) as

$$y = y_p + c_1y_1 + c_2y_2.$$

Since we need only one nontrivial solution of (5.7.2) to find the general solution of (5.7.1) by reduction of order, it's natural to ask why we're interested in variation of parameters, which requires two linearly independent solutions of (5.7.2) to achieve the same goal. Here's the answer:

- If we already know two linearly independent solutions of (5.7.2) then variation of parameters will probably be simpler than reduction of order.

- Variation of parameters generalizes naturally to a method for finding particular solutions of higher order linear equations (Section 9.4) and linear systems of equations (Section 10.7), while reduction of order doesn't.

- Variation of parameters is a powerful theoretical tool used by researchers in differential equations. Although a detailed discussion of this is beyond the scope of this book, you can get an idea of what it means from Exercises 37–39.

We'll now derive the method. As usual, we consider solutions of (5.7.1) and (5.7.2) on an interval (a, b) where P_0, P_1, P_2, and F are continuous and P_0 has no zeros. Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation (5.7.2). We look for a particular solution of (5.7.1) in the form

$$y_p = u_1 y_1 + u_2 y_2 \tag{5.7.3}$$

where u_1 and u_2 are functions to be determined so that y_p satisfies (5.7.1). You may not think this is a good idea, since there are now two unknown functions to be determined, rather than one. However, since u_1 and u_2 have to satisfy only one condition (that y_p is a solution of (5.7.1)), we can impose a second condition that produces a convenient simplification, as follows.

Differentiating (5.7.3) yields

$$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2. \tag{5.7.4}$$

As our second condition on u_1 and u_2 we require that

$$u_1' y_1 + u_2' y_2 = 0. \tag{5.7.5}$$

Then (5.7.4) becomes

$$y_p' = u_1 y_1' + u_2 y_2'; \tag{5.7.6}$$

that is, (5.7.5) permits us to differentiate y_p (once!) as if u_1 and u_2 are constants. Differentiating (5.7.4) yields

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'. \tag{5.7.7}$$

(There are no terms involving u_1'' and u_2'' here, as there would be if we hadn't required (5.7.5).) Substituting (5.7.3), (5.7.6), and (5.7.7) into (5.7.1) and collecting the coefficients of u_1 and u_2 yields

$$u_1(P_0 y_1'' + P_1 y_1' + P_2 y_1) + u_2(P_0 y_2'' + P_1 y_2' + P_2 y_2) + P_0(u_1' y_1' + u_2' y_2') = F.$$

As in the derivation of the method of reduction of order, the coefficients of u_1 and u_2 here are both zero because y_1 and y_2 satisfy the complementary equation. Hence, we can rewrite the last equation as

$$P_0(u_1' y_1' + u_2' y_2') = F. \tag{5.7.8}$$

Therefore y_p in (5.7.3) satisfies (5.7.1) if

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= \frac{F}{P_0}, \end{aligned} \tag{5.7.9}$$

where the first equation is the same as (5.7.5) and the second is from (5.7.8).

We'll now show that you can always solve (5.7.9) for u_1' and u_2'. (The method that we use here will always work, but simpler methods usually work when you're dealing with specific equations.) To obtain u_1', multiply the first equation in (5.7.9) by y_2' and the second equation by y_2. This yields

$$\begin{aligned} u_1' y_1 y_2' + u_2' y_2 y_2' &= 0 \\ u_1' y_1' y_2 + u_2' y_2' y_2 &= \frac{F y_2}{P_0}. \end{aligned}$$

Subtracting the second equation from the first yields

$$u_1'(y_1y_2' - y_1'y_2) = -\frac{Fy_2}{P_0}. \tag{5.7.10}$$

Since $\{y_1, y_2\}$ is a fundamental set of solutions of (5.7.2) on (a, b), Theorem 5.1.6 implies that the Wronskian $y_1y_2' - y_1'y_2$ has no zeros on (a, b). Therefore we can solve (5.7.10) for u_1', to obtain

$$u_1' = -\frac{Fy_2}{P_0(y_1y_2' - y_1'y_2)}. \tag{5.7.11}$$

We leave it to you to start from (5.7.9) and show by a similar argument that

$$u_2' = \frac{Fy_1}{P_0(y_1y_2' - y_1'y_2)}. \tag{5.7.12}$$

We can now obtain u_1 and u_2 by integrating u_1' and u_2'. The constants of integration can be taken to be zero, since any choice of u_1 and u_2 in (5.7.3) will suffice.

