CHAPTER 9

Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

INTRODUCTORY REMARK

Thus far we have concentrated on first-order differential equations. We will now turn our attention to the second-order case. After investigating solution techniques, we will discuss applications of these differential equations (see Chapter 14).

THE CHARACTERISTIC EQUATION

Corresponding to the differential equation

$$y'' + a_1 y' + a_0 y = 0 \tag{9.1}$$

in which a_1 and a_0 are constants, is the algebraic equation

$$\lambda^2 + a_1\lambda + a_0 = 0 \tag{9.2}$$

which is obtained from Eq. (*9.1*) by replacing y'', y' and y by λ^2, λ^1, and $\lambda^0 = 1$, respectively. Equation (*9.2*) is called the *characteristic equation* of (*9.1*).

Example 9.1. The characteristic equation of $y'' + 3y' - 4y = 0$ is $\lambda^2 + 3\lambda - 4 = 0$; the characteristic equation of $y'' - 2y' + y = 0$ is $\lambda^2 - 2\lambda + 1 = 0$.

Characteristic equations for differential equations having dependent variables other than y are obtained analogously, by replacing the jth derivative of the dependent variable by λ^j ($j = 0, 1, 2$).

The characteristic equation can be factored into

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0 \tag{9.3}$$

THE GENERAL SOLUTION

The general solution of (*9.1*) is obtained directly from the roots of (*9.3*). There are three cases to consider.

Case 1. λ_1 and λ_2 both real and distinct. Two linearly independent solutions are $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$, and the general solution is (Theorem 8.2)

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \tag{9.4}$$

In the special case $\lambda_2 = -\lambda_1$, the solution (*9.4*) can be rewritten as $y = k_1 \cosh \lambda_1 x + k_2 \sinh \lambda_1 x$.

Case 2. $\lambda_1 = a + ib$, a complex number. Since a_1 and a_0 in (*9.1*) and (*9.2*) are assumed real, the roots of (*9.2*) must appear in conjugate pairs; thus, the other root is $\lambda_2 = a - ib$. Two linearly independent solutions are $e^{(a+ib)x}$ and $e^{(a-ib)x}$, and the general complex solution is

$$y = d_1 e^{(a+ib)x} + d_2 e^{(a-ib)x} \tag{9.5}$$

which is algebraically equivalent to (see Problem 9.16)

$$y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx \tag{9.6}$$

Case 3. $\lambda_1 = \lambda_2$. Two linearly independent solutions are $e^{\lambda_1 x}$ and $xe^{\lambda_1 x}$, and the general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} \tag{9.7}$$

Warning: The above solutions *are not valid* if the differential equation is not linear or does not have constant coefficients. Consider, for example, the equation $y'' - x^2 y = 0$. The roots of the characteristic equation are $\lambda_1 = x$ and $\lambda_2 = -x$, but the solution is *not*

$$y = c_1 e^{(x)x} + c_2 e^{(-x)x} = c_1 e^{x^2} + c_2 e^{-x^2}$$

Linear equations with variable coefficients are considered in Chapters 27, 28 and 29.

Solved Problems

9.1. Solve $y'' - y' - 2y = 0$.

The characteristic equation is $\lambda^2 - \lambda - 2 = 0$, which can be factored into $(\lambda + 1)(\lambda - 2) = 0$. Since the roots $\lambda_1 = -1$ and $\lambda_2 = 2$ are real and distinct, the solution is given by (*9.4*) as

$$y = c_1 e^{-x} + c_2 e^{2x}$$

9.2. Solve $y'' - 7y' = 0$.

The characteristic equation is $\lambda^2 - 7\lambda = 0$, which can be factored into $(\lambda - 0)(\lambda - 7) = 0$. Since the roots $\lambda_1 = 0$ and $\lambda_2 = 7$ are real and distinct, the solution is given by (*9.4*) as

$$y = c_1 e^{0x} + c_2 e^{7x} = c_1 + c_2 e^{7x}$$

9.3. Solve $y'' - 5y = 0$.

The characteristic equation is $\lambda^2 - 5 = 0$, which can be factored into $(\lambda - \sqrt{5})(\lambda + \sqrt{5}) = 0$. Since the roots $\lambda_1 = \sqrt{5}$ and $\lambda_2 = -\sqrt{5}$ are real and distinct, the solution is given by (*9.4*) as

$$y = c_1 e^{\sqrt{5}x} + c_2 e^{-\sqrt{5}x}$$

9.4. Rewrite the solution of Problem 9.3 in terms of hyperbolic functions.

Using the results of Problem 9.3 with the identities

$$e^{\lambda x} = \cosh \lambda x + \sinh \lambda x \quad \text{and} \quad e^{-\lambda x} = \cosh \lambda x - \sinh \lambda x$$

we obtain,

$$\begin{aligned} y &= c_1 e^{\sqrt{5}x} + c_2 e^{-\sqrt{5}x} \\ &= c_1(\cosh \sqrt{5}x + \sinh \sqrt{5}x) + c_2(\cosh \sqrt{5}x - \sinh \sqrt{5}x) \\ &= (c_1 + c_2)\cosh \sqrt{5}x + (c_1 - c_2)\sinh \sqrt{5}x \\ &= k_1 \cosh \sqrt{5}x + k_2 \sinh \sqrt{5}x \end{aligned}$$

where $k_1 = c_1 + c_2$ and $k_2 = c_1 - c_2$.

9.5. Solve $\ddot{y} + 10\dot{y} + 21y = 0$.

Here the independent variable is t. The characteristic equation is

$$\lambda^2 + 10\lambda + 21 = 0$$

which can be factored into

$$(\lambda + 3)(\lambda + 7) = 0$$

The roots $\lambda_1 = -3$ and $\lambda_2 = -7$ are real and distinct, so the general solution is

$$y = c_1 e^{-3t} + c_2 e^{-7t}$$

9.6. Solve $\ddot{x} - 0.01x = 0$.

The characteristic equation is

$$\lambda^2 - 0.01 = 0$$

which can be factored into

$$(\lambda - 0.1)(\lambda + 0.1) = 0$$

The roots $\lambda_1 = 0.1$ and $\lambda_2 = -0.1$ are real and distinct, so the general solution is

$$y = c_1 e^{0.1t} + c_2 e^{-0.1t}$$

or, equivalently,

$$y = k_1 \cosh 0.1t + k_2 \sinh 0.1t$$

9.7. Solve $y'' + 4y' + 5y = 0$.

The characteristic equation is

$$\lambda^2 + 4\lambda + 5 = 0$$

Using the quadratic formula, we find its roots to be

$$\lambda = \frac{-(4) \pm \sqrt{(4)^2 - 4(5)}}{2} = -2 \pm i$$

These roots are a complex conjugate pair, so the general solution is given by (*9.6*) (with $a = -2$ and $b = 1$) as

$$y = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$$

9.8. Solve $y'' + 4y = 0$.

The characteristic equation is

$$\lambda^2 + 4\lambda = 0$$

which can be factored into

$$(\lambda - 2i)(\lambda + 2i) = 0$$

These roots are a complex conjugate pair, so the general solution is given by (*9.6*) (with $a = 0$ and $b = 2$) as

$$y = c_1 \cos 2x + c_2 \sin 2x$$

9.9. Solve $y'' - 3y' + 4y = 0$.

The characteristic equation is

$$\lambda^2 - 3\lambda + 4 = 0$$

Using the quadratic formula, we find its roots to be

$$\lambda = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

These roots are a complex conjugate pair, so the general solution is given by (*9.6*) as

$$y = c_1 e^{(3/2)x} \cos\frac{\sqrt{7}}{2}x + c_2 e^{(3/2)x} \sin\frac{\sqrt{7}}{2}x$$

9.10. Solve $\ddot{y} - 6\dot{y} + 25y = 0$.

The characteristic equation is

$$\lambda^2 - 6\lambda + 25 = 0$$

Using the quadratic formula, we find its roots to be

$$\lambda = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(25)}}{2} = 3 \pm i4$$

These roots are a complex conjugate pair, so the general solution is

$$y = c_1 e^{3t} \cos 4t + c_2 e^{3t} \sin 4t$$

9.11. Solve $\dfrac{d^2I}{dt^2} + 20\dfrac{dI}{dt} + 200I = 0$.

The characteristic equation is

$$\lambda^2 - 20\lambda + 200 = 0$$

Using the quadratic formula, we find its roots to be

$$\lambda = \frac{-(20) \pm \sqrt{(20)^2 - 4(200)}}{2} = -10 \pm i10$$

These roots are a complex conjugate pair, so the general solution is

$$I = c_1 e^{-10t} \cos 10t + c_2 e^{-10t} \sin 10t$$

9.12. Solve $y'' - 8y' + 16y = 0$.

The characteristic equation is

$$\lambda^2 - 8\lambda + 16 = 0$$

which can be factored into

$$(\lambda - 4)^2 = 0$$

The roots $\lambda_1 = \lambda_2 = 4$ are real and equal, so the general solution is given by (*9.7*) as

$$y = c_1 e^{4x} + c_2 x e^{4x}$$

9.13. Solve $y'' = 0$.

The characteristic equation is $\lambda^2 = 0$, which has roots $\lambda_1 = \lambda_2 = 0$. The solution is given by (*9.7*) as

$$y = c_1 e^{0x} + c_2 x e^{0x} = c_1 + c_2 x$$

9.14. Solve $\ddot{x} + 4\dot{x} + 4x = 0$.

The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0$$

which can be factored into

$$(\lambda + 2)^2 = 0$$

The roots $\lambda_1 = \lambda_2 = -2$ are real and equal, so the general solution is

$$x = c_1 e^{-2t} + c_2 t e^{-2t}$$

9.15. Solve $100\dfrac{d^2N}{dt^2} - 20\dfrac{dN}{dt} + N = 0$.

Dividing both sides of the differential equation by 100, to force the coefficient of the highest derivative to be unity, we obtain

$$\frac{d^2N}{dt^2} - 0.2\frac{dN}{dt} + 0.01N = 0$$

Its characteristic equation is

$$\lambda^2 - 0.2\lambda + 0.01 = 0$$

which can be factored into

$$(\lambda - 0.1)^2 = 0$$

The roots $\lambda_1 = \lambda_2 = 0.1$ are real and equal, so the general solution is

$$N = c_1 e^{-0.1t} + c_2 t e^{-0.1t}$$

9.16. Prove that (*9.6*) is algebraically equivalent to (*9.5*).

Using Euler's relations

$$e^{ibx} = \cos bx + i \sin bx \quad e^{-ibx} = \cos bx - i \sin bx$$

we can rewrite (*9.5*) as

$$\begin{aligned} y &= d_1 e^{ax} e^{ibx} + d_2 e^{ax} e^{-ibx} = e^{ax}(d_1 e^{ibx} + d_2 e^{-ibx}) \\ &= e^{ax}[d_1(\cos bx + i \sin bx) + d_2(\cos bx - i \sin bx)] \\ &= e^{ax}[(d_1 + d_2)\cos bx + i(d_1 - d_2)\sin bx] \\ &= c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx \end{aligned} \tag{1}$$

where $c_1 = d_1 + d_2$ and $c_2 = i(d_1 - d_2)$.

Equation (*1*) is real if and only if c_1 and c_2 are both real, which occurs, if and only if d_1 and d_2 are complex conjugates. Since we are interested in the general *real* solution to (*9.1*), we restrict d_1 and d_2 to be a conjugate pair.

Supplementary Problems

Solve the following differential equations.

9.17. $y'' - y = 0$

9.18. $y'' - y' - 30y = 0$

9.19. $y'' - 2y' + y = 0$

9.20. $y'' + y = 0$

9.21. $y'' + 2y' + 2y = 0$

9.22. $y'' - 7y = 0$

9.23. $y'' + 6y' + 9y = 0$

9.24. $y'' + 2y' + 3y = 0$

9.25. $y'' - 3y' - 5y = 0$

9.26. $y'' + y' + \frac{1}{4}y = 0$

9.27. $\ddot{x} - 20\dot{x} + 64x = 0$

9.28. $\ddot{x} + 60\dot{x} + 500x = 0$

9.29. $\ddot{x} - 3\dot{x} + x = 0$

9.30. $\ddot{x} - 10\dot{x} + 25x = 0$

9.31. $\ddot{x} + 25x = 0$

9.32. $\ddot{x} + 25\dot{x} = 0$

9.33. $\ddot{x} + \dot{x} + 2x = 0$

9.34. $\ddot{u} - 2\dot{u} + 4u = 0$

9.35. $\ddot{u} - 4\dot{u} + 2u = 0$

9.36. $\ddot{u} - 36\dot{u} = 0$

9.37. $\ddot{u} - 36u = 0$

9.38. $\frac{d^2Q}{dt^2} - 5\frac{dQ}{dt} + 7Q = 0$

9.39. $\frac{d^2Q}{dt^2} - 7\frac{dQ}{dt} + 5Q = 0$

9.40. $\frac{d^2P}{dt^2} - 18\frac{dP}{dt} + 81P = 0$

9.41. $\frac{d^2P}{dx^2} + 2\frac{dP}{dx} + 9P = 0$

9.42. $\frac{d^2N}{dx^2} + 5\frac{dN}{dx} - 24N = 0$

9.43. $\frac{d^2N}{dx^2} + 5\frac{dN}{dx} + 24N = 0$

9.44. $\frac{d^2T}{d\theta^2} + 30\frac{dT}{d\theta} + 225T = 0$

9.45. $\frac{d^2R}{d\theta^2} + 5\frac{dR}{d\theta} = 0$

CHAPTER 10

*n*th-Order Linear Homogeneous Differential Equations with Constant Coefficients

THE CHARACTERISTIC EQUATION

The characteristic equation of the differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0 \tag{10.1}$$

with constant coefficients a_j $(j = 0, 1, \ldots, n-1)$ is

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \tag{10.2}$$

The characteristic equation (*10.2*) is obtained from (*10.1*) by replacing $y^{(j)}$ by λ^j $(j = 0, 1, \ldots, n-1)$. Characteristic equations for differential equations having dependent variables other than y are obtained analogously, by replacing the jth derivative of the dependent variable by λ^j $(j = 0, 1, \ldots, n-1)$.

Example 10.1. The characteristic equation of $y^{(4)} - 3y''' + 2y'' - y = 0$ is $\lambda^4 - 3\lambda^3 + 2\lambda^2 - 1 = 0$. The characteristic equation of

$$\frac{d^5x}{dt^5} - 3\frac{d^3x}{dt^3} + 5\frac{dx}{dt} - 7x = 0$$

is

$$\lambda^5 - 3\lambda^3 + 5\lambda - 7 = 0$$

Caution: Characteristic equations are only defined for linear homogeneous differential equations with constant coefficients.

THE GENERAL SOLUTION

The roots of the characteristic equation determine the solution of (*10.1*). If the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all real and distinct, the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x} \tag{10.3}$$

If the roots are distinct, but some are complex, then the solution is again given by (*10.3*). As in Chapter 9, those terms involving complex exponentials can be combined to yield terms involving sines and cosines. If λ_k is a root of multiplicity p [that is, if $(\lambda - \lambda_k)^p$ is a factor of the characteristic equation, but $(\lambda - \lambda_k)^{p+1}$ is not] then there will be p linearly independent solutions associated with λ_k given by $e^{\lambda_k x}, xe^{\lambda_k x}, x^2 e^{\lambda_k x}, \ldots, x^{p-1} e^{\lambda_k x}$. These solutions are combined in the usual way with the solutions associated with the other roots to obtain the complete solution.

In theory it is always possible to factor the characteristic equation, but in practice this can be extremely difficult, especially for differential equations of high order. In such cases, one must often use numerical techniques to approximate the solutions. See Chapters 18, 19 and 20.

Solved Problems

10.1. Solve $y''' - 6y'' + 11y' - 6y = 0$.

The characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$, which can be factored into

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

The roots are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$; hence the solution is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

10.2. Solve $y^{(4)} - 9y'' + 20y = 0$.

The characteristic equation is $\lambda^4 - 9\lambda^2 + 20 = 0$, which can be factored into

$$(\lambda - 2)(\lambda + 2)(\lambda - \sqrt{5})(\lambda + \sqrt{5}) = 0$$

The roots are $\lambda_1 = 2$, $\lambda_2 = -2$, $\lambda_3 = \sqrt{5}$, and $\lambda_4 = -\sqrt{5}$; hence the solution is

$$\begin{aligned} y &= c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{\sqrt{5}x} + c_4 e^{-\sqrt{5}x} \\ &= k_1 \cosh 2x + k_2 \sinh 2x + k_3 \cosh \sqrt{5}x + k_4 \sinh \sqrt{5}x \end{aligned}$$

10.3. Solve $y' - 5y = 0$.

The characteristic equation is $\lambda - 5 = 0$, which has the single root $\lambda_1 = 5$. The solution is then $y = c_1 e^{5x}$. (Compare this result with Problem 6.9.)