You should not memorize (5.7.11) and (5.7.12). On the other hand, you don't want to rederive the whole procedure for every specific problem. We recommend the a compromise:

(a) Write

$$y_p = u_1y_1 + u_2y_2 \tag{5.7.13}$$

to remind yourself of what you're doing.

(b) Write the system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \frac{F}{P_0} \end{aligned} \tag{5.7.14}$$

for the specific problem you're trying to solve.

(c) Solve (5.7.14) for u_1' and u_2' by any convenient method.

(d) Obtain u_1 and u_2 by integrating u_1' and u_2', taking the constants of integration to be zero.

(e) Substitute u_1 and u_2 into (5.7.13) to obtain y_p.

Example 5.7.1 Find a particular solution y_p of

$$x^2y'' - 2xy' + 2y = x^{9/2}, \tag{5.7.15}$$

given that $y_1 = x$ and $y_2 = x^2$ are solutions of the complementary equation

$$x^2y'' - 2xy' + 2y = 0.$$

Then find the general solution of (5.7.15).

Solution We set

$$y_p = u_1x + u_2x^2,$$

where

$$\begin{aligned} u_1'x + u_2'x^2 &= 0 \\ u_1' + 2u_2'x &= \frac{x^{9/2}}{x^2} = x^{5/2}. \end{aligned}$$

From the first equation, $u_1' = -u_2'x$. Substituting this into the second equation yields $u_2'x = x^{5/2}$, so $u_2' = x^{3/2}$ and therefore $u_1' = -u_2'x = -x^{5/2}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -\frac{2}{7}x^{7/2} \quad \text{and} \quad u_2 = \frac{2}{5}x^{5/2}.$$

Therefore

$$y_p = u_1x + u_2x^2 = -\frac{2}{7}x^{7/2}x + \frac{2}{5}x^{5/2}x^2 = \frac{4}{35}x^{9/2},$$

and the general solution of (5.7.15) is

$$y = \frac{4}{35}x^{9/2} + c_1x + c_2x^2.$$

Example 5.7.2 Find a particular solution y_p of

$$(x-1)y'' - xy' + y = (x-1)^2, \tag{5.7.16}$$

given that $y_1 = x$ and $y_2 = e^x$ are solutions of the complementary equation

$$(x-1)y'' - xy' + y = 0.$$

Then find the general solution of (5.7.16).

Solution We set

$$y_p = u_1x + u_2e^x,$$

where

$$\begin{aligned} u_1'x + u_2'e^x &= 0 \\ u_1' \quad + u_2'e^x &= \frac{(x-1)^2}{x-1} = x-1. \end{aligned}$$

Subtracting the first equation from the second yields $-u_1'(x-1) = x-1$, so $u_1' = -1$. From this and the first equation, $u_2' = -xe^{-x}u_1' = xe^{-x}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -x \quad \text{and} \quad u_2 = -(x+1)e^{-x}.$$

Therefore

$$y_p = u_1x + u_2e^x = (-x)x + (-(x+1)e^{-x})e^x = -x^2 - x - 1,$$

so the general solution of (5.7.16) is

$$y = y_p + c_1x + c_2e^x = -x^2 - x - 1 + c_1x + c_2e^x = -x^2 - 1 + (c_1 - 1)x + c_2e^x. \tag{5.7.17}$$

However, since c_1 is an arbitrary constant, so is $c_1 - 1$; therefore, we improve the appearance of this result by renaming the constant and writing the general solution as

$$y = -x^2 - 1 + c_1x + c_2e^x. \ \blacksquare \tag{5.7.18}$$

There's nothing *wrong* with leaving the general solution of (5.7.16) in the form (5.7.17); however, we think you'll agree that (5.7.18) is preferable. We can also view the transition from (5.7.17) to (5.7.18) differently. In this example the particular solution $y_p = -x^2 - x - 1$ contained the term $-x$, which satisfies the complementary equation. We can drop this term and redefine $y_p = -x^2 - 1$, since $-x^2 - x - 1$

is a solution of (5.7.16) and x is a solution of the complementary equation; hence, $-x^2-1=(-x^2-x-1)+x$ is also a solution of (5.7.16). In general, it's always legitimate to drop linear combinations of $\{y_1, y_2\}$ from particular solutions obtained by variation of parameters. (See Exercise 36 for a general discussion of this question.) We'll do this in the following examples and in the answers to exercises that ask for a particular solution. Therefore, don't be concerned if your answer to such an exercise differs from ours only by a solution of the complementary equation.

Example 5.7.3 Find a particular solution of

$$y''+3y'+2y=\frac{1}{1+e^x}. \tag{5.7.19}$$

Then find the general solution.

Solution

The characteristic polynomial of the complementary equation

$$y''+3y'+2y=0 \tag{5.7.20}$$

is $p(r)=r^2+3r+2=(r+1)(r+2)$, so $y_1=e^{-x}$ and $y_2=e^{-2x}$ form a fundamental set of solutions of (5.7.20). We look for a particular solution of (5.7.19) in the form

$$y_p=u_1e^{-x}+u_2e^{-2x},$$

where

$$\begin{aligned} u_1'e^{-x}+\ u_2'e^{-2x} &= 0\\ -u_1'e^{-x}-2u_2'e^{-2x} &= \frac{1}{1+e^x}. \end{aligned}$$

Adding these two equations yields

$$-u_2'e^{-2x}=\frac{1}{1+e^x}, \quad \text{so} \quad u_2'=-\frac{e^{2x}}{1+e^x}.$$

From the first equation,

$$u_1'=-u_2'e^{-x}=\frac{e^x}{1+e^x}.$$

Integrating by means of the substitution $v=e^x$ and taking the constants of integration to be zero yields

$$u_1=\int\frac{e^x}{1+e^x}\,dx=\int\frac{dv}{1+v}=\ln(1+v)=\ln(1+e^x)$$

and

$$\begin{aligned} u_2 &= -\int\frac{e^{2x}}{1+e^x}\,dx=-\int\frac{v}{1+v}\,dv=\int\left[\frac{1}{1+v}-1\right]dv\\ &= \ln(1+v)-v=\ln(1+e^x)-e^x. \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= u_1e^{-x}+u_2e^{-2x}\\ &= [\ln(1+e^x)]e^{-x}+[\ln(1+e^x)-e^x]\,e^{-2x}, \end{aligned}$$

so

$$y_p = \left(e^{-x} + e^{-2x}\right)\ln(1+e^x) - e^{-x}.$$

Since the last term on the right satisfies the complementary equation, we drop it and redefine

$$y_p = \left(e^{-x} + e^{-2x}\right)\ln(1+e^x).$$

The general solution of (5.7.19) is

$$y = y_p + c_1e^{-x} + c_2e^{-2x} = \left(e^{-x} + e^{-2x}\right)\ln(1+e^x) + c_1e^{-x} + c_2e^{-2x}.$$