10.4. Solve $y''' - 6y'' + 2y' + 36y = 0$.

The characteristic equation, $\lambda^3 - 6\lambda^2 + 2\lambda + 36 = 0$, has roots $\lambda_1 = -2$, $\lambda_2 = 4 + i\sqrt{2}$, and $\lambda_3 = 4 - i\sqrt{2}$. The solution is

$$y = c_1 e^{-2x} + d_2 e^{(4+i\sqrt{2})x} + d_3 e^{(4-i\sqrt{2})x}$$

which can be rewritten, using Euler's relations (see Problem 9.16) as

$$y = c_1e^{-2x} + c_2e^{4x}\cos\sqrt{2}x + c_3e^{4x}\sin\sqrt{2}x$$

10.5. Solve $\dfrac{d^4x}{dt^4} - 4\dfrac{d^3x}{dt^3} + 7\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 6x = 0.$

The characteristic equation, $\lambda^4 - 4\lambda^3 + 7\lambda^2 - 4\lambda + 6 = 0$, has roots $\lambda_1 = 2 + i\sqrt{2}$, $\lambda_2 = 2 - i\sqrt{2}$, $\lambda_3 = i$, and $\lambda_4 = -i$. The solution is

$$x = d_1e^{(2+i\sqrt{2})t} + d_2e^{(2-i\sqrt{2})t} + d_3e^{it} + d_4e^{-it}$$

If, using Euler's relations, we combine the first two terms and then similarly combine the last two terms, we can rewrite the solution as

$$x = c_1e^{2t}\cos\sqrt{2}t + c_2e^{2x}\sin\sqrt{2}t + c_3\cos t + c_4\sin t$$

10.6. Solve $y^{(4)} + 8y''' + 24y'' + 32y' + 16y = 0.$

The characteristic equation, $\lambda^4 + 8\lambda^3 + 24\lambda^2 + 32\lambda + 16 = 0$, can be factored into $(\lambda + 2)^4 = 0$. Here $\lambda_1 = -2$ is a root of multiplicity four; hence the solution is

$$y = c_1e^{-2x} + c_2xe^{-2x} + c_3x^2e^{-2x} + c_4x^3e^{-2x}$$

10.7. Solve $\dfrac{d^5P}{dt^5} - \dfrac{d^4P}{dt^4} - 2\dfrac{d^3P}{dt^3} + 2\dfrac{d^2P}{dt^2} + \dfrac{dP}{dt} - P = 0.$

The characteristic equation can be factored into $(\lambda - 1)^3(\lambda + 1)^2 = 0$; hence, $\lambda_1 = 1$ is a root of multiplicity three and $\lambda_2 = -1$ is a root of multiplicity two. The solution is

$$P = c_1e^t + c_2te^t + c_3t^2e^t + c_4e^{-t} + c_5te^{-t}$$

10.8. Solve $\dfrac{d^4Q}{dx^4} - 8\dfrac{d^3Q}{dx^3} + 32\dfrac{d^2Q}{dx^2} - 64\dfrac{dQ}{dx} + 64Q = 0.$

The characteristic equation has roots $2 \pm i2$ and $2 \pm i2$; hence $\lambda_1 = 2 + i2$ and $\lambda_2 = 2 - i2$ are both roots of multiplicity two. The solution is

$$\begin{aligned} Q &= d_1e^{(2+i2)x} + d_2xe^{(2+i2)x} + d_3e^{(2-i2)x} + d_4xe^{(2-i2)x} \\ &= e^{2x}(d_1e^{i2x} + d_3e^{-i2x}) + xe^{2x}(d_2e^{i2x} + d_4e^{-i2x}) \\ &= e^{2x}(c_1\cos 2x + c_3\sin 2x) + xe^{2x}(c_2\cos 2x + c_4\sin 2x) \\ &= (c_1 + c_2x)\,e^{2x}\cos 2x + (c_3 + c_4x)\,e^{2x}\sin 2x \end{aligned}$$

10.9. **Find the general solution to a fourth-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if one solution is known to be x^3e^{4x}.**

If x^3e^{4x} is a solution, then so too are x^2e^{4x}, xe^{4x}, and e^{4x}. We now have four linearly independent solutions to a fourth-order linear, homogeneous differential equation, so we can write the general solution as

$$y(x) = c_4x^3e^{4x} + c_3x^2e^{4x} + c_2xe^{4x} + c_1e^{4x}$$

10.10. **Determine the differential equation described in Problem 10.9.**

The characteristic equation of a fourth-order differential equation is a fourth-degree polynomial having exactly four roots. Because x^3e^{4x} is a solution, we know that $\lambda = 4$ is a root of multiplicity four of the corresponding characteristic equation, so the characteristic equation must be $(\lambda - 4)^4 = 0$, or

$$\lambda^4 - 16\lambda^3 + 96\lambda^2 - 256\lambda + 256 = 0$$

The associated differential equation is

$$y^{(4)} - 16y''' + 96y'' - 256y' + 256y = 0$$

10.11. Find the general solution to a third-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if two solutions are known to be e^{-2x} and $\sin 3x$.

If $\sin 3x$ is a solution, then so too is $\cos 3x$. Together with e^{-2x}, we have three linearly independent solutions to a third-order linear, homogeneous differential equation, and we can write the general solution as

$$y(x) = c_1 e^{-2x} + c_2 \cos 3x + c_3 \sin 3x$$

10.12. Determine the differential equation described in Problem 10.11.

The characteristic equation of a third-order differential equation must have three roots. Because e^{-2x} and $\sin 3x$ are solutions, we know that $\lambda = -2$ and $\lambda = \pm i3$ are roots of the corresponding characteristic equation, so this equation must be

$$(\lambda + 2)(\lambda - i3)(\lambda + i3) = 0$$

or

$$\lambda^3 + 2\lambda^2 + 9\lambda + 18 = 0$$

The associated differential equation is

$$y''' + 2y'' + 9y' + 18y = 0$$

10.13. Find the general solution to a sixth-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if one solution is known to be $x^2 e^{7x} \cos 5x$.

If $x^2 e^{7x} \cos 5x$ is a solution, then so too are $xe^{7x} \cos 5x$ and $e^{7x} \cos 5x$. Furthermore, because complex roots of a characteristic equation come in conjugate pairs, every solution containing a cosine term is matched with another solution containing a sine term. Consequently, $x^2 e^{7x} \sin 5x$, $xe^{7x} \sin 5x$, and $e^{7x} \sin 5x$ are also solutions. We now have six linearly independent solutions to a sixth-order linear, homogeneous differential equation, so we can write the general solution as

$$y(x) = c_1 x^2 e^{7x} \cos 5x + c_2 x^2 e^{7x} \sin 5x + c_3 xe^{7x} \cos 5x + c_4 xe^{7x} \sin 5x + c_5 e^{7x} \cos 5x + c_6 e^{7x} \sin 5x$$

10.14. Redo Problem 10.13 if the differential equation has order 8.

An eighth-order linear differential equation possesses eight linearly independent solutions, and since we can only identify six of them, as we did in Problem 10.13, we do not have enough information to solve the problem. We can say that the solution to Problem 10.13 will be *part* of the solution to this problem.

10.15. Solve $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 36\frac{dy}{dx} - 36y = 0$ if one solution is xe^{2x}.

If xe^{2x} is a solution, then so too is e^{2x} which implies that $(\lambda - 2)^2$ is a factor of the characteristic equation $\lambda^4 - 4\lambda^3 - 5\lambda^2 + 36\lambda - 36 = 0$. Now,

$$\frac{\lambda^4 - 4\lambda^3 - 5\lambda^2 + 36\lambda - 36}{(\lambda - 2)^2} = \lambda^2 - 9$$

so two other roots of the characteristic equation are $\lambda = \pm 3$, with corresponding solutions e^{3x} and e^{-3x}. Having identified four linearly independent solutions to the given fourth-order linear differential equation, we can write the general solution as

$$y(x) = c_1 e^{2x} + c_2 xe^{2x} + c_3 e^{3x} + c_4 e^{-3x}$$

Supplementary Problems

In Problems 10.16 through 10.34, solve the given differential equations.

10.16. $y''' - 2y'' - y' + 2y = 0$

10.17. $y''' - y'' - y' + y = 0$

10.18. $y''' - 3y'' + 3y' - y = 0$

10.19. $y''' - y'' + y' - y = 0$

10.20. $y^{(4)} + 2y'' + y = 0$

10.21. $y^{(4)} - y = 0$

10.22. $y^{(4)} + 2y''' - 2y' - y = 0$

10.23. $y^{(4)} - 4y'' + 16y' + 32y = 0$

10.24. $y^{(4)} + 5y''' = 0$

10.25. $y^{(4)} + 2y''' + 3y'' + 2y' + y = 0$

10.26. $y^{(6)} - 5y^{(4)} + 16y''' + 36y'' - 16y' - 32y = 0$

10.27. $\frac{d^4x}{dt^4} + 4\frac{d^3x}{dt^3} + 6\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 0$

10.28. $\frac{d^3x}{dt^3} = 0$

10.29. $\frac{d^4x}{dt^4} + 10\frac{d^2x}{dt^2} + 9x = 0$

10.30. $\frac{d^3x}{dt^3} - 5\frac{d^2x}{dt^2} + 25\frac{dx}{dt} - 125x = 0$

10.31. $q^{(4)} + q'' - 2q = 0$

10.32. $q^{(4)} - 3q'' + 2q = 0$

10.33. $N''' - 12N'' - 28N + 480N = 0$

10.34. $\frac{d^5r}{d\theta^5} + 5\frac{d^4r}{d\theta^4} + 10\frac{d^3r}{d\theta^3} + 10\frac{d^2r}{d\theta^2} + 5\frac{dr}{d\theta} + r = 0$

In Problems 10.35 through 10.41, a complete set of roots is given for the characteristic equation of an *n*th-order near homogeneous differential equation in $y(x)$ with real numbers as coefficients. Determine the general solution of the differential equation.

10.35. $2, 8, -14$

10.36. $0, \pm i19$

10.37. $0, 0, 2 \pm i9$

10.38. $2 \pm i9, 2 \pm i9$

10.39. $5, 5, 5, -5, -5$

10.40. $\pm i6, \pm i6, \pm i6$

10.41. $-3 \pm i, -3 \pm i, 3 \pm i, 3 \pm i$

10.42. Determine the differential equation associated with the roots given in Problem 10.35.

10.43. Determine the differential equation associated with the roots given in Problem 10.36.

10.44. Determine the differential equation associated with the roots given in Problem 10.37.

10.45. Determine the differential equation associated with the roots given in Problem 10.38.

10.46. Determine the differential equation associated with the roots given in Problem 10.39.

10.47. Find the general solution to a fourth-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if one solution is known to be x^3e^{-x}.

10.48. Find the general solution to a fourth-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if two solutions are $\cos 4x$ and $\sin 3x$.

10.49. Find the general solution to a fourth-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if one solution is $x \cos 4x$.

10.50. Find the general solution to a fourth-order linear homogeneous differential equation for $y(x)$ with real numbers as coefficients if two solutions are xe^{2x} and xe^{5x}.

CHAPTER 11

The Method of Undetermined Coefficients

The general solution to the linear differential equation $\mathbf{L}(y) = \phi(x)$ is given by Theorem 8.4 as $y = y_h + y_p$ where y_p denotes one solution to the differential equation and y_h is the general solution to the associated homogeneous equation, $\mathbf{L}(y) = 0$. Methods for obtaining y_h when the differential equation has constant coefficients are given in Chapters 9 and 10. In this chapter and the next, we give methods for obtaining a particular solution y_p *once* y_h *is known.*

SIMPLE FORM OF THE METHOD

The *method of undetermined coefficients* is applicable only if $\phi(x)$ and *all* of its derivatives can be written in terms of the same *finite* set of linearly independent functions, which we denote by $\{y_1(x), y_2(x), \ldots, y_n(x)\}$. The method is initiated by assuming a particular solution of the form

$$y_p(x) = A_1 y_1(x) + A_2 y_2(x) + \cdots + A_n y_n(x)$$

where $A_1, A_2, \ldots, A_n$ denote arbitrary multiplicative constants. These arbitrary constants are then evaluated by substituting the proposed solution into the given differential equation and equating the coefficients of like terms.

Case 1. $\phi(x) = p_n(x)$, an nth-degree polynomial in x. Assume a solution of the form

$$y_p = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0 \tag{11.1}$$

where A_j $(j = 0, 1, 2, \ldots, n)$ is a constant to be determined.

Case 2. $\phi(x) = ke^{\alpha x}$ where k and α are known constants. Assume a solution of the form

$$y_p = Ae^{\alpha x} \tag{11.2}$$

where A is a constant to be determined.

Case 3. $\phi(x) = k_1 \sin \beta x + k_2 \cos \beta x$ where k_1, k_2, and β are known constants. Assume a solution of the form

$$y_p = A \sin \beta x + B \cos \beta x \tag{11.3}$$

where A and B are constants to be determined.

Note: (*11.3*) in its entirety is assumed even when k_1 or k_2 is zero, because the derivatives of sines or cosines involve both sines and cosines.

GENERALIZATIONS

If $\phi(x)$ is the product of terms considered in Cases 1 through 3, take y_p to be the product of the corresponding assumed solutions and algebraically combine arbitrary constants where possible. In particular, if $\phi(x) = e^{\alpha x}p_n(x)$ is the product of a polynomial with an exponential, assume

$$y_p = e^{\alpha x}(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0) \tag{11.4}$$

where A_j is as in Case 1. If, instead, $\phi(x) = e^{\alpha x}p_n(x) \sin \beta x$ is the product of a polynomial, exponential, and sine term, or if $\phi(x) = e^{\alpha x}p_n(x) \cos \beta x$ is the product of a polynomial, exponential, and cosine term, then assume

$$y_p = e^{\alpha x} \sin \beta x\,(A_n x^n + \cdots + A_1 x + A_0) + e^{\alpha x} \cos \beta x\,(B_n x^n + \cdots + B_1 x + B_0) \tag{11.5}$$

where A_j and B_j ($j = 0, 1, \ldots, n$) are constants which still must be determined.

If $\phi(x)$ is the sum (or difference) of terms already considered, then we take y_p to be the sum (or difference) of the corresponding assumed solutions and algebraically combine arbitrary constants where possible.

MODIFICATIONS

If any term of the assumed solution, disregarding multiplicative constants, is also a term of y_h (the homogeneous solution), then the assumed solution must be modified by multiplying it by x^m, where m is the smallest positive integer such that the product of x^m with the assumed solution has no terms in common with y_h.

LIMITATIONS OF THE METHOD

In general, if $\phi(x)$ is not one of the types of functions considered above, or if the differential equation *does not have constant coefficients*, then the method given in Chapter 12 applies.