Example 5.7.4 Solve the initial value problem

$$(x^2-1)y'' + 4xy' + 2y = \frac{2}{x+1}, \quad y(0) = -1, \quad y'(0) = -5, \tag{5.7.21}$$

given that

$$y_1 = \frac{1}{x-1} \quad \text{and} \quad y_2 = \frac{1}{x+1}$$

are solutions of the complementary equation

$$(x^2-1)y'' + 4xy' + 2y = 0.$$

Solution We first use variation of parameters to find a particular solution of

$$(x^2-1)y'' + 4xy' + 2y = \frac{2}{x+1}$$

on $(-1, 1)$ in the form

$$y_p = \frac{u_1}{x-1} + \frac{u_2}{x+1},$$

where

$$\begin{aligned} \frac{u_1'}{x-1} + \frac{u_2'}{x+1} &= 0 \\ -\frac{u_1'}{(x-1)^2} - \frac{u_2'}{(x+1)^2} &= \frac{2}{(x+1)(x^2-1)}. \end{aligned} \tag{5.7.22}$$

Multiplying the first equation by $1/(x-1)$ and adding the result to the second equation yields

$$\left[\frac{1}{x^2-1} - \frac{1}{(x+1)^2}\right]u_2' = \frac{2}{(x+1)(x^2-1)}. \tag{5.7.23}$$

Since

$$\left[\frac{1}{x^2-1} - \frac{1}{(x+1)^2}\right] = \frac{(x+1)-(x-1)}{(x+1)(x^2-1)} = \frac{2}{(x+1)(x^2-1)},$$

(5.7.23) implies that $u_2' = 1$. From (5.7.22),

$$u_1' = -\frac{x-1}{x+1}u_2' = -\frac{x-1}{x+1}.$$

Integrating and taking the constants of integration to be zero yields

$$\begin{aligned} u_1 &= -\int \frac{x-1}{x+1}\,dx = -\int \frac{x+1-2}{x+1}\,dx \\ &= \int \left[\frac{2}{x+1} - 1\right] dx = 2\ln(x+1) - x \end{aligned}$$

and

$$u_2 = \int dx = x.$$

Therefore

$$\begin{aligned} y_p &= \frac{u_1}{x-1} + \frac{u_2}{x+1} = [2\ln(x+1) - x]\frac{1}{x-1} + x\frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1} + x\left[\frac{1}{x+1} - \frac{1}{x-1}\right] = \frac{2\ln(x+1)}{x-1} - \frac{2x}{(x+1)(x-1)}. \end{aligned}$$

However, since

$$\frac{2x}{(x+1)(x-1)} = \left[\frac{1}{x+1} + \frac{1}{x-1}\right]$$

is a solution of the complementary equation, we redefine

$$y_p = \frac{2\ln(x+1)}{x-1}.$$

Therefore the general solution of (5.7.24) is

$$y = \frac{2\ln(x+1)}{x-1} + \frac{c_1}{x-1} + \frac{c_2}{x+1}. \tag{5.7.24}$$

Differentiating this yields

$$y' = \frac{2}{x^2-1} - \frac{2\ln(x+1)}{(x-1)^2} - \frac{c_1}{(x-1)^2} - \frac{c_2}{(x+1)^2}.$$

Setting $x = 0$ in the last two equations and imposing the initial conditions $y(0) = -1$ and $y'(0) = -5$ yields the system

$$\begin{aligned} -c_1 + c_2 &= -1 \\ -2 - c_1 - c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = 2$, $c_2 = 1$. Substituting these into (5.7.24) yields

$$\begin{aligned} y &= \frac{2\ln(x+1)}{x-1} + \frac{2}{x-1} + \frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1} \end{aligned}$$

as the solution of (5.7.21). Figure 5.7.1 is a graph of the solution.

Comparison of Methods

We've now considered three methods for solving nonhomogeneous linear equations: undetermined coefficients, reduction of order, and variation of parameters. It's natural to ask which method is best for a given

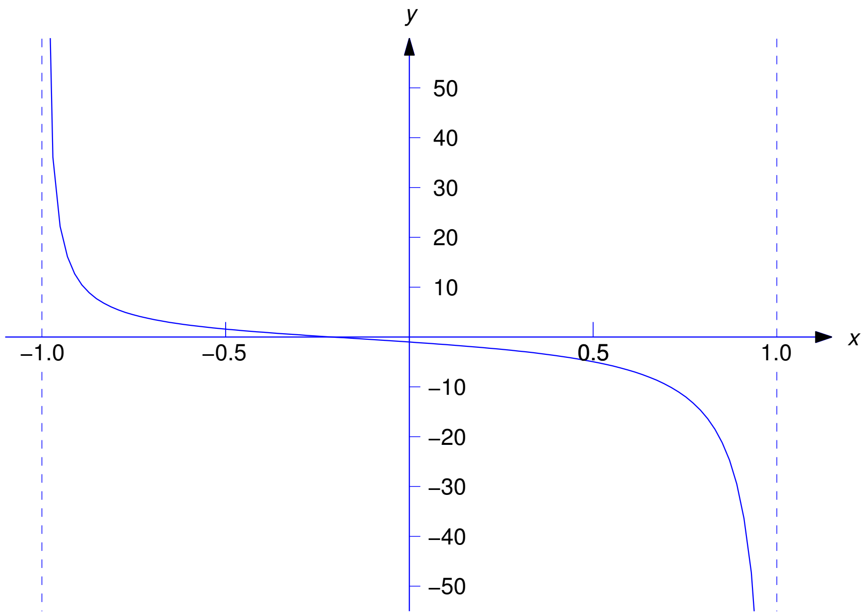

Figure 5.7.1 $y = \dfrac{2\ln(x+1)}{x-1} + \dfrac{3x+1}{x^2-1}$

problem. The method of undetermined coefficients should be used for constant coefficient equations with forcing functions that are linear combinations of polynomials multiplied by functions of the form $e^{\alpha x}$, $e^{\lambda x}\cos\omega x$, or $e^{\lambda x}\sin\omega x$. Although the other two methods can be used to solve such problems, they will be more difficult except in the most trivial cases, because of the integrations involved.

If the equation isn't a constant coefficient equation or the forcing function isn't of the form just specified, the method of undetermined coefficients does not apply and the choice is necessarily between the other two methods. The case could be made that reduction of order is better because it requires only one solution of the complementary equation while variation of parameters requires two. However, variation of parameters will probably be easier if you already know a fundamental set of solutions of the complementary equation.

5.7 Exercises

In Exercises 1–6 use variation of parameters to find a particular solution.

1. $y'' + 9y = \tan 3x$

2. $y'' + 4y = \sin 2x \sec^2 2x$

3. $y'' - 3y' + 2y = \dfrac{4}{1+e^{-x}}$

4. $y'' - 2y' + 2y = 3e^x \sec x$

5. $y'' - 2y' + y = 14x^{3/2}e^x$

6. $y'' - y = \dfrac{4e^{-x}}{1-e^{-2x}}$

In Exercises 7–29 use variation of parameters to find a particular solution, given the solutions y_1, y_2 of the complementary equation.

7. $x^2y'' + xy' - y = 2x^2 + 2; \quad y_1 = x, \quad y_2 = \dfrac{1}{x}$

8. $xy'' + (2-2x)y' + (x-2)y = e^{2x}; \quad y_1 = e^x, \quad y_2 = \dfrac{e^x}{x}$

9. $4x^2y'' + (4x - 8x^2)y' + (4x^2 - 4x - 1)y = 4x^{1/2}e^x, \quad x > 0;$
$y_1 = x^{1/2}e^x,\ y_2 = x^{-1/2}e^x$

10. $y'' + 4xy' + (4x^2 + 2)y = 4e^{-x(x+2)}; \quad y_1 = e^{-x^2}, \quad y_2 = xe^{-x^2}$

11. $x^2y'' - 4xy' + 6y = x^{5/2},\ x > 0; \quad y_1 = x^2,\ y_2 = x^3$

12. $x^2y'' - 3xy' + 3y = 2x^4 \sin x; \quad y_1 = x,\ y_2 = x^3$

13. $(2x+1)y'' - 2y' - (2x+3)y = (2x+1)^2e^{-x}; \quad y_1 = e^{-x}, \quad y_2 = xe^x$