Solved Problems

11.1. Solve $y'' - y' - 2y = 4x^2$.

From Problem 9.1, $y_h = c_1e^{-x} + c_2e^{2x}$. Here $\phi(x) = 4x^2$, a second-degree polynomial. Using (*11.1*), we assume that

$$y_p = A_2x^2 + A_1x + A_0 \tag{1}$$

Thus, $y_p' = 2A_2x + A_1$ and $y_p'' = 2A_2$. Substituting these results into the differential equation, we have

$$2A_2 - (2A_2x + A_1) - 2(A_2x^2 + A_1x + A_0) = 4x^2$$

or, equivalently,

$$(-2A_2)x^2 + (-2A_2 - 2A_1)x + (2A_2 - A_1 - 2A_0) = 4x^2 + (0)x + 0$$

Equating the coefficients of like powers of x, we obtain

$$-2A_2 = 4 \qquad -2A_2 - 2A_1 = 0 \qquad 2A_2 - A_1 - 2A_0 = 0$$

Solving this system, we find that $A_2 = -2$, $A_1 = 2$, and $A_0 = -3$. Hence (*1*) becomes

$$y_p = -2x^2 + 2x - 3$$

and the general solution is

$$y = y_h + y_p = c_1e^{-x} + c_2e^{2x} - 2x^2 + 2x - 3$$

11.2. Solve $y'' - y' - 2y = e^{3x}$.

From Problem 9.1, $y_h = c_1e^{-x} + c_2e^{2x}$. Here $\phi(x)$ has the form displayed in Case 2 with $k = 1$ and $\alpha = 3$. Using (*11.2*), we assume that

$$y_p = Ae^{3x} \tag{1}$$

Thus, $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$. Substituting these results into the differential equation, we have

$$9Ae^{3x} - 3Ae^{3x} - 2Ae^{3x} = e^{3x} \quad \text{or} \quad 4Ae^{3x} = e^{3x}$$

It follows that $4A = 1$, or $A = \frac{1}{4}$, so that (*1*) becomes $y_p = \frac{1}{4}e^{3x}$. The general solution then is

$$y = c_1e^{-x} + c_2e^{2x} + \frac{1}{4}e^{3x}$$

11.3. Solve $y'' - y' - 2y = \sin 2x$.

Again by Problem 9.1, $y_h = c_1e^{-x} + c_2e^{2x}$. Here $\phi(x)$ has the form displayed in Case 3 with $k_1 = 1$, $k_2 = 0$, and $\beta = 2$. Using (*11.3*), we assume that

$$y_p = A \sin 2x + B \cos 2x \tag{1}$$

Thus, $y_p' = 2A \cos 2x - 2B \sin 2x$ and $y_p'' = -4A \sin 2x - 4B \cos 2x$. Substituting these results into the differential equation, we have

$$(-4A \sin 2x - 4B \cos 2x) - (2A \cos 2x - 2B \sin 2x) - 2(A \sin 2x + B \cos 2x) = \sin 2x$$

or, equivalently,

$$(-6A + 2B) \sin 2x + (-6B - 2A) \cos 2x = (1) \sin 2x + (0) \cos 2x$$

Equating coefficients of like terms, we obtain

$$-6A + 2B = 1 \qquad -2A - 6B = 0$$

Solving this system, we find that $A = -3/20$ and $B = 1/20$. Then from (*1*),

$$y_p = -\frac{3}{20}\sin 2x + \frac{1}{20}\cos 2x$$

and the general solution is

$$y = c_1e^{-x} + c_2e^{2x} - \frac{3}{20}\sin 2x + \frac{1}{20}\cos 2x$$

11.4. Solve $\ddot{y} - 6\dot{y} + 25y = 2\sin\frac{t}{2} - \cos\frac{t}{2}$.

From Problem 9.10,

$$y_h = c_1e^{3t}\cos 4t + c_2e^{3t}\sin 4t$$

Here $\phi(t)$ has the form displayed in Case 3 with the independent variable t replacing x, $k_1 = 2$, $k_2 = -1$, and $\beta = \frac{1}{2}$. Using (*11.3*), with t replacing x, we assume that

$$y_p = A\sin\frac{t}{2} + B\cos\frac{t}{2} \tag{1}$$

Consequently,

$$\dot{y}_p = \frac{A}{2}\cos\frac{t}{2} - \frac{B}{2}\sin\frac{t}{2}$$

and

$$\ddot{y}_p = -\frac{A}{4}\sin\frac{t}{2} - \frac{B}{4}\cos\frac{t}{2}$$

Substituting these results into the differential equation, we obtain

$$\left(-\frac{A}{4}\sin\frac{t}{2} - \frac{B}{4}\cos\frac{t}{4}\right) - 6\left(\frac{A}{2}\cos\frac{t}{2} - \frac{B}{2}\sin\frac{t}{2}\right) + 25\left(A\sin\frac{t}{2} + B\cos\frac{t}{2}\right) = 2\sin\frac{t}{2} - \cos\frac{t}{2}$$

or, equivalently

$$\left(\frac{99}{4}A + 3B\right)\sin\frac{t}{2} + \left(-3A + \frac{99}{4}B\right)\cos\frac{t}{2} = 2\sin\frac{t}{2} - \cos\frac{t}{2}$$

Equating coefficients of like terms, we have

$$\frac{99}{4}A + 3B = 2; \quad -3A + \frac{99}{4}B = -1$$

It follows that $A = 56/663$ and $B = -20/663$, so that (*1*) becomes

$$y_p = \frac{56}{663}\sin\frac{t}{2} - \frac{20}{663}\cos\frac{t}{2}$$

The general solution is

$$y = y_h + y_p = c_1e^{3t}\cos 4t + c_2e^{3t}\sin 4t + \frac{56}{663}\sin\frac{t}{2} - \frac{20}{663}\cos\frac{t}{2}$$

11.5. Solve $\ddot{y} - 6\dot{y} + 25y = 64e^{-t}$.

From Problem 9.10,

$$y_h = c_1e^{3t}\cos 4t + c_2e^{3t}\sin 4t$$

Here $\phi(t)$ has the form displayed in Case 2 with the independent variable t replacing x, $k = 64$ and $\alpha = -1$. Using (*11.2*), with t replacing x, we assume that

$$y_p = Ae^{-t} \tag{1}$$

Consequently, $\dot{y}_p = -Ae^{-t}$ and $\ddot{y}_p = Ae^{-t}$. Substituting these results into the differential equation, we obtain

$$Ae^{-t} - 6(-Ae^{-t}) + 25(Ae^{-t}) = 64e^{-t}$$

or, equivalently, $32Ae^{-t} = 64e^{-t}$. It follows that $32A = 64$ or $A = 2$, so that (*1*) becomes $y_p = 2e^{-t}$. The general solution is

$$y = y_h + y_p = c_1e^{3t}\cos 4t + c_2e^{3t}\sin 4t + 2e^{-t}$$

11.6. Solve $\ddot{y} - 6\dot{y} + 25y = 50t^3 - 36t^2 - 63t + 18$.

Again by Problem 9.10,

$$y_h = c_1 e^{3t} \cos 4t + c_2 e^{3t} \sin 4t$$

Here $\phi(t)$ is a third-degree polynomial in t. Using (*11.1*) with t replacing x, we assume that

$$y_p = A_3 t^3 + A_2 t^2 + A_1 t + A_0 \tag{1}$$

Consequently,

$$\dot{y}_p = 3A_3 t^2 + 2A_2 t + A_1$$

and

$$\ddot{y}_p = 6A_3 t + 2A_2$$

Substituting these results into the differential equation, we obtain

$$(6A_3 t + 2A_2) - 6(3A_3 t^2 + 2A_2 t + A_1) + 25(A_3 t^3 + A_2 t^2 + A_1 t + A_0) = 50t^3 - 36t^2 - 63t + 18$$

or, equivalently,

$$(25A_3)t^3 + (-18A_3 + 25A_2)t^2 + (6A_3 - 12A_2 + 25A_1) + (2A_2 - 6A_1 + 25A_0) = 50t^3 - 36t^2 - 63t + 18$$

Equating coefficients of like powers of t, we have

$$25A_3 = 50; \quad -18A_3 + 25A_2 = -36; \quad 6A_3 - 12A_2 + 25A_1 = -63; \quad 2A_2 - 6A_1 + 25A_0 = 18$$

Solving these four algebraic equations simultaneously, we obtain $A_3 = 2$, $A_2 = 0$, $A_1 = -3$, and $A_0 = 0$, so that (*1*) becomes

$$y_p = 2t^3 - 3t$$

The general solution is

$$y = y_h + y_p = c_1 e^{3t} \cos 4t + c_2 e^{3t} \sin 4t + 2t^3 - 3t$$

11.7. Solve $y''' - 6y'' + 11y' - 6y = 2xe^{-x}$.

From Problem 10.1, $y_h = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$. Here $\phi(x) = e^{\alpha x} p_n(x)$, where $\alpha = -1$ and $p_n(x) = 2x$, a first-degree polynomial. Using Eq. (*11.4*), we assume that $y_p = e^{-x}(A_1 x + A_0)$, or

$$y_p = A_1 x e^{-x} + A_0 e^{-x} \tag{1}$$

Thus,

$$y_p' = -A_1 x e^{-x} + A_1 e^{-x} - A_0 e^{-x}$$

$$y_p'' = A_1 x e^{-x} - 2A_1 e^{-x} + A_0 e^{-x}$$

$$y_p''' = -A_1 x e^{-x} + 3A_1 e^{-x} - A_0 e^{-x}$$

Substituting these results into the differential equation and simplifying, we obtain

$$-24A_1 x e^{-x} + (26A_1 - 24A_0)e^{-x} = 2xe^{-x} + (0)e^{-x}$$

Equating coefficients of like terms, we have

$$-24A_1 = 2 \quad 26A_1 - 24A_0 = 0$$

from which $A_1 = -1/12$ and $A_0 = -13/144$.

Equation (*1*) becomes

$$y_p = -\frac{1}{12} x e^{-x} - \frac{13}{144} e^{-x}$$

and the general solution is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{12} x e^{-x} - \frac{13}{144} e^{-x}$$

11.8. Determine the form of a particular solution for $y'' = 9x^2 + 2x - 1$.

Here $\phi(x) = 9x^2 + 2x - 1$, and the solution of the associated homogeneous differential equation $y'' = 0$ is $y_h = c_1x + c_0$. Since $\phi(x)$ is a second-degree polynomial, we first try $y_p = A_2x^2 + A_1x + A_0$. Note, however, that this assumed solution has terms, disregarding multiplicative constants, in common with y_h: in particular, the first-power term and the constant term. Hence, we must determine the smallest positive integer m such that $x^m(A_2x^2 + A_1x + A_0)$ has no terms in common with y_h.

For $m = 1$, we obtain

$$x(A_2x^2 + A_1x + A_0) = A_2x^3 + A_1x^2 + A_0x$$

which still has a first-power term in common with y_h. For $m = 2$, we obtain

$$x^2(A_2x^2 + A_1x + A_0) = A_2x^4 + A_1x^3 + A_0x^2$$

which has no terms in common with y_h; therefore, we assume an expression of this form for y_p.

11.9. Solve $y'' = 9x^2 + 2x - 1$.

Using the results of Problem 11.8, we have $y_h = c_1x + c_0$ and we assume

$$y_p = A_2x^4 + A_1x^3 + A_0x^2 \tag{1}$$

Substituting (*1*) into the differential equation, we obtain

$$12A_2x^2 + 6A_1x + 2A_0 = 9x^2 + 2x - 1$$

from which $A_2 = 3/4$, $A_1 = 1/3$, and $A_0 = -1/2$. Then (*1*) becomes

$$y_p = \frac{3}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2$$

and the general solution is

$$y = c_1x + c_0 + \frac{3}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2$$

The solution also can be obtained simply by twice integrating both sides of the differential equation with respect to x.

11.10. Solve $y' - 5y = 2e^{5x}$.

From Problem 10.3, $y_h = c_1e^{5x}$. Since $\phi(x) = 2e^{5x}$, it would follow from Eq. (*11.2*) that the guess for y_p should be $y_p = A_0e^{5x}$. Note, however, that this y_p has exactly the same form as y_h; therefore, we must modify y_p. Multiplying y_p by x ($m = 1$), we obtain

$$y_p = A_0xe^{5x} \tag{1}$$

As this expression has no terms in common with y_h; it is a candidate for the particular solution. Substituting (*1*) and $y_p' = A_0e^{5x} + 5A_0xe^{5x}$ into the differential equation and simplifying, we obtain $A_0e^{5x} = 2e^{5x}$, from which $A_0 = 2$. Equation (*1*) becomes $y_p = 2xe^{5x}$, and the general solution is $y = (c_1 + 2x)e^{5x}$.

11.11. Determine the form of a particular solution of

$$y' - 5y = (x - 1)\sin x + (x + 1)\cos x$$

Here $\phi(x) = (x - 1)\sin x + (x + 1)\cos x$, and from Problem 10.3, we know that the solution to the associated homogeneous problem $y' - 5y = 0$ is $y_h = c_1e^{5x}$. An assumed solution for $(x - 1)\sin x$ is given by Eq. (*11.5*) (with $\alpha = 0$) as

$$(A_1x + A_0)\sin x + (B_1x + B_0)\cos x$$

and an assumed solution for $(x + 1)\cos x$ is given also by Eq. (*11.5*) as

$$(C_1x + C_0)\sin x + (D_1x + D_0)\cos x$$

(Note that we have used C and D in the last expression, since the constants A and B already have been used.) We therefore take

$$y_p = (A_1x + A_0)\sin x + (B_1x + B_0)\cos x + (C_1x + C_0)\sin x + (D_1x + D_0)\cos x$$

Combining like terms, we arrive at

$$y_p = (E_1x + E_0)\sin x + (F_1x + F_0)\cos x$$

as the assumed solution, where $E_j = A_j + C_j$ and $F_j = B_j + D_j$ $(j = 0, 1)$.

11.12. Solve $y' - 5y = (x - 1)\sin x + (x + 1)\cos x$.

From Problem 10.3, $y_h = c_1e^{5x}$. Using the results of Problem 11.11, we assume that

$$y_p = (E_1x + E_0)\sin x + (F_1x + F_0)\cos x \tag{1}$$

Thus, $$y_p' = (E_1 - F_1x - F_0)\sin x + (E_1x + E_0 + E_1)\cos x$$

Substituting these values into the differential equation and simplifying, we obtain

$$(-5E_1 - F_1)x\sin x + (-5E_0 + E_1 - F_0)\sin x + (-5F_1 + E_1)x\cos x + (-5F_0 + E_0 + F_1)\cos x$$
$$= (1)x\sin x + (-1)\sin x + (1)x\cos x + (1)\cos x$$

Equating coefficients of like terms, we have

$$\begin{aligned} -5E_1 - F_1 &= 1 \\ -5E_0 + E_1 - F_0 &= -1 \\ E_1 - 5F_1 &= 1 \\ E_0 - 5F_0 + F_1 &= 1 \end{aligned}$$

Solving, we obtain $E_1 = -2/13$, $E_0 = 71/338$, $F_1 = -3/13$, and $F_0 = -69/338$. Then, from (*1*),

$$y_p = \left(-\frac{2}{13}x + \frac{71}{338}\right)\sin x + \left(-\frac{3}{13}x + \frac{69}{338}\right)\cos x$$

and the general solution is

$$y = c_1e^{5x} + \left(\frac{-2}{13}x + \frac{71}{338}\right)\sin x - \left(\frac{3}{13}x + \frac{69}{338}\right)\cos x$$

11.13. Solve $y' - 5y = 3e^x - 2x + 1$.

From Problem 10.3, $y_h = c_1e^{5x}$. Here, we can write $\phi(x)$ as the sum of two manageable functions: $\phi(x) = (3e^x) + (-2x + 1)$. For the term $3e^x$ we would assume a solution of the form Ae^x; for the term $-2x + 1$ we would assume a solution of the form $B_1x + B_0$. Thus, we try

$$y_p = Ae^x + B_1x + B_0 \tag{1}$$

Substituting (*1*) into the differential equation and simplifying, we obtain

$$(-4A)e^x + (-5B_1)x + (B_1 - 5B_0) = (3)e^x + (-2)x + (1)$$

Equating coefficients of like terms, we find that $A = -3/4$, $B_1 = 2/5$, and $B_0 = -3/25$. Hence, (*1*) becomes

$$y_p = -\frac{3}{4}e^x + \frac{2}{5}x - \frac{3}{25}$$

and the general solution is

$$y = c_1 e^{5x} - \frac{3}{4}e^x + \frac{2}{5}x - \frac{3}{25}$$

11.14. Solve $y' - 5y = x^2e^x - xe^{5x}$.

From Problem 10.3, $y_h = c_1e^{5x}$. Here $\phi(x) = x^2e^x - xe^{5x}$, which is the difference of two terms, each in manageable form. For x^2e^x we would assume a solution of the form

$$e^x(A_2x^2 + A_1x + A_0) \tag{1}$$

For xe^{5x} we would try initially a solution of the form

$$e^{5x}(B_1x+B_0) = B_1xe^{5x} + B_0e^{5x}$$

But this supposed solution would have, disregarding multiplicative constants, the term e^{5x} in common with y_h. We are led, therefore, to the modified expression

$$xe^{5x}(B_1x + B_0) = e^{5x}(B_1x^2 + B_0x) \tag{2}$$

We now take y_p to be the sum of (*1*) and (*2*):

$$y_p = e^x(A_2x^2 + A_1x + A_0) + e^{5x}(B_1x^2 + B_0x) \tag{3}$$

Substituting (*3*) into the differential equation and simplifying, we obtain

$$\begin{aligned} &e^x[(-4A_2)x^2 + (2A_2 - 4A_1)x + (A_1 - 4A_0)] + e^{5x}[(2B_1)x + B_0] \\ &= e^x[(1)x^2 + (0)x + (0)] + e^{5x}[(-1)x + (0)] \end{aligned}$$

Equating coefficients of like terms, we have

$$-4A_2 = 1 \quad 2A_2 - 4A_1 = 0 \quad A_1 - 4A_0 = 0 \quad 2B_1 = -1 \quad B_0 = 0$$

from which

$$A_2 = -\frac{1}{4} \quad A_1 = -\frac{1}{8} \quad A_0 = -\frac{1}{32}$$

$$B_1 = -\frac{1}{2} \quad B_0 = 0$$

Equation (*3*) then gives

$$y_p = e^x\left(-\frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{32}\right) - \frac{1}{2}x^2e^{5x}$$

and the general solution is

$$y = c_1e^{5x} + e^x\left(-\frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{32}\right) - \frac{1}{2}x^2e^{5x}$$

Supplementary Problems

In Problems 11.15 through 11.26, determine the form of a particular solution to $\mathbf{L}(y) = \phi(x)$ for $\phi(x)$ as given if the solution to the associated homogeneous equation $\mathbf{L}(y) = 0$ is $y_h = c_1e^{2x} + c_2e^{3x}$.

11.15. $\phi(x) = 2x - 7$

11.16. $\phi(x) = -3x^2$

11.17. $\phi(x) = 132x^2 - 388x + 1077$

11.18. $\phi(x) = 0.5e^{-2x}$

11.19. $\phi(x) = 13e^{5x}$

11.20. $\phi(x) = 4e^{2x}$

11.21. $\phi(x) = 2\cos 3x$

11.22. $\phi(x) = \frac{1}{2}\cos 3x - 3\sin 3x$

11.23. $\phi(x) = x\cos 3x$

11.24. $\phi(x) = 2x + 3e^{8x}$

11.25. $\phi(x) = 2xe^{5x}$

11.26. $\phi(x) = 2xe^{3x}$

In Problems 11.27 through 11.36, determine the form of a particular solution to $\mathbf{L}(y) = \phi(x)$ for $\phi(x)$ as given if the solution to the associated homogeneous equation $\mathbf{L}(y) = 0$ is $y_h = c_1e^{5x}\cos 3x + c_2e^{5x}\sin 3x$.

11.27. $\phi(x) = 2e^{3x}$

11.28. $\phi(x) = xe^{3x}$

11.29. $\phi(x) = -23e^{5x}$

11.30. $\phi(x) = (x^2 - 7)e^{5x}$

11.31. $\phi(x) = 5\cos\sqrt{2}x$

11.32. $\phi(x) = x^2\sin\sqrt{2}x$

11.33. $\phi(x) = -\cos 3x$

11.34. $\phi(x) = 2\sin 4x - \cos 7x$

11.35. $\phi(x) = 31e^{-x}\cos 3x$

11.36. $\phi(x) = -\frac{1}{6}e^{5x}\cos 3x$

In Problems 11.37 through 11.43, determine the form of a particular solution to $\mathbf{L}(x) = \phi(t)$ for $\phi(t)$ as given if the solution to the associated homogeneous equation $\mathbf{L}(x) = 0$ is $x_h = c_1 + c_2e^t + c_3te^t$.

11.37. $\phi(t) = t$

11.38. $\phi(t) = 2t^2 - 3t + 82$

11.39. $\phi(t) = te^{-2t} + 3$

11.40. $\phi(t) = -6e^t$

11.41. $\phi(t) = te^t$

11.42. $\phi(t) = 3 + t\cos t$

11.43. $\phi(t) = te^{2t}\cos 3t$

In Problems 11.44 through 11.52, find the general solutions to the given differential equations.

11.44. $y'' - 2y' + y = x^2 - 1$

11.45. $y'' - 2y' + y = 3e^{2x}$

11.46. $y'' - 2y' + y = 4\cos x$

11.47. $y'' - 2y' + y = 3e^x$

11.48. $y'' - 2y' + y = xe^x$

11.49. $y' - y = e^x$

11.50. $y' - y = xe^{2x} + 1$

11.51. $y' - y = \sin x + \cos 2x$

11.52. $y''' - 3y'' + 3y' - y = e^x + 1$

CHAPTER 12

Variation of Parameters

Variation of parameters is another method (see Chapter 11) for finding a particular solution of the nth-order linear differential equation

$$\mathbf{L}(y) = \phi(x) \tag{12.1}$$

once the solution of the associated homogeneous equation $\mathbf{L}(y) = 0$ is known. Recall from Theorem 8.2 that if $y_1(x), y_2(x), \ldots, y_n(x)$ are n linearly independent solutions of $\mathbf{L}(y) = 0$, then the general solution of $\mathbf{L}(y) = 0$ is

$$y_h = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \tag{12.2}$$

THE METHOD

A particular solution of $\mathbf{L}(y) = \phi(x)$ has the form

$$y_p = v_1 y_1 + v_2 y_2 + \cdots + v_n y_n \tag{12.3}$$

where $y_i = y_i(x)$ ($i = 1, 2, \ldots, n$) is given in Eq. (*12.2*) and v_i ($i = 1, 2, \ldots, n$) is an unknown function of x which still must be determined.

To find v_i, first solve the following linear equations simultaneously for v_i':

$$\begin{aligned} v_1' y_1 + v_2' y_2 + \cdots + v_n' y_n &= 0 \\ v_1' y_1' + v_2' y_2' + \cdots + v_n' y_n' &= 0 \\ &\vdots \\ v_1' y_1^{(n-2)} + v_2' y_2^{(n-2)} + \cdots + v_n' y_n^{(n-2)} &= 0 \\ v_1' y_1^{(n-1)} + v_2' y_2^{(n-1)} + \cdots + v_n' y_n^{(n-1)} &= \phi(x) \end{aligned} \tag{12.4}$$

Then integrate each v_i' to obtain v_i, disregarding all constants of integration. This is permissible because we are seeking only *one* particular solution.

Example 12.1. For the special case $n = 3$, Eqs. (*12.4*) reduce to

$$\begin{aligned} v_1' y_1 + v_2' y_2 + v_3' y_3 &= 0 \\ v_1' y_1' + v_2' y_2' + v_3' y_3' &= 0 \\ v_1' y_1'' + v_2' y_2'' + v_3' y_3'' &= \phi(x) \end{aligned} \tag{12.5}$$

For the case $n = 2$, Eqs. (*12.4*) become

$$v_1'y_1 + v_2'y_2 = 0$$
$$v_1'y_1' + v_2'y_2' = \phi(x) \qquad (12.6)$$

and for the case $n = 1$, we obtain the single equation

$$v_1'y_1 = \phi(x) \qquad (12.7)$$

Since $y_1(x), y_2(x), \ldots, y_n(x)$ are n linearly independent solutions of the same equation $\mathbf{L}(y) = 0$, their Wronskian is not zero (Theorem 8.3). This means that the system (*12.4*) has a nonzero determinant and can be solved uniquely for $v_1'(x), v_2'(x), \ldots, v_n'(x)$.

SCOPE OF THE METHOD

The method of variation of parameters can be applied to *all* linear differential equations. It is therefore more powerful than the method of undetermined coefficients, which is restricted to linear differential equations with constant coefficients and particular forms of $\phi(x)$. Nonetheless, in those cases where both methods are applicable, the method of undetermined coefficients is usually the more efficient and, hence, preferable.

As a practical matter, the integration of $v_i'(x)$ may be impossible to perform. In such an event, other methods (in particular, numerical techniques) must be employed.

Solved Problems

12.1. Solve $y''' + y' = \sec x$.

This is a third-order equation with

$$y_h = c_1 + c_2 \cos x + c_3 \sin x$$

(see Chapter 10); it follows from Eq. (*12.3*) that

$$y_p = v_1 + v_2 \cos x + v_3 \sin x \qquad (1)$$

Here $y_1 = 1$, $y_2 = \cos x$, $y_3 = \sin x$, and $\phi(x) = \sec x$, so (*12.5*) becomes

$$v_1'(1) + v_2'(\cos x) + v_3'(\sin x) = 0$$
$$v_1'(0) + v_2'(-\sin x) + v_3'(\cos x) = 0$$
$$v_1'(0) + v_2'(-\cos x) + v_3'(-\sin x) = \sec x$$

Solving this set of equations simultaneously, we obtain $v_1' = \sec x$, $v_2' = -1$, and $v_3' = -\tan x$. Thus,

$$v_1 = \int v_1'\,dx = \int \sec x\,dx = \ln|\sec x + \tan x|$$
$$v_2 = \int v_2'\,dx = \int -1\,dx = -x$$
$$v_3 = \int v_3'\,dx = \int -\tan x\,dx = -\int \frac{\sin x}{\cos x}\,dx = \ln|\cos x|$$

Substituting these values into (*1*), we obtain

$$y_p = \ln|\sec x + \tan x| - x\cos x + (\sin x)\ln|\cos x|$$

The general solution is therefore

$$y = y_h + y_p = c_1 + c_2\cos x + c_3 \sin x + \ln|\sec x + \tan x| - x\cos x + (\sin x)\ln|\cos x|$$

12.2. Solve $y''' - 3y'' + 2y' = \dfrac{e^x}{1+e^{-x}}$.

This is a third-order equation with

$$y_h = c_1 + c_2e^x + c_3e^{2x}$$

(see Chapter 10); it follows from Eq. (*12.3*) that

$$y_p = v_1 + v_2e^x + v_3e^{2x} \tag{1}$$

Here $y_1 = 1$, $y_2 = e^x$, $y_3 = e^{2x}$, and $\phi(x) = e^x/(1 + e^{-x})$, so Eq. (*12.5*) becomes

$$v_1'(1) + v_2'(e^x) + v_3'(e^{2x}) = 0$$
$$v_1'(0) + v_2'(e^x) + v_3'(2e^{2x}) = 0$$
$$v_1'(0) + v_2'(e^x) + v_3'(4e^{2x}) = \frac{e^x}{1+e^{-x}}$$

Solving this set of equations simultaneously, we obtain

$$v_1' = \frac{1}{2}\left(\frac{e^x}{1+e^{-x}}\right)$$
$$v_2' = \frac{-1}{1+e^{-x}}$$
$$v_3' = \frac{1}{2}\left(\frac{e^{-x}}{1+e^{-x}}\right)$$

Thus, using the substitutions $u = e^x + 1$ and $w = 1 + e^{-x}$, we find that

$$v_1 = \frac{1}{2}\int\frac{e^x}{1+e^{-x}}\,dx = \frac{1}{2}\int\frac{e^x}{e^x+1}\,e^x dx$$
$$= \frac{1}{2}\int\frac{u-1}{u}\,du = \frac{1}{2}u - \frac{1}{2}\ln|u|$$
$$= \frac{1}{2}(e^x+1) - \frac{1}{2}\ln(e^x+1)$$
$$v_2 = \int\frac{-1}{1+e^{-x}}\,dx = -\int\frac{e^x}{e^x+1}\,dx$$
$$= -\int\frac{du}{u} = -\ln|u| = -\ln(e^x+1)$$
$$v_3 = \frac{1}{2}\int\frac{e^{-x}}{1+e^{-x}}\,dx = -\frac{1}{2}\int\frac{dw}{w} = -\frac{1}{2}\ln|w| = -\frac{1}{2}\ln(1+e^{-x})$$

Substituting these values into (*1*), we obtain

$$y_p = \left[\frac{1}{2}(e^x+1) - \frac{1}{2}\ln(e^x+1)\right] + [-\ln(e^x+1)]e^x + \left[-\frac{1}{2}\ln(1+e^{-x})\right]e^{2x}$$

The general solution is

$$y = y_h + y_p = c_1 + c_2e^x + c_3e^{2x} + \frac{1}{2}(e^x+1) - \frac{1}{2}\ln(e^x+1) - e^x\ln(e^x+1) - \frac{1}{2}e^{2x}\ln(1+e^{-x})$$

This solution can be simplified. We first note that

$$\ln(1+e^{-x}) = \ln[e^{-x}(e^x+1)] = \ln e^{-x} + \ln(e^x+1) = -1 + \ln(e^x+1)$$

so
$$-\frac{1}{2}e^{2x}\ln(1+e^{-x}) = -\frac{1}{2}e^{2x}[-1+\ln(e^x+1)] = \frac{1}{2}e^{2x} - \frac{1}{2}e^{2x}\ln(e^x+1)$$

Then, combining like terms, we have

$$y = \left(c_1 + \frac{1}{2}\right) + \left(c_2 + \frac{1}{2}\right)e^x + \left(c_3 + \frac{1}{2}\right)e^{2x} + \left[-\frac{1}{2} - e^x - \frac{1}{2}e^{2x}\right]\ln(e^x + 1)$$

$$= c_4 + c_5e^x + c_6e^{2x} - \frac{1}{2}[1 + 2e^x + (e^x)^2]\ln(e^x + 1)$$

$$= c_4 + c_5e^x + c_6e^{2x} - \frac{1}{2}(e^x + 1)^2\ln(e^x + 1)\left(\text{with } c_4 = c_1 + \frac{1}{2},\quad c_5 = c_2 + \frac{1}{2},\quad c_6 = c_3 + \frac{1}{2}\right)$$

12.3. Solve $y'' - 2y' + y = \dfrac{e^x}{x}$.

Here $n = 2$ and $y_h = c_1e^x + c_2xe^x$; hence,

$$y_p = v_1e^x + v_2xe^x \tag{1}$$

Since $y_1 = e^x$, $y_2 = xe^x$, and $\phi(x) = e^x/x$, it follows from Eq. (*12.6*) that

$$v_1'(e^x) + v_2'(xe^x) = 0$$

$$v_1'(e^x) + v_2'(e^x + xe^x) = \frac{e^x}{x}$$

Solving this set of equations simultaneously, we obtain $v_1' = -1$ and $v_2' = 1/x$. Thus,

$$v_1 = \int v_1'\,dx = \int -1\,dx = -x$$

$$v_2 = \int v_2'\,dx = \int \frac{1}{x}\,dx = \ln|x|$$

Substituting these values into (*1*), we obtain

$$y_p = -xe^x + xe^x \ln|x|$$

The general solution is therefore,

$$y = y_h + y_p = c_1e^x + c_2xe^x - xe^x + xe^x\ln|x|$$

$$= c_1e^x + c_3xe^x + xe^x\ln|x| \quad (c_3 = c_2 - 1)$$

12.4. Solve $y'' - y' - 2y = e^{3x}$.

Here $n = 2$ and $y_h = c_1e^{-x} + c_2e^{2x}$; hence,

$$y_p = v_1e^{-x} + v_2e^{2x} \tag{1}$$

Since $y_1 = e^{-x}$, $y_2 = e^{2x}$, and $\phi(x) = e^{3x}$, it follows from Eq. (*12.6*) that

$$v_1'(e^{-x}) + v_2'(e^{2x}) = 0$$

$$v_1'(-e^{-x}) + v_2'(2e^{2x}) = e^{3x}$$

Solving this set of equations simultaneously, we obtain $v_1' = -e^{4x}/3$ and $v_2' = e^x/3$, from which $v_1 = -e^{4x}/12$ and $v_2 = e^x/3$. Substituting these results into (*1*), we obtain

$$y_p = -\frac{1}{12}e^{4x}e^{-x} + \frac{1}{3}e^xe^{2x} = -\frac{1}{12}e^{3x} + \frac{1}{3}e^{3x} = \frac{1}{4}e^{3x}$$

The general solution is, therefore,

$$y = c_1e^{-x} + c_2e^{2x} + \frac{1}{4}e^{3x}$$

(Compare with Problem 11.2.)

12.5. Solve $\ddot{x} + 4x = \sin^2 2t$.

This is a second-order equation for $x(t)$ with

$$x_h = c_1 \cos 2t + c_2 \sin 2t$$

It follows from Eq. (*12.3*) that

$$x_p = v_1 \cos 2t + v_2 \sin 2t \tag{1}$$

where v_1 and v_2 are now functions of t. Here $x_1 = \cos 2t$, $x_2 = \sin 2t$ are two linearly independent solutions of the associated homogeneous differential equation and $\phi(t) = \sin^2 2t$, so Eq. (*12.6*), with x replacing y, becomes

$$v_1' \cos 2t + v_2' \sin 2t = 0$$

$$v_1'(-2 \sin 2t) + v_2'(2 \cos 2t) = \sin^2 2t$$

The solution of this set of equations is

$$v_1' = -\frac{1}{2}\sin^3 2t$$

$$v_2' = \frac{1}{2}\sin^2 2t \cos 2t$$

Thus,

$$v_1 = -\frac{1}{2}\int \sin^3 2t\, dt = \frac{1}{4}\cos 2t - \frac{1}{12}\cos^3 2t$$

$$v_2 = \frac{1}{2}\int \sin^2 2t \cos 2t\, dt = \frac{1}{12}\sin^3 2t$$

Substituting these values into (*1*), we obtain

$$\begin{aligned} x_p &= \left[\frac{1}{4}\cos 2t - \frac{1}{12}\cos^3 2t\right]\cos 2t + \left[\frac{1}{12}\sin^3 2t\right]\sin 2t \\ &= \frac{1}{4}\cos^2 2t - \frac{1}{12}(\cos^4 2t - \sin^4 2t) \\ &= \frac{1}{4}\cos^2 2t - \frac{1}{12}(\cos^2 2t - \sin^2 2t)(\cos^2 2t + \sin^2 2t) \\ &= \frac{1}{6}\cos^2 2t + \frac{1}{12}\sin^2 2t \end{aligned}$$

because $\cos^2 2t + \sin^2 2t = 1$. The general solution is

$$x = x_h + x_p = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{6}\cos^2 2t + \frac{1}{12}\sin^2 2t$$

12.6. Solve $t^2 \dfrac{d^2N}{dt^2} - 2t\dfrac{dN}{dt} + 2N = t \ln t$ if it is known that two linearly independent solutions of the associated homogeneous differential equation are t and t^2.

We first write the differential equation in standard form, with unity as the coefficient of the highest derivative. Dividing the equation by t^2, we obtain

$$\frac{d^2N}{dt^2} - \frac{2}{t}\frac{dN}{dt} + \frac{2}{t^2}N = \frac{1}{t}\ln t$$

with $\phi(t) = (1/t) \ln t$. We are given $N_1 = t$ and $N_2 = t^2$ as two linearly independent solutions of the associated second-order homogeneous equation. It follows from Theorem 8.2 that

$$N_h = c_1 t + c_2 t^2$$

We assume, therefore, that

$$N_p = v_1 t + v_2 t^2 \tag{1}$$

Equations (*12.6*), with N replacing y, become

$$v_1'(t) + v_2'(t^2) = 0$$

$$v_1'(1) + v_2'(2t) = \frac{1}{t}\ln t$$

The solution of this set of equations is

$$v_1' = -\frac{1}{t}\ln t \quad \text{and} \quad v_2' = \frac{1}{t^2}\ln t$$

Thus,

$$v_1 = -\int \frac{1}{t}\ln t\, dt = -\frac{1}{2}\ln^2 t$$

$$v_2 = \int \frac{1}{t^2}\ln t\, dt = -\frac{1}{t}\ln t - \frac{1}{t}$$

and (*1*) becomes

$$N_p = \left[-\frac{1}{2}\ln^2 t\right]t + \left[-\frac{1}{t}\ln t - \frac{1}{t}\right]t^2 = -\frac{t}{2}\ln^2 t - t\ln t - t$$

The general solution is

$$N = N_h + N_p = c_1 t + c_2 t^2 - \frac{t}{2}\ln^2 t - t\ln t - t$$

$$= c_3 t + c_2 t^2 - \frac{t}{2}\ln^2 t - t\ln t \quad (\text{with } c_3 = c_1 - 1)$$

12.7. Solve $y' + \dfrac{4}{x}y = x^4$.

Here $n = 1$ and (from Chapter 6) $y_h = c_1 x^{-4}$; hence,

$$y_p = v_1 x^{-4} \tag{1}$$

Since $y_1 = x^{-4}$ and $\phi(x) = x^4$, Eq. (*12.7*) becomes $v_1' x^{-4} = x^4$, from which we obtain $v_1' = x^8$ and $v_1 = x^9/9$. Equation (*1*) now becomes $y_p = x^5/9$, and the general solution is therefore

$$y = c_1 x^{-4} + \frac{1}{9}x^5$$

(Compare with Problem 6.6.)

12.8. Solve $y^{(4)} = 5x$ by variation of parameters.

Here $n = 4$ and $y_h = c_1 + c_2 x + c_3 x^2 + c_4 x^3$; hence,

$$y_p = v_1 + v_2 x + v_3 x^2 + v_4 x^3 \tag{1}$$

Since $y_1 = 1$, $y_2 = x$, $y_3 = x^2$, $y_4 = x^3$, and $\phi(x) = 5x$, it follows from Eq. (*12.4*), with $n = 4$, that

$$v_1'(1) + v_2'(x) + v_3'(x^2) + v_4'(x^3) = 0$$

$$v_1'(0) + v_2'(1) + v_3'(2x) + v_4'(3x^2) = 0$$

$$v_1'(0) + v_2'(0) + v_3'(2) + v_4'(6x) = 0$$

$$v_1'(0) + v_2'(0) + v_3'(0) + v_4'(6) = 5x$$

Solving this set of equations simultaneously, we obtain

$$v_1' = -\frac{5}{6}x^4 \qquad v_2' = \frac{5}{2}x^3 \qquad v_3' = -\frac{5}{2}x^2 \qquad v_4' = \frac{5}{6}x$$

whence
$$v_1 = -\frac{1}{6}x^5 \qquad v_2 = \frac{5}{8}x^4 \qquad v_3 = -\frac{5}{6}x^3 \qquad v_4 = \frac{5}{12}x^2$$

Then, from (*1*),

$$y_p = -\frac{1}{6}x^5 + \frac{5}{8}x^4(x) - \frac{5}{6}x^3(x^2) + \frac{5}{12}x^2(x^3) = \frac{1}{24}x^5$$

and the general solution is

$$y_h = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{1}{24}x^5$$

The solution also can be obtained simply by integrating both sides of the differential equation four times with respect to x.

Supplementary Problems

Use variation of parameters to find the general solutions of the following differential equations:

12.9. $y'' - 2y' + y = \dfrac{e^x}{x^5}$

12.10. $y'' + y = \sec x$

12.11. $y'' - y' - 2y = e^{3x}$

12.12. $y'' - 60y' - 900y = 5e^{10x}$

12.13. $y'' - 7y' = -3$

12.14. $y'' + \dfrac{1}{x}y' - \dfrac{1}{x^2}y = \ln x$ if two solutions to the associated homogeneous problem are known to be x and $1/x$.

12.15. $x^2y'' - xy' = x^3e^x$ if two solutions to the associated homogeneous problem are known to be 1 and x^2.

12.16. $y' - \dfrac{1}{x}y = x^2$

12.17. $y' + 2xy = x$

12.18. $y''' = 12$

12.19. $\ddot{x} - 2\dot{x} + x = \dfrac{e^t}{t^3}$

12.20. $\ddot{x} - 6\dot{x} + 9x = \dfrac{e^{3t}}{t^2}$

12.21. $\ddot{x} + 4x = 4\sec^2 2t$

12.22. $\ddot{x} - 4\dot{x} + 3x = \dfrac{e^t}{1+e^t}$

12.23. $(t^2 - 1)\ddot{x} - 2t\dot{x} + 2x = (t^2 - 1)^2$ if two solutions to the associated homogeneous equations are known to be t and $t^2 + 1$.

12.24. $(t^2 + t)\ddot{x} + (2 - t^2)\dot{x} - (2 + t)x = t(t + 1)^2$ if two solutions to the associated homogeneous equations are known to be e^t and $1/t$.

12.25. $\dddot{r} - 3\ddot{r} + 3\dot{r} - r = \dfrac{e^t}{t}$

12.26. $\dddot{r} + 6\ddot{r} + 12\dot{r} + 8r = 12e^{-2t}$

12.27. $\dddot{z} - 5\ddot{z} + 25\dot{z} - 125z = 1000$

12.28. $\dfrac{d^3z}{d\theta^3} - 3\dfrac{d^2z}{d\theta^2} + 2\dfrac{dz}{d\theta} = \dfrac{e^{3\theta}}{1+e^\theta}$

12.29. $t^3\dddot{y} + 3t^2\ddot{y} = 1$ if three linearly independent solutions to the associated homogeneous equations are known to be $1/t$, 1, and t.

12.30. $y^{(5)} - 4y^{(3)} = 32e^{2x}$

CHAPTER 13

Initial-Value Problems for Linear Differential Equations

Initial-value problems are solved by applying the initial conditions to the general solution of the differential equation. It must be emphasized that the initial conditions are applied *only* to the general solution and *not* to the homogeneous solution y_h, even though it is y_h that possesses all the arbitrary constants that must be evaluated. The one exception is when the general solution is the homogeneous solution; that is, when the differential equation under consideration is itself homogeneous.

Solved Problems

13.1. Solve $y'' - y' - 2y = 4x^2$; $y(0) = 1$, $y'(0) = 4$.

The general solution of the differential equation is given in Problem 11.1 as

$$y = c_1e^{-x} + c_2e^{2x} - 2x^2 + 2x - 3 \tag{1}$$

Therefore,

$$y' = -c_1e^{-x} + 2c_2e^{2x} - 4x + 2 \tag{2}$$

Applying the first initial condition to (*1*), we obtain

$$y(0) = c_1e^{-(0)} + c_2e^{2(0)} - 2(0)^2 + 2(0) - 3 = 1 \qquad \text{or} \qquad c_1 + c_2 = 4 \tag{3}$$

Applying the second initial condition to (*2*), we obtain

$$y'(0) = -c_1e^{-(0)} + 2c_2e^{2(0)} - 4(0) + 2 = 4 \qquad \text{or} \qquad -c_1 + 2c_2 = 2 \tag{4}$$

Solving (*3*) and (*4*) simultaneously, we find that $c_1 = 2$ and $c_2 = 2$. Substituting these values into (*1*), we obtain the solution of the initial-value problem as

$$y = 2e^{-x} + 2e^{2x} - 2x^2 + 2x - 3$$

13.2. Solve $y'' - 2y' + y = \dfrac{e^x}{x}$; $y(1) = 0$, $y'(1) = 1$.

The general solution of the differential equation is given in Problem 12.3 as

$$y = c_1 e^x + c_3 x e^x + x e^x \ln |x| \tag{1}$$

Therefore,

$$y' = c_1 e^x + c_3 e^x + c_3 x e^x + e^x \ln |x| + x e^x \ln |x| + e^x \tag{2}$$

Applying the first initial condition to (*1*), we obtain

$$y(1) = c_1 e^1 + c_3(1)e^1 + (1)e^1 \ln 1 = 0$$

or (noting that $\ln 1 = 0$),

$$c_1 e + c_3 e = 0 \tag{3}$$

Applying the second initial condition to (2), we obtain

$$y'(1) = c_1 e^1 + c_3 e^1 + c_3(1)e^1 + e^1 \ln 1 + (1)e^1 \ln 1 + e^1 = 1$$

or

$$c_1 e + 2c_3 e = 1 - e \tag{4}$$

Solving (*3*) and (*4*) simultaneously, we find that $c_1 = -c_3 = (e - 1)/e$. Substituting these values into (*1*), we obtain the solution of the initial-value problem as

$$y = e^{x-1}(e - 1)(1 - x) + x e^x \ln |x|$$

13.3. Solve $y'' + 4y' + 8y = \sin x$; $y(0) = 1$, $y'(0) = 0$.

Here $y_h = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x)$, and, by the method of undetermined coefficients,

$$y_p = \frac{7}{65}\sin x - \frac{4}{65}\cos x$$

Thus, the general solution to the differential equation is

$$y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{7}{65}\sin x - \frac{4}{65}\cos x \tag{1}$$

Therefore,

$$y' = -2e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + e^{-2x}(-2c_1 \sin 2x + 2c_2 \cos 2x) + \frac{7}{65}\cos x + \frac{4}{65}\sin x \tag{2}$$

Applying the first initial condition to (*1*), we obtain

$$c_1 = \frac{69}{65} \tag{3}$$

Applying the second initial condition to (2), we obtain

$$-2c_1 + 2c_2 = -\frac{7}{65} \tag{4}$$

Solving (*3*) and (*4*) simultaneously, we find that $c_1 = 69/65$ and $c_2 = 131/130$. Substituting these values into (*1*), we obtain the solution of the initial-value problem as

$$y = e^{-2x}\left(\frac{69}{65}\cos 2x + \frac{131}{130}\sin 2x\right) + \frac{7}{65}\sin x - \frac{4}{65}\cos x$$

13.4. Solve $y''' - 6y'' + 11y' - 6y = 0$; $y(\pi) = 0$, $y'(\pi) = 0$, $y''(\pi) = 1$.

From Problem 10.1, we have

$$y_h = c_1e^x + c_2e^{2x} + c_3e^{3x} \tag{1}$$
$$y_h' = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}$$
$$y_h'' = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}$$

Since the given differential equation is homogeneous, y_h is also the general solution. Applying each initial condition separately, we obtain

$$y(\pi) = c_1e^{\pi} + c_2e^{2\pi} + c_3e^{3\pi} = 0$$
$$y'(\pi) = c_1e^{\pi} + 2c_2e^{2\pi} + 3c_3e^{3\pi} = 0$$
$$y''(\pi) = c_1e^{\pi} + 4c_2e^{2\pi} + 9c_3e^{3\pi} = 1$$

Solving these equations simultaneously, we find

$$c_1 = \frac{1}{2}e^{-\pi} \quad c_2 = -e^{-2\pi} \quad c_3 = \frac{1}{2}e^{-3\pi}$$

Substituting these values into the first equation (*1*), we obtain

$$y = \frac{1}{2}e^{(x-\pi)} - e^{2(x-\pi)} + \frac{1}{2}e^{3(x-\pi)}$$

13.5. Solve $\ddot{x} + 4x = \sin^2 2t$; $x(0) = 0$, $\dot{x}(0) = 0$.

The general solution of the differential equation is given in Problem 12.5 as

$$x = c_1\cos 2t + c_2\sin 2t + \frac{1}{6}\cos^2 2t + \frac{1}{12}\sin^2 2t \tag{1}$$

Therefore,
$$\dot{x} = -2c_1\sin 2t + 2c_2\cos 2t - \frac{1}{3}\cos 2t\sin 2t \tag{2}$$

Applying the first initial condition to (*1*), we obtain

$$x(0) = c_1 + \frac{1}{6} = 0$$

Hence $c_1 = -1/6$. Applying the second initial condition to (*2*), we obtain

$$\dot{x}(0) = 2c_2 = 0$$

Hence $c_2 = 0$. The solution to the initial-value problem is

$$x = -\frac{1}{6}\cos 2t + \frac{1}{6}\cos^2 2t + \frac{1}{12}\sin^2 2t$$

13.6. Solve $\ddot{x} + 4x = \sin^2 2t$; $x(\pi/8) = 0$, $\dot{x}(\pi/8) = 0$.

The general solution of the differential equation and the derivative of the solution are as given in (*1*) and (*2*) of Problem 13.5. Applying the first initial condition, we obtain

$$0 = x\left(\frac{\pi}{8}\right) = c_1\cos\frac{\pi}{4} + c_2\sin\frac{\pi}{4} + \frac{1}{6}\cos^2\frac{\pi}{4} + \frac{1}{12}\sin^2\frac{\pi}{4}$$
$$= c_1\frac{\sqrt{2}}{2} + c_2\frac{\sqrt{2}}{2} + \frac{1}{6}\left(\frac{1}{2}\right) + \frac{1}{12}\left(\frac{1}{2}\right)$$

or
$$c_1 + c_2 = -\frac{\sqrt{2}}{8} \tag{1}$$

Applying the second initial condition, we obtain

$$0 = \dot{x}\left(\frac{\pi}{8}\right) = -2c_1 \sin\frac{\pi}{4} + 2c_2 \cos\frac{\pi}{4} - \frac{1}{3}\cos\frac{\pi}{4}\sin\frac{\pi}{4}$$

$$= -2c_1\frac{\sqrt{2}}{2} + 2c_2\frac{\sqrt{2}}{2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$

or

$$-c_1 + c_2 = \frac{\sqrt{2}}{12} \tag{2}$$

Solving (*1*) and (*2*) simultaneously, we find that

$$c_1 = -\frac{5}{48}\sqrt{2} \quad \text{and} \quad c_2 = -\frac{1}{48}\sqrt{2}$$

whereupon, the solution to the initial-value problem becomes

$$x = -\frac{5}{48}\sqrt{2}\cos 2t - \frac{1}{48}\sqrt{2}\sin 2t + \frac{1}{6}\cos^2 2t + \frac{1}{12}\sin^2 2t$$

Supplementary Problems

Solve the following initial-value problems.

13.7. $y'' - y' - 2y = e^{3x}$; $y(0) = 1$, $y'(0) = 2$

13.8. $y'' - y' - 2y = e^{3x}$; $y(0) = 2$, $y'(0) = 1$

13.9. $y'' - y' - 2y = 0$; $y(0) = 2$, $y'(0) = 1$

13.10. $y'' - y' - 2y = e^{3x}$; $y(1) = 2$, $y'(1) = 1$

13.11. $y'' + y = x$; $y(1) = 0$, $y'(1) = 1$

13.12. $y'' + 4y = \sin^2 2x$; $y(\pi) = 0$, $y'(\pi) = 0$

13.13. $y'' + y = 0$; $y(2) = 0$, $y'(2) = 0$

13.14. $y''' = 12$; $y(1) = 0$, $y'(1) = 0$, $y''(1) = 0$

13.15. $\ddot{y} = 2\dot{y} + 2y = \sin 2t + \cos 2t$; $y(0) = 0$, $\dot{y}(0) = 1$

CHAPTER 14

Applications of Second-Order Linear Differential Equations

SPRING PROBLEMS

The simple spring system shown in Fig. 14-1 consists of a mass m attached to the lower end of a spring that is itself suspended vertically from a mounting. The system is in its *equilibrium position* when it is at rest. The mass is set in motion by one or more of the following means: displacing the mass from its equilibrium position, providing it with an initial velocity, or subjecting it to an external force $F(t)$.

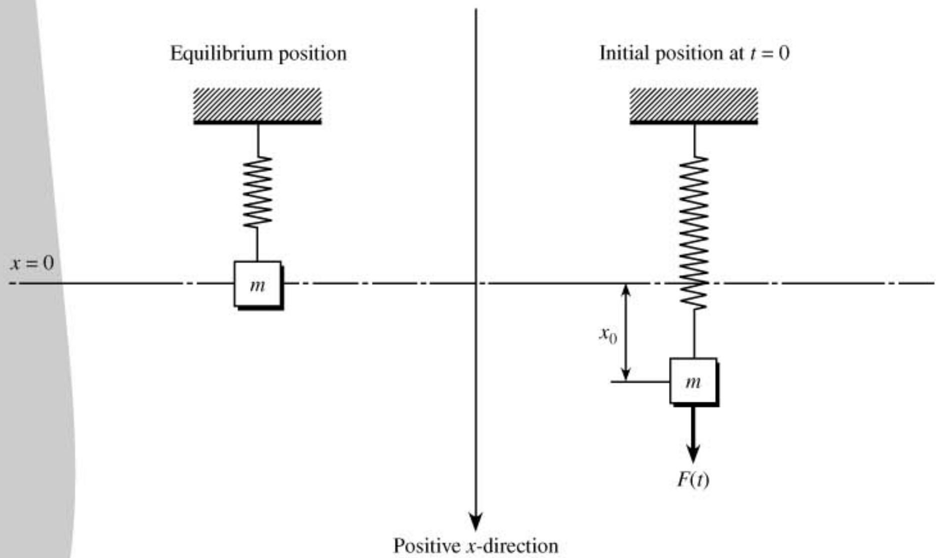

Fig. 14.1

Hooke's law: *The restoring force F of a spring is equal and opposite to the forces applied to the spring and is proportional to the extension (contraction) l of the spring as a result of the applied force; that is, $F = -kl$, where k denotes the constant of proportionality, generally called the spring constant.*

Example 14.1. A steel ball weighing 128 lb is suspended from a spring, whereupon the spring is stretched 2 ft from its natural length. The applied force responsible for the 2-ft displacement is the weight of the ball, 128 lb. Thus, $F = -128$ lb. Hooke's law then gives $-128 = -k(2)$, or $k = 64$ lb/ft.

For convenience, we choose the downward direction as the positive direction and take the origin to be the center of gravity of the mass in the equilibrium position. We assume that the mass of the spring is negligible and can be neglected and that air resistance, when present, is proportional to the velocity of the mass. Thus, at any time t, there are three forces acting on the system: (1) $F(t)$, measured in the positive direction; (2) a restoring force given by Hooke's law as $F_s = -kx$, $k > 0$; and (3) a force due to air resistance given by $F_a = -a\dot{x}$, $a > 0$, where a is the constant of proportionality. Note that the restoring force F_s always acts in a direction that will tend to return the system to the equilibrium position: if the mass is below the equilibrium position, then x is positive and $-kx$ is negative; whereas if the mass is above the equilibrium position, then x is negative and $-kx$ is positive. Also note that because $a > 0$ the force F_a due to air resistance acts in the opposite direction of the velocity and thus tends to retard, or damp, the motion of the mass.

It now follows from Newton's second law (see Chapter 7) that $m\ddot{x} = -kx - a\dot{x} + F(t)$, or

$$\ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m}x = \frac{F(t)}{m} \tag{14.1}$$

If the system starts at $t = 0$ with an initial velocity v_0 and from an initial position x_0, we also have the initial conditions

$$x(0) = x_0 \quad \dot{x}(0) = v_0 \tag{14.2}$$

(See Problems 14.1–14.10.)

The force of gravity does not explicitly appear in (*14.1*), but it is present nonetheless. We automatically compensated for this force by measuring distance from the equilibrium position of the spring. If one wishes to exhibit gravity explicitly, then distance must be measured from the bottom end of the *natural length* of the spring. That is, the motion of a vibrating spring can be given by

$$\ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m}x = g + \frac{F(t)}{m}$$

if the origin, $x = 0$, is the terminal point of the unstretched spring before the mass m is attached.

ELECTRICAL CIRCUIT PROBLEMS

The simple electrical circuit shown in Fig. 14-2 consists of a resistor R in ohms; a capacitor C in farads; an inductor L in henries; and an electromotive force (emf) $E(t)$ in volts, usually a battery or a generator, all

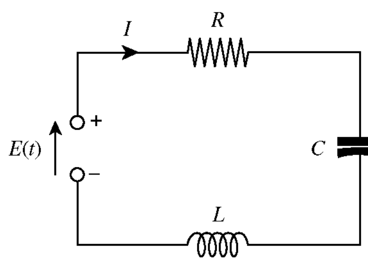

Fig. 14.2

connected in series. The current I flowing through the circuit is measured in amperes and the charge q on the capacitor is measured in coulombs.

Kirchhoff's loop law: *The algebraic sum of the voltage drops in a simple closed electric circuit is zero.*

It is known that the voltage drops across a resistor, a capacitor, and an inductor are respectively RI, $(1/C)q$, and $L(dI/dt)$ where q is the charge on the capacitor. The voltage drop across an emf is $-E(t)$. Thus, from Kirchhoff's loop law, we have

$$RI + L\frac{dI}{dt} + \frac{1}{C}q - E(t) = 0 \tag{14.3}$$

The relationship between q and I is

$$I = \frac{dq}{dt} \quad \frac{dI}{dt} = \frac{d^2q}{dt^2} \tag{14.4}$$

Substituting these values into (*14.3*), we obtain

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{1}{L}E(t) \tag{14.5}$$

The initial conditions for q are

$$q(0) = q_0 \quad \left.\frac{dq}{dt}\right|_{t=0} = I(0) = I_0 \tag{14.6}$$

To obtain a differential equation for the current, we differentiate Eq. (*14.3*) with respect to t and then substitute Eq. (*14.4*) directly into the resulting equation to obtain

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}\frac{dE(t)}{dt} \tag{14.7}$$

The first initial condition is $I(0) = I_0$. The second initial condition is obtained from Eq. (*14.3*) by solving for dI/dt and then setting $t = 0$. Thus,

$$\left.\frac{dI}{dt}\right|_{t=0} = \frac{1}{L}E(0) - \frac{R}{L}I_0 - \frac{1}{LC}q_0 \tag{14.8}$$

An expression for the current can be gotten either by solving Eq. (*14.7*) directly or by solving Eq. (*14.5*) for the charge and then differentiating that expression. (See Problems 14.12–14.16.)

BUOYANCY PROBLEMS

Consider a body of mass m submerged either partially or totally in a liquid of weight density ρ. Such a body experiences two forces, a downward force due to gravity and a counter force governed by:

Archimedes' principle: *A body in liquid experiences a buoyant upward force equal to the weight of the liquid displaced by that body.*

Equilibrium occurs when the buoyant force of the displaced liquid equals the force of gravity on the body. Figure 14-3 depicts the situation for a cylinder of radius r and height H where h units of cylinder height are submerged at equilibrium. At equilibrium, the volume of water displaced by the cylinder is $\pi r^2 h$, which provides a buoyant force of $\pi r^2 h\rho$ that must equal the weight of the cylinder mg. Thus,

$$\pi r^2 h\rho = mg \tag{14.9}$$

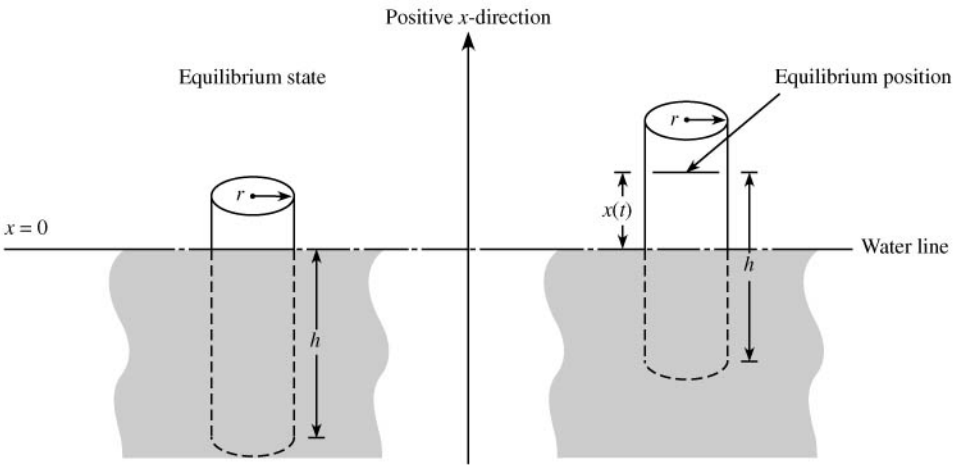

Fig. 14.3

Motion will occur when the cylinder is displaced from its equilibrium position. We arbitrarily take the upward direction to be the positive x-direction. If the cylinder is raised out of the water by $x(t)$ units, as shown in Fig. 14-3, then it is no longer in equilibrium. The downward or negative force on such a body remains mg but the buoyant or positive force is reduced to $\pi r^2[h - x(t)]\rho$. It now follows from Newton's second law that

$$m\ddot{x} = \pi r^2[h - x(t)]\rho - mg$$

Substituting (*14.9*) into this last equation, we can simplify it to

$$m\ddot{x} = -\pi r^2 x(t)\rho$$

or

$$\ddot{x} + \frac{\pi r^2 \rho}{m} x = 0 \qquad (14.10)$$

(See Problems 14.19–14.24.)

CLASSIFYING SOLUTIONS

Vibrating springs, simple electrical circuits, and floating bodies are all governed by second-order linear differential equations with constant coefficients of the form

$$\ddot{x} + a_1\dot{x} + a_0 x = f(t) \qquad (14.11)$$

For vibrating spring problems defined by Eq. (*14.1*), $a_1 = a/m$, $a_0 = k/m$, and $f(t) = F(t)/m$. For buoyancy problems defined by Eq. (*14.10*), $a_1 = 0$, $a_0 = \pi r^2\rho/m$, and $f(t) \equiv 0$. For electrical circuit problems, the independent variable x is replaced either by q in Eq. (*14.5*) or I in Eq. (*14.7*).

The motion or current in all of these systems is classified as *free* and *undamped* when $f(t) \equiv 0$ and $a_1 = 0$. It is classified as *free* and *damped* when $f(t)$ is identically zero but a_1 is not zero. For damped motion, there are three separate cases to consider, depending on whether the roots of the associated characteristic equation (see Chapter 9) are (1) real and distinct, (2) equal, or (3) complex conjugate. These cases are respectively classified as (1) *overdamped*, (2) *critically damped*, and (3) *oscillatory damped* (or, in electrical problems, *underdamped*). If $f(t)$ is not identically zero, the motion or current is classified as *forced*.

A motion or current is *transient* if it "dies out" (that is, goes to zero) as $t \to \infty$. A *steady-state* motion or current is one that is not transient and does not become unbounded. Free damped systems always yield transient motions, while forced damped systems (assuming the external force to be sinusoidal) yield both transient and steady-state motions.

Free undamped motion defined by Eq. (*14.11*) with $a_1 = 0$ and $f(t) \equiv 0$ always has solutions of the form

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t \qquad (14.12)$$

which defines *simple harmonic motion*. Here c_1, c_2, and ω are constants with ω often referred to as *circular frequency*. The *natural frequency* f is

$$f = \frac{\omega}{2\pi}$$

and it represents the number of complete oscillations per time unit undertaken by the solution. The *period* of the system of the time required to complete one oscillation is

$$T = \frac{1}{f}$$

Equation (*14.12*) has the alternate form

$$x(t) = (-1)^k A \cos(\omega t - \phi) \qquad (14.13)$$

where the *amplitude* $A = \sqrt{c_1^2 + c_2^2}$, the *phase angle* $\phi = \arctan(c_2/c_1)$, and k is zero when c_1 is positive and unity when c_1 is negative.

Solved Problems

14.1. A steel ball weighing 128 lb is suspended from a spring, whereupon the spring is stretched 2 ft from its natural length. The ball is started in motion with no initial velocity by displacing it 6 in above the equilibrium position. Assuming no air resistance, find (*a*) an expression for the position of the ball at any time t, and (*b*) the position of the ball at $t = \pi/12$ sec.

(*a*) The equation of motion is governed by Eq. (*14.1*). There is no externally applied force, so $F(t) = 0$, and no resistance from the surrounding medium, so $a = 0$. The motion is free and undamped. Here $g = 32$ ft/sec^2, $m = 128/32 = 4$ slugs, and it follows from Example 14.1 that $k = 64$ lb/ft. Equation (*14.1*) becomes $\ddot{x} + 16x = 0$. The roots of its characteristic equation are $\lambda = \pm 4i$, so its solution is

$$x(t) = c_1 \cos 4t + c_2 \sin 4t \qquad (1)$$

At $t = 0$, the position of the ball is $x_0 = -\frac{1}{2}$ ft (the minus sign is required because the ball is initially displaced *above* the equilibrium position, which is in the *negative* direction). Applying this initial condition to (*1*), we find that

$$-\frac{1}{2} = x(0) = c_1 \cos 0 + c_2 \sin 0 = c_1$$

so (*1*) becomes

$$x(t) = -\frac{1}{2}\cos 4t + c_2 \sin 4t \qquad (2)$$

The initial velocity is given as $v_0 = 0$ ft/sec. Differentiating (*2*), we obtain

$$v(t) = \dot{x}(t) = 2\sin 4t + 4c_2 \cos 4t$$

whereupon $$0 = v(0) = 2\sin 0 + 4c_2\cos 0 = 4c_2$$

Thus, $c_2 = 0$, and (*2*) simplifies to

$$x(t) = -\frac{1}{2}\cos 4t \tag{3}$$

as the equation of motion of the steel ball at any time t.

(*b*) At $t = \pi/12$,

$$x\left(\frac{\pi}{12}\right) = -\frac{1}{2}\cos\frac{4\pi}{12} = -\frac{1}{4}\text{ ft}$$

14.2. A mass of 2 kg is suspended from a spring with a known spring constant of 10 N/m and allowed to come to rest. It is then set in motion by giving it an initial velocity of 150 cm/sec. Find an expression for the motion of the mass, assuming no air resistance.

The equation of motion is governed by Eq. (*14.1*) and represents free undamped motion because there is no externally applied force on the mass, $F(t) = 0$, and no resistance from the surrounding medium, $a = 0$. The mass and the spring constant are given as $m = 2$ kg and $k = 10$ N/m, respectively, so Eq. (*14.1*) becomes $\ddot{x} + 5x = 0$. The roots of its characteristic equation are purely imaginary, so its solution is

$$x(t) = c_1\cos\sqrt{5}t + c_2\sin\sqrt{5}t \tag{1}$$

At $t = 0$, the position of the ball is at the equilibrium position $x_0 = 0$ m. Applying this initial condition to (*1*), we find that

$$0 = x(0) = c_1\cos 0 + c_2\sin 0 = c_1$$

whereupon (*1*) becomes

$$x(t) = c_2\sin\sqrt{5}t \tag{2}$$

The initial velocity is given as $v_0 = 150$ cm/sec $= 1.5$ m/sec. Differentiating (*2*), we obtain

$$v(t) = \dot{x}(t) = \sqrt{5}c_2\cos\sqrt{5}t$$

whereupon, $$1.5 = v(0) = \sqrt{5}c_2\cos 0 = \sqrt{5}c_2 \quad c_2 = \frac{1.5}{\sqrt{5}} = 0.6708$$

and (*2*) simplifies to

$$x(t) = 0.6708\sin\sqrt{5}t \tag{3}$$

as the position of the mass at any time t.

14.3. Determine the circular frequency, natural frequency, and period for the simple harmonic motion described in Problem 14.2.

Circular frequency: $\omega = \sqrt{5} = 2.236\text{ cycles/sec} = 2.236\text{ Hz}$

Natural frequency: $f = \omega/2\pi = \dfrac{\sqrt{5}}{2\pi} = 0.3559\text{ Hz}$

Period: $T = 1/f = \dfrac{2\pi}{\sqrt{5}} = 2.81\text{ sec}$

14.4. Determine the circular frequency, natural frequency, and period for the simple harmonic motion described in Problem 14.1.

Circular frequency:	$\omega = 4$ cycles/sec $= 4$ Hz
Natural frequency:	$f = 4/2\pi = 0.6366$ Hz
Period:	$T = 1/f = \pi/2 = 1.57$ sec

14.5. A 10-kg mass is attached to a spring, stretching it 0.7 m from its natural length. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction. Find the subsequent motion, if the force due to air resistance is $-90\dot{x}$ N.

Taking $g = 9.8$ m/sec^2, we have $w = mg = 98$ N and $k = w/l = 140$ N/m. Furthermore, $a = 90$ and $F(t) \equiv 0$ (there is no external force). Equation (*14.1*) becomes

$$\ddot{x} + 9\dot{x} + 14x = 0 \tag{1}$$

The roots of the associated characteristic equation are $\lambda_1 = -2$ and $\lambda_2 = -7$, which are real and distinct; hence this problem is an example of overdamped motion. The solution of (*1*) is

$$x = c_1e^{-2t} + c_2e^{-7t}$$

The initial conditions are $x(0) = 0$ (the mass starts at the equilibrium position) and $\dot{x}(0) = -1$ (the initial velocity is in the negative direction). Applying these conditions, we find that $c_1 = -c_2 = -\frac{1}{5}$, so that $x = \frac{1}{5}(e^{-7t} - e^{-2t})$. Note that $x \to 0$ as $t \to \infty$; thus, the motion is transient.

14.6. A mass of 1/4 slug is attached to a spring, whereupon the spring is stretched 1.28 ft from its natural length. The mass is started in motion from the equilibrium position with an initial velocity of 4 ft/sec in the downward direction. Find the subsequent motion of the mass if the force due to air resistance is $-2\dot{x}$ lb.

Here $m = 1/4$, $a = 2$, $F(t) \equiv 0$ (there is no external force), and, from Hooke's law, $k = mg/l = (1/4)(32)/1.28 = 6.25$. Equation (*14.1*) becomes

$$\ddot{x} + 8\dot{x} + 25x = 0 \tag{1}$$

The roots of the associated characteristic equation are $\lambda_1 = -4 + i3$ and $\lambda_2 = -4 - i3$, which are complex conjugates; hence this problem is an example of oscillatory damped motion. The solution of (*1*) is

$$x = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t)$$

The initial conditions are $x(0) = 0$ and $\dot{x}(0) = 4$. Applying these conditions, we find that $c_1 = 0$ and $c_2 = \frac{4}{3}$; thus, $x = \frac{4}{3}e^{-4t} \sin 3t$. Since $x \to 0$ as $t \to \infty$, the motion is transient.

14.7. A mass of 1/4 slug is attached to a spring having a spring constant of 1 lb/ft. The mass is started in motion by initially displacing it 2 ft in the downward direction and giving it an initial velocity of 2 ft/sec in the upward direction. Find the subsequent motion of the mass, if the force due to air resistance is $-1\dot{x}$ lb.

Here $m = 1/4$, $a = 1$, $k = 1$, and $F(t) \equiv 0$. Equation (*14.1*) becomes

$$\ddot{x} + 4\dot{x} + 4x = 0 \tag{1}$$

The roots of the associated characteristic equation are $\lambda_1 = \lambda_2 = -2$, which are equal; hence this problem is an example of critically damped motion. The solution of (*1*) is

$$x = c_1e^{-2t} + c_2te^{-2t}$$

The initial conditions are $x(0) = 2$ and $\dot{x}(0) = -2$ (the initial velocity is in the negative direction). Applying these conditions, we find that $c_1 = c_2 = 2$. Thus,

$$x = 2e^{-2t} + 2te^{-2t}$$

Since $x \to 0$ as $t \to \infty$, the motion is transient.

14.8. Show that the types of motions that result from free damped problems are completely determined by the quantity $a^2 - 4\,km$.

For free damped motions $F(t) \equiv 0$ and Eq. (*14.1*) becomes

$$\ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m}x = 0$$

The roots of the associated characteristic equation are

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4km}}{2m} \qquad \lambda_2 = \frac{-a - \sqrt{a^2 - 4km}}{2m}$$

If $a^2 - 4\,km > 0$, the roots are real and distinct; if $a^2 - 4\,km = 0$, the roots are equal; if $a^2 - 4\,km < 0$, the roots are complex conjugates. The corresponding motions are, respectively, overdamped, critically damped, and oscillatory damped. Since the real parts of both roots are always negative, the resulting motion in all three cases is transient. (For overdamped motion, we need only note that $\sqrt{a^2 - 4km} < a$, whereas for the other two cases the real parts are both $-a/2m$.)

14.9. A 10-kg mass is attached to a spring having a spring constant of 140 N/m. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force $F(t) = 5 \sin t$. Find the subsequent motion of the mass if the force due to air resistance is $-90\dot{x}$ N.

Here $m = 10$, $k = 140$, $a = 90$, and $F(t) = 5 \sin t$. The equation of motion, (*14.1*), becomes

$$\ddot{x} + 9\dot{x} + 14x = \frac{1}{2}\sin t \tag{1}$$

The general solution to the associated homogeneous equation $\ddot{x} + 9\dot{x} + 14x = 0$ is (see Problem 14.5)

$$x_h = c_1 e^{-2t} + c_2 e^{-7t}$$

Using the method of undetermined coefficients (see Chapter 11), we find

$$x_p = \frac{13}{500}\sin t - \frac{9}{500}\cos t \tag{2}$$

The general solution of (*1*) is therefore

$$x = x_h + x_p = c_1 e^{-2t} + c_2 e^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t$$

Applying the initial conditions, $x(0) = 0$ and $\dot{x}(0) = -1$, we obtain

$$x = \frac{1}{500}(-90e^{-2t} + 99e^{-7t} + 13\sin t - 9\cos t)$$

Note that the exponential terms, which come from x_h and hence represent an associated free overdamped motion, quickly die out. These terms are the transient part of the solution. The terms coming from x_p, however, do not die out as $t \to \infty$; they are the steady-state part of the solution.

14.10. A 128-lb weight is attached to a spring having a spring constant of 64 lb/ft. The weight is started in motion with no initial velocity by displacing it 6 in above the equilibrium position and by simultaneously applying to the weight an external force $F(t) = 8 \sin 4t$. Assuming no air resistance, find the subsequent motion of the weight.

Here $m = 4$, $k = 64$, $a = 0$, and $F(t) = 8 \sin 4t$; hence, Eq. (*14.1*) becomes

$$\ddot{x} + 16x = 2\sin 4t \tag{1}$$

This problem is, therefore, an example of forced undamped motion. The solution to the associated homogeneous equation is

$$x_h = c_1 \cos 4t + c_2 \sin 4t$$

A particular solution is found by the method of undetermined coefficients (the modification described in Chapter 11 is necessary here): $x_p = -\frac{1}{4}\cos 4t$. The solution to (*1*) is then

$$x = c_1 \cos 4t + c_2 \sin 4t - \frac{1}{4} t \cos 4t$$

Applying the initial conditions, $x(0) = -\frac{1}{2}$ and $\dot{x}(0) = 0$, we obtain

$$x = -\frac{1}{2}\cos 4t + \frac{1}{16}\sin 4t - \frac{1}{4} t \cos 4t$$

Note that $|x| \to \infty$ as $t \to \infty$. This phenomenon is called *pure resonance*. It is due to the forcing function $F(t)$ having the same circular frequency as that of the associated free undamped system.

14.11. Write the steady-state motion found in Problem 14.9 in the form specified by Eq. (*14.13*).

The steady-state displacement is given by (*2*) of Problem 14.9 as

$$x(t) = -\frac{9}{500}\cos t + \frac{13}{500}\sin t$$

Its circular frequency is $\omega = 1$. Here

$$A = \sqrt{\left(\frac{13}{500}\right)^2 + \left(-\frac{9}{500}\right)^2} = 0.0316$$

and $$\phi = \arctan\frac{13/500}{-9/500} = -0.965 \text{ radians}$$

The coefficient of the cosine term in the steady-state displacement is negative, so $k = 1$, and Eq. (*14.13*) becomes

$$x(t) = -0.0316 \cos (t + 0.965)$$

14.12. An RCL circuit connected in series has $R = 180$ ohms, $C = 1/280$ farad, $L = 20$ henries, and an applied voltage $E(t) = 10 \sin t$. Assuming no initial charge on the capacitor, but an initial current of 1 ampere at $t = 0$ when the voltage is first applied, find the subsequent charge on the capacitor.

Substituting the given quantities into Eq. (*14.5*), we obtain

$$\ddot{q} + 9\dot{q} + 14q = \frac{1}{2}\sin t$$

This equation is identical in form to (*1*) of Problem 14.9; hence, the solution must be identical in form to the solution of that equation. Thus,

$$q = c_1 e^{-2t} + c_2 e^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t$$

Applying the initial conditions $q(0) = 0$ and $\dot{q}(0) = 1$, we obtain $c_1 = 110/500$ and $c_2 = -101/500$. Hence,

$$q = \frac{1}{500}(110e^{-2t} - 101e^{-7t} + 13\sin t - 9\cos t)$$

As in Problem 14.9, the solution is the sum of transient and steady-state terms.

14.13. An RCL circuit connected in series has $R = 10$ ohms, $C = 10^{-2}$ farad, $L = \frac{1}{2}$ henry, and an applied voltage $E = 12$ volts. Assuming no initial current and no initial charge at $t = 0$ when the voltage is first applied, find the subsequent current in the system.

Substituting the given values into Eq. (*14.7*), we obtain the homogeneous equation [since $E(t) = 12$, $dE/dt = 0$]

$$\frac{d^2I}{dt^2} + 20\frac{dI}{dt} + 200I = 0$$

The roots of the associated characteristic equation are $\lambda_1 = -10 + 10i$ and $\lambda_2 = -10 - 10i$; hence, this is an example of a free underdamped system for the current. The solution is

$$I = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t) \tag{1}$$

The initial conditions are $I(0) = 0$ and, from Eq. (*14.8*),

$$\left.\frac{dI}{dt}\right|_{t=0} = \frac{12}{1/2} - \left(\frac{10}{1/2}\right)(0) - \frac{1}{(1/2)(10^{-2})}(0) = 24$$

Applying these conditions to (*1*), we obtain $c_1 = 0$ and $c_2 = \frac{12}{5}$; thus, $I = \frac{12}{5}e^{-10t}\sin 10t$, which is completely transient.

14.14. Solve Problem 14.13 by first finding the charge on the capacitor.

We first solve for the charge q and then use $I = dq/dt$ to obtain the current. Substituting the values given in Problem 14.13 into Eq. (*14.5*), we have $\ddot{q} + 20\dot{q} + 200q = 24$, which represents a forced system for the charge, in contrast to the free damped system obtained in Problem 14.3 for the current. Using the method of undetermined coefficients to find a particular solution, we obtain the general solution

$$q = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t) + \frac{3}{25}$$

Initial conditions for the charge are $q(0) = 0$ and $\dot{q}(0) = 0$; applying them, we obtain $c_1 = c_2 = -3/25$. Therefore,

$$q = -e^{-10t}\left(\frac{3}{25}\cos 10t + \frac{3}{25}\sin 10t\right) + \frac{3}{25}$$

and

$$I = \frac{dq}{dt} = \frac{12}{5}e^{-10t}\sin 10t$$

as before.

Note that although the current is completely transient, the charge on the capacitor is the sum of both transient and steady-state terms.

14.15. An RCL circuit connected in series has a resistance of 5 ohms, an inductance of 0.05 henry, a capacitor of 4×10^{-4} farad, and an applied alternating emf of 200 cos 100*t* volts. Find an expression for the current flowing through this circuit if the initial current and the initial charge on the capacitor are both zero.

Here $R/L = 5/0.05 = 100$, $1/(LC) = 1/[0.05(4 \times 10^{-4})] = 50{,}000$, and

$$\frac{1}{L}\frac{dE(t)}{dt} = \frac{1}{0.05}200(-100 \sin 100t) = -400{,}000 \sin 100t$$

so Eq. (*14.7*) becomes

$$\frac{d^2I}{dt^2} + 100\frac{dI}{dt} + 50{,}000I = -400{,}000 \sin 100t$$

The roots of its characteristic equation are $-50 \pm 50\sqrt{19}i$, hence the solution to the associated homogeneous problem is

$$I_h = c_1e^{-50t}\cos 50\sqrt{19}t + c_2e^{-50t}\sin 50\sqrt{19}t$$

Using the method of undetermined coefficients, we find a particular solution to be

$$I_p = \frac{40}{17}\cos 100t - \frac{160}{17}\sin 100t$$

so the general solution is

$$I = I_h + I_p = c_1 e^{-50t}\cos 50\sqrt{19}t + c_2 e^{-50t}\sin 50\sqrt{19}t + \frac{40}{17}\cos 100t - \frac{160}{17}\sin 100t \qquad (1)$$

The initial conditions are $I(0) = 0$ and, from Eq. (*14.8*),

$$\left.\frac{dI}{dt}\right|_{t=0} = \frac{200}{0.05} - \frac{5}{0.05}(0) - \frac{1}{0.05(4\times 10^{-4})}(0) = 4000$$

Applying the first of these conditions to (*1*) directly, we obtain

$$0 = I(0) = c_1(1) + c_2(0) + \frac{40}{17}$$

or $c_1 = -40/17 = -2.35$. Substituting this value into (*1*) and then differentiating, we find that

$$\frac{dI}{dt} = -2.35(-50e^{-50t}\cos 50\sqrt{19}t - 50\sqrt{19}e^{-50t}\sin 50\sqrt{19}t)$$
$$+ c_2(-50e^{-50t}\sin 50\sqrt{19}t + 50\sqrt{19}e^{-50t}\cos 50\sqrt{19}t) - \frac{4000}{17}\sin 100t - \frac{16,000}{17}\cos 100t$$

whereupon
$$4000 = \left.\frac{dI}{dt}\right|_{t=0} = -2.35(-50) + c_2(50\sqrt{19}) - \frac{16,000}{17}$$

and $c_2 = 22.13$. Equation (*1*) becomes

$$I = -2.35e^{-50t}\cos 50\sqrt{19}t + 22.13e^{-50t}\sin 50\sqrt{19}t + \frac{40}{17}\cos 100t - \frac{160}{17}\sin 100t$$

14.16. Solve Problem 14.15 by first finding the charge on the capacitor.

Substituting the values given in Problem 14.15 into Eq. (*14.5*), we obtain

$$\frac{d^2q}{dt^2} + 100\frac{dq}{dt} + 50,000q = 4000\cos 100t$$

The associated homogeneous equation is identical in form to the one in Problem 14.15, so it has the same solution (with I_h replaced by q_h). Using the method of undetermined coefficients, we find a particular solution to be

$$q_p = \frac{16}{170}\cos 100t + \frac{4}{170}\sin 100t$$

so the general solution is

$$q = q_h + q_p = c_1 e^{-50t}\cos 50\sqrt{19}t + c_2 e^{-50t}\sin 50\sqrt{19}t + \frac{16}{170}\cos 100t + \frac{4}{170}\sin 100t \qquad (1)$$

The initial conditions on the charge are $q(0) = 0$ and

$$\left.\frac{dq}{dt}\right|_{t=0} = I(0) = 0$$

Applying the first of these conditions to (*1*) directly, we obtain

$$0 = q(0) = c_1(1) + c_2(0) + \frac{16}{170}$$

or $c_1 = -16/170 = -0.0941$. Substituting this value into (*1*) and then differentiating, we find that

$$\frac{dq}{dt} = -0.0941(-50e^{-50t}\cos 50\sqrt{19}t - 50\sqrt{19}e^{-50t}\sin 50\sqrt{19}t)$$
$$+ c_2(-50e^{-50t}\sin 50\sqrt{19}t + 50\sqrt{19}e^{-50t}\cos 50\sqrt{19}t) - \frac{160}{17}\sin 100t + \frac{40}{17}\cos 100t \qquad (2)$$

whereupon
$$0 = \left.\frac{dq}{dt}\right|_{t=0} = -0.0941(-50) + c_2(50\sqrt{19}) + \frac{40}{17}$$

and $c_2 = -0.0324$. Substituting this value into (*2*) and simplifying, we obtain as before

$$I(t) = \frac{dq}{dt} = -2.35e^{-50t}\cos 50\sqrt{19}t + 22.13e^{-50t}\sin 50\sqrt{19}t + \frac{40}{17}\cos 100t - \frac{160}{17}\sin 100t \qquad (3)$$

14.17. Determine the circular frequency, the natural frequency, and the period of the steady-state current found in Problem 14.16.

The current is given by (*3*) of Problem 14.16. As $t \to \infty$, the exponential terms tend to zero, so the steady-state current is

$$I(t) = \frac{40}{17}\cos 100t - \frac{160}{17}\sin 100t$$

Circular frequency: $\omega = 100$ Hz

Natural frequency: $f = \omega/2\pi = 100/2\pi = 15.92$ Hz

Period: $T = 1/f = 2\pi/100 = 0.063$ sec

14.18. Write the steady-state current found in Problem 14.17 in the form specified by Eq. (*14.13*).

The amplitude is

$$A = \sqrt{\left(\frac{40}{17}\right)^2 + \left(-\frac{160}{17}\right)^2} = 9.701$$

and the phase angle is

$$\phi = \arctan\frac{-160/17}{40/17} = -1.326 \text{ radians}$$

The circular frequency is $\omega = 100$. The coefficient of the cosine term is positive, so $k = 0$ and Eq. (*14.13*) becomes

$$I_s(t) = 9.701\cos(100t + 1.326)$$

14.19. Determine whether a cylinder of radius 4 in, height 10 in, and weight 15 lb can float in a deep pool of water of weight density 62.5 lb/ft^3.

Let h denote the length (in feet) of the submerged portion of the cylinder at equilibrium. With $r = \frac{1}{3}$ft, it follows from Eq. (*14.9*) that

$$h = \frac{mg}{\pi r^2 \rho} = \frac{15}{\pi\left(\frac{1}{3}\right)^2 62.5} = 0.688 \text{ ft} = 8.25 \text{ in}$$

Thus, the cylinder will float with $10 - 8.25 = 1.75$ in of length above the water line at equilibrium.

14.20. Determine an expression for the motion of the cylinder described in Problem 14.19 if it is released with 20 percent of its length above the water line with a velocity of 5 ft/sec in the downward direction.

Here $r = \frac{1}{3}$ft, $\rho = 62.5$ lb/ft^3, $m = 15/32$ slugs and Eq. (*14.10*) becomes

$$\ddot{x} + 46.5421x = 0$$

The roots of the associated characteristic equation are $\pm\sqrt{46.5421}i = \pm\, 6.82i$; the general solution of the differential equation is

$$x(t) = c_1 \cos 6.82t + c_2 \sin 6.82t \qquad (1)$$

At $t = 0$, 20 percent of the 10-in length of the cylinder, or 2 in, is out of the water. Using the results of Problem 14.19, we know that the equilibrium position has 1.75 in above the water, so at $t = 0$, the cylinder is raised 1/4 in or 1/48 ft above its equilibrium position. In the context of Fig. 14-3, $x(0) = 1/48$ ft. The initial velocity is 5 ft/sec in the downward or *negative* direction in the coordinate system of Fig. 14-3, so $\dot{x}(0) = -5$. Applying these initial conditions to (*1*), we find that

$$c_1 = \frac{1}{48} = 0.021 \quad \text{and} \quad c_2 = \frac{-5}{6.82} = -0.73$$

Equation (*1*) becomes

$$x(t) = 0.021 \cos 6.82t - 0.73 \sin 6.82t$$

14.21. Determine whether a cylinder of diameter 10 cm, height 15 cm, and weight 19.6 N can float in a deep pool of water of weight density 980 dynes/cm^3.

Let h denote the length (in centimeters) of the submerged portion of the cylinder at equilibrium. With $r = 5$ cm and $mg = 19.6$ N $= 1.96 \times 10^6$ dynes, it follows from Eq. (*14.9*) that

$$h = \frac{mg}{\pi r^2 \rho} = \frac{1.96 \times 10^6}{\pi (5)^2 (980)} = 25.5 \text{ cm}$$

Since this is more height than the cylinder possesses, the cylinder cannot displace sufficient water to float and will sink to the bottom of the pool.

14.22. Determine whether a cylinder of diameter 10 cm, height 15 cm, and weight 19.6 N can float in a deep pool of liquid having weight density 2450 dynes/cm^3.

Let h denote the length of the submerged portion of the cylinder at equilibrium. With $r = 5$ cm and $mg = 19.6$ N $= 1.96 \times 10^6$ dynes, it follows from Eq. (*14.9*) that

$$h = \frac{mg}{\pi r^2 \rho} = \frac{1.96 \times 10^6}{\pi (5)^2 (2450)} = 10.2 \text{ cm}$$

Thus, the cylinder will float with $15 - 10.2 = 4.8$ cm of length above the liquid at equilibrium.

14.23. Determine an expression for the motion of the cylinder described in Problem 14.22 if it is released at rest with 12 cm of its length fully submerged.

Here $r = 5$ cm, $\rho = 2450$ dynes/cm^3, $m = 19.6/9.8 = 2$ kg $= 2000$ g, and Eq. (*14.10*) becomes

$$\ddot{x} + 96.21x = 0$$

The roots of the associated characteristic equation are $\pm\sqrt{96.21}i = \pm\, 9.8i$; the general solution of the differential equation is

$$x(t) = c_1 \cos 9.81t + c_2 \sin 9.81t \qquad (1)$$

At $t = 0$, 12 cm of the length of the cylinder is submerged. Using the results of Problem 14.22, we know that the equilibrium position has 10.2 cm submerged, so at $t = 0$, the cylinder is submerged $12 - 10.2 = 1.8$ cm *below* its equilibrium position. In the context of Fig. 14-3, $x(0) = -1.8$ cm with a negative sign indicating that the equilibrium line is submerged. The cylinder begins at rest, so its initial velocity is $\dot{x}(0) = 0$. Applying these initial conditions to (*1*), we find that $c_1 = -1.8$ and $c_2 = 0$. Equation (*1*) becomes

$$x(t) = -1.8 \cos 9.81t$$

14.24. A solid cylinder partially submerged in water having weight density 62.5 lb/ft^3, with its axis vertical, oscillates up and down within a period of 0.6 sec. Determine the diameter of the cylinder if it weighs 2 lb.

With $\rho = 62.5$ lb/ft^3 and $m = 2/32$ slugs, Eq. (*14.10*) becomes

$$\ddot{x} + 1000\pi r^2 x = 0$$

which has as its general solution

$$x(t) = c_1 \cos\sqrt{1000\pi}rt + c_2 \sin\sqrt{1000\pi}rt \tag{1}$$

Its circular frequency is $\omega = r\sqrt{1000\pi}$; its natural frequency is $f = \omega/2\pi = r\sqrt{250/\pi} = 8.92r$; its period is $T = 1/f = 1/8.92r$. We are given $0.6 = T = 1/8.92r$, thus $r = 0.187$ ft = 2.24 in with a diameter of 4.48 in.

14.25. A prism whose cross section is an equilateral triangle with sides of length l floats in a pool of liquid of weight density ρ with its height parallel to the vertical axis. The prism is set in motion by displacing it from its equilibrium position (see Fig. 14-4) and giving it an initial velocity. Determine the differential equation governing the subsequent motion of this prism.

Equilibrium occurs when the buoyant force of the displaced liquid equals the force of gravity on the body. The area of an equilateral triangle with sides of length l is $A = \sqrt{3}l^2/4$. For the prism depicted in Fig. 14-4, with h units of height submerged at equilibrium, the volume of water displaced at equilibrium is $\sqrt{3}l^2h/4$, providing a buoyant force of $\sqrt{3}l^2h\rho/4$. By Archimedes' principle, this buoyant force at equilibrium must equal the weight of the prism mg; hence,

$$\sqrt{3}l^2h\rho/4 = mg \tag{1}$$

We arbitrarily take the upward direction to be the positive x-direction. If the prism is raised out of the water by $x(t)$ units, as shown in Fig. 14-4, then it is no longer in equilibrium. The downward or negative force on such a body remains mg but the buoyant or positive force is reduced to $\sqrt{3}l^2[h - x(t)]\rho/4$. It now follows from Newton's second law that

$$m\ddot{x} = \frac{\sqrt{3}l^2[h - x(t)]\rho}{4} - mg$$

Substituting (*1*) into this last equation, we simplify it to

$$\ddot{x} + \frac{\sqrt{3}l^2\rho}{4m}x = 0$$

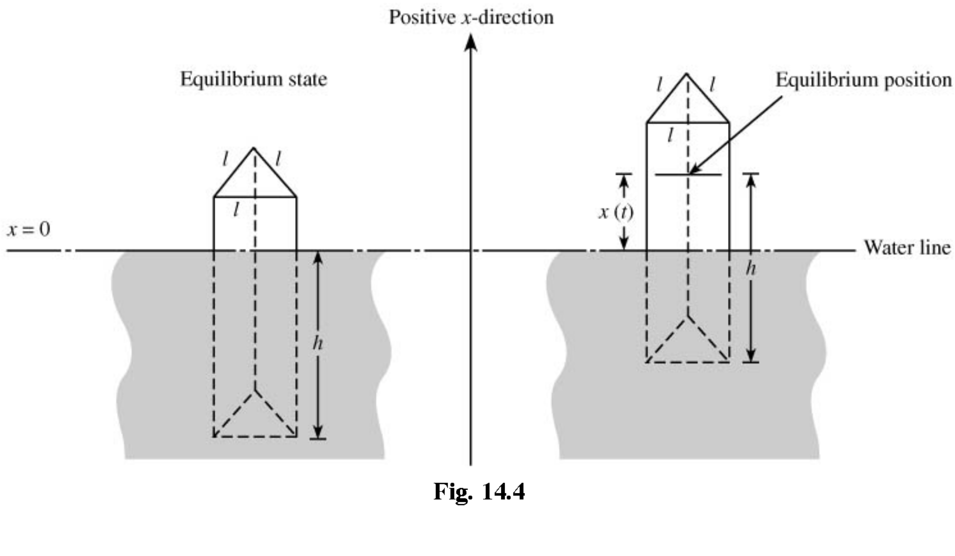

Fig. 14.4

Supplementary Problems

14.26. A 10-lb weight is suspended from a spring and stretches it 2 in from its natural length. Find the spring constant.

14.27. A mass of 0.4 slug is hung onto a spring and stretches it 9 in from its natural length. Find the spring constant.

14.28. A mass of 0.4 g is hung onto a spring and stretches it 3 cm from its natural length. Find the spring constant.

14.29. A mass of 0.3 kg is hung onto a spring and stretches it 15 cm from its natural length. Find the spring constant.

14.30. A 20-lb weight is suspended from the end of a vertical spring having a spring constant of 40 lb/ft and is allowed to reach equilibrium. It is then set into motion by stretching the spring 2 in from its equilibrium position and releasing the mass from rest. Find the position of the weight at any time t if there is no external force and no air resistance.

14.31. Solve Problem 14.30 if the weight is set in motion by compressing the spring by 2 in from its equilibrium position and giving it an initial velocity of 2 ft/sec in the downward direction.

14.32. A 20-g mass is suspended from the end of a vertical spring having a spring constant of 2880 dynes/cm and is allowed to reach equilibrium. It is then set into motion by stretching the spring 3 cm from its equilibrium position and releasing the mass with an initial velocity of 10 cm/sec in the downward direction. Find the position of the mass at any time t if there is no external force and no air resistance.

14.33. A 32-lb weight is attached to a spring, stretching it 8 ft from its natural length. The weight is started in motion by displacing it 1 ft in the upward direction and by giving it an initial velocity of 2 ft/sec in the downward direction. Find the subsequent motion of the weight, if the medium offers negligible resistance.

14.34. Determine (*a*) the circular frequency, (*b*) the natural frequency, and (*c*) the period for the vibrations described in Problem 14.31.

14.35. Determine (*a*) the circular frequency, (*b*) the natural frequency, and (*c*) the period for the vibrations described in Problem 14.32.

14.36. Determine (*a*) the circular frequency, (*b*) the natural frequency, and (*c*) the period for the vibrations described in Problem 14.33.

14.37. Find the solution to Eq. (*14.1*) with initial conditions given by Eq. (*14.2*) when the vibrations are free and undamped.

14.38. A $\frac{1}{4}$-slug mass is hung onto a spring, whereupon the spring is stretched 6 in from its natural length. The mass is then started in motion from the equilibrium position with an initial velocity of 4 ft/sec in the upward direction. Find the subsequent motion of the mass, if the force due to air resistance is $-2\dot{x}$ lb.

14.39. A $\frac{1}{2}$-slug mass is attached to a spring so that the spring is stretched 2 ft from its natural length. The mass is started in motion with no initial velocity by displacing it $\frac{1}{2}$ft in the upward direction. Find the subsequent motion of the mass, if the medium offers a resistance of $-4\dot{x}$ lb.

14.40. A $\frac{1}{2}$-slug mass is attached to a spring having a spring constant of 6 lb/ft. The mass is set into motion by displacing it 6 in below its equilibrium position with no initial velocity. Find the subsequent motion of the mass, if the force due to the medium is $-4\dot{x}$ lb.

14.41. A $\frac{1}{2}$-kg mass is attached to a spring having a spring constant of 8 N/m. The mass is set into motion by displacing it 10 cm above its equilibrium position with an initial velocity of 2 m/sec in the upward direction. Find the subsequent motion of the mass if the surrounding medium offers a resistance of $-4\dot{x}$N.

14.42. Solve Problem 14.41 if instead the spring constant is 8.01 N/m.

14.43. Solve Problem 14.41 if instead the spring constant is 7.99 N/m.

14.44. A 1-slug mass is attached to a spring having a spring constant of 8 lb/ft. The mass is initially set into motion from the equilibrium position with no initial velocity by applying an external force $F(t) = 16 \cos 4t$. Find the subsequent motion of the mass, if the force due to air resistance is $-4\dot{x}$ lb.

14.45. A 64-lb weight is attached to a spring whereupon the spring is stretched 1.28 ft and allowed to come to rest. The weight is set into motion by applying an external force $F(t) = 4 \sin 2t$. Find the subsequent motion of the weight if the surrounding medium offers a negligible resistance.

14.46. A 128-lb weight is attached to a spring whereupon the spring is stretched 2 ft and allowed to come to rest. The weight is set into motion from rest by displacing the spring 6 in above its equilibrium position and also by applying an external force $F(t) = 8 \sin 4t$. Find the subsequent motion of the weight if the surrounding medium offers a negligible resistance.

14.47. Solve Problem 14.38 if, in addition, the mass is subjected to an externally applied force $F(t) = 16 \sin 8t$.

14.48. A 16-lb weight is attached to a spring whereupon the spring is stretched 1.6 ft and allowed to come to rest. The weight is set into motion from rest by displacing the spring 9 in above its equilibrium position and also by applying an external force $F(t) = 5 \cos 2t$. Find the subsequent motion of the weight if the surrounding medium offers a resistance of $-2\dot{x}$ lb.

14.49. Write the steady-state portion of the motion found in Problem 14.48 in the form specified by Eq. (*14.13*).

14.50. A $\frac{1}{2}$-kg mass is attached to a spring having a spring constant of 6 N/m and allowed to come to rest. The mass is set into motion by applying an external force $F(t) = 24 \cos 3t - 33 \sin 3t$. Find the subsequent motion of the mass if the surrounding medium offers a resistance of $-3\dot{x}$ N.

14.51. Write the steady-state portion of the motion found in Problem 14.50 in the form of Eq. (*14.13*).

14.52. An RCL circuit connected in series with $R = 6$ ohms, $C = 0.02$ farad, and $L = 0.1$ henry has an applied voltage $E(t) = 6$ volts. Assuming no initial current and no initial charge at $t = 0$ when the voltage is first applied, find the subsequent charge on the capacitor and the current in the circuit.

14.53. An RCL circuit connected in series with a resistance of 5 ohms, a condenser of capacitance 4×10^{-4} farad, and an inductance of 0.05 henry has an applied emf $E(t) = 110$ volts. Assuming no initial current and no initial charge on the capacitor, find expressions for the current flowing through the circuit and the charge on the capacitor at any time t.

14.54. An RCL circuit connected in series with $R = 6$ ohms, $C = 0.02$ farad, and $L = 0.1$ henry has no applied voltage. Find the subsequent current in the circuit if the initial charge on the capacitor is $\frac{1}{10}$ coulomb and the initial current is zero.

14.55. An RCL circuit connected in series with a resistance of 1000 ohm, a condenser of capacitance 4×10^{-6} farad, and an inductance of 1 henry has an applied emf $E(t) = 24$ volts. Assuming no initial current and no initial charge on the capacitor, find an expression for the current flowing through the circuit at any time t.

14.56. An RCL circuit connected in series with a resistance of 4 ohms, a capacitor of 1/26 farad, and an inductance of 1/2 henry has an applied voltage $E(t) = 16 \cos 2t$. Assuming no initial current and no initial charge on the capacitor, find an expression for the current flowing through the circuit at any time t.

14.57. Determine the steady-state current in the circuit described in Problem 14.56 and write it in the form of Eq. (*14.13*).

14.58. An RCL circuit connected in series with a resistance of 16 ohms, a capacitor of 0.02 farad, and an inductance of 2 henries has an applied voltage $E(t) = 100 \sin 3t$. Assuming no initial current and no initial charge on the capacitor, find an expression for the current flowing through the circuit at any time t.

14.59. Determine the steady-state current in the circuit described in Problem 14.56 and write it in the form of Eq. (*14.13*).

14.60. An RCL circuit connected in series with a resistance of 20 ohms, a capacitor of 10^{-4} farad, and an inductance of 0.05 henry has an applied voltage $E(t) = 100 \cos 200t$. Assuming no initial current and no initial charge on the capacitor, find an expression for the current flowing through the circuit at any time t.

14.61. Determine the steady-state current in the circuit described in Problem 14.60 and write it in the form of Eq. (*14.13*).

14.62. An RCL circuit connected in series with a resistance of 2 ohms, a capacitor of 1/260 farad, and an inductance of 0.1 henry has an applied voltage $E(t) = 100 \sin 60t$. Assuming no initial current and no initial charge on the capacitor