14. $4xy'' + 2y' + y = \sin\sqrt{x}; \quad y_1 = \cos\sqrt{x}, \quad y_2 = \sin\sqrt{x}$

15. $xy'' - (2x+2)y' + (x+2)y = 6x^3e^x; \quad y_1 = e^x, \quad y_2 = x^3e^x$

16. $x^2y'' - (2a-1)xy' + a^2y = x^{a+1}; \quad y_1 = x^a, \quad y_2 = x^a \ln x$

17. $x^2y'' - 2xy' + (x^2+2)y = x^3 \cos x; \quad y_1 = x\cos x, \quad y_2 = x \sin x$

18. $xy'' - y' - 4x^3y = 8x^5; \quad y_1 = e^{x^2},\ y_2 = e^{-x^2}$

19. $(\sin x)y'' + (2\sin x - \cos x)y' + (\sin x - \cos x)y = e^{-x}; \quad y_1 = e^{-x}, \quad y_2 = e^{-x}\cos x$

20. $4x^2y'' - 4xy' + (3 - 16x^2)y = 8x^{5/2}; \quad y_1 = \sqrt{x}e^{2x},\ y_2 = \sqrt{x}e^{-2x}$

21. $4x^2y'' - 4xy' + (4x^2+3)y = x^{7/2}; \quad y_1 = \sqrt{x}\sin x,\ y_2 = \sqrt{x}\cos x$

22. $x^2y'' - 2xy' - (x^2-2)y = 3x^4; \quad y_1 = xe^x,\ y_2 = xe^{-x}$

23. $x^2y'' - 2x(x+1)y' + (x^2+2x+2)y = x^3e^x; \quad y_1 = xe^x, \quad y_2 = x^2e^x$

24. $x^2y'' - xy' - 3y = x^{3/2}; \quad y_1 = 1/x, \quad y_2 = x^3$

25. $x^2y'' - x(x+4)y' + 2(x+3)y = x^4e^x; \quad y_1 = x^2, \quad y_2 = x^2e^x$

26. $x^2y'' - 2x(x+2)y' + (x^2+4x+6)y = 2xe^x; \quad y_1 = x^2e^x, \quad y_2 = x^3e^x$

27. $x^2y'' - 4xy' + (x^2+6)y = x^4; \quad y_1 = x^2\cos x, \quad y_2 = x^2 \sin x$

28. $(x-1)y'' - xy' + y = 2(x-1)^2e^x; \quad y_1 = x, \quad y_2 = e^x$

29. $4x^2y'' - 4x(x+1)y' + (2x+3)y = x^{5/2}e^x; \quad y_1 = \sqrt{x}, \quad y_2 = \sqrt{x}e^x$

In Exercises 30–32 use variation of parameters to solve the initial value problem, given y_1, y_2 are solutions of the complementary equation.

30. $(3x-1)y'' - (3x+2)y' - (6x-8)y = (3x-1)^2e^{2x}, \quad y(0) = 1,\ y'(0) = 2;$
$y_1 = e^{2x},\ y_2 = xe^{-x}$

31. $(x-1)^2y'' - 2(x-1)y' + 2y = (x-1)^2, \quad y(0) = 3, \quad y'(0) = -6;$
$y_1 = x-1,\ y_2 = x^2 - 1$

32. $(x-1)^2y'' - (x^2-1)y' + (x+1)y = (x-1)^3e^x, \quad y(0) = 4, \quad y'(0) = -6;$
$y_1 = (x-1)e^x, \quad y_2 = x - 1$

In Exercises 33–35 use variation of parameters to solve the initial value problem and graph the solution, given that y_1, y_2 are solutions of the complementary equation.

33. C/G $(x^2-1)y''+4xy'+2y=2x, \quad y(0)=0,\ y'(0)=-2; \quad y_1=\dfrac{1}{x-1},\ y_2=\dfrac{1}{x+1}$

34. C/G $x^2y''+2xy'-2y=-2x^2, \quad y(1)=1,\ y'(1)=-1; \quad y_1=x,\ y_2=\dfrac{1}{x^2}$

35. C/G $(x+1)(2x+3)y''+2(x+2)y'-2y=(2x+3)^2, \quad y(0)=0, \quad y'(0)=0;$
$y_1=x+2, \quad y_2=\dfrac{1}{x+1}$

36. Suppose

$$y_p=\overline{y}+a_1y_1+a_2y_2$$

is a particular solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F(x), \tag{A}$$

where y_1 and y_2 are solutions of the complementary equation

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0.$$

Show that $\overline{y}$ is also a solution of (A).

37. Suppose p, q, and f are continuous on (a,b) and let x_0 be in (a,b). Let y_1 and y_2 be the solutions of

$$y''+p(x)y'+q(x)y=0$$

such that

$$y_1(x_0)=1, \quad y_1'(x_0)=0, \quad y_2(x_0)=0, \quad y_2'(x_0)=1.$$

Use variation of parameters to show that the solution of the initial value problem

$$y''+p(x)y'+q(x)y=f(x), \quad y(x_0)=k_0,\ y'(x_0)=k_1,$$

is

$$\begin{aligned} y(x) &= k_0y_1(x)+k_1y_2(x) \\ &\quad + \int_{x_0}^{x}\left(y_1(t)y_2(x)-y_1(x)y_2(t)\right) f(t)\exp\left(\int_{x_0}^{t}p(s)\,ds\right)dt. \end{aligned}$$

HINT: *Use Abel's formula for the Wronskian of $\{y_1,y_2\}$, and integrate u_1' and u_2' from x_0 to x.* Show also that

$$\begin{aligned} y'(x) &= k_0y_1'(x)+k_1y_2'(x) \\ &\quad + \int_{x_0}^{x}\left(y_1(t)y_2'(x)-y_1'(x)y_2(t)\right) f(t)\exp\left(\int_{x_0}^{t}p(s)\,ds\right)dt. \end{aligned}$$

38. Suppose f is continuous on an open interval that contains $x_0=0$. Use variation of parameters to find a formula for the solution of the initial value problem

$$y''-y=f(x), \quad y(0)=k_0, \quad y'(0)=k_1.$$

39. Suppose f is continuous on (a,∞), where $a<0$, so $x_0=0$ is in (a,∞).

(a) Use variation of parameters to find a formula for the solution of the initial value problem

$$y'' + y = f(x), \quad y(0) = k_0, \quad y'(0) = k_1.$$

HINT: *You will need the addition formulas for the sine and cosine:*

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B\\ \cos(A+B) &= \cos A\cos B - \sin A\sin B.\end{aligned}$$

For the rest of this exercise assume that the improper integral $\int_0^\infty f(t)\,dt$ is absolutely convergent.

(b) Show that if y is a solution of

$$y'' + y = f(x) \tag{A}$$

on (a,∞), then

$$\lim_{x\to\infty}\left(y(x) - A_0\cos x - A_1\sin x\right) = 0 \tag{B}$$

and

$$\lim_{x\to\infty}\left(y'(x) + A_0\sin x - A_1\cos x\right) = 0, \tag{C}$$

where

$$A_0 = k_0 - \int_0^\infty f(t)\sin t\,dt \quad \text{and} \quad A_1 = k_1 + \int_0^\infty f(t)\cos t\,dt.$$

HINT: *Recall from calculus that if* $\int_0^\infty f(t)\,dt$ *converges absolutely, then* $\lim_{x\to\infty}\int_x^\infty |f(t)|\,dt = 0$.

(c) Show that if A_0 and A_1 are arbitrary constants, then there's a unique solution of $y'' + y = f(x)$ on (a,∞) that satisfies (B) and (C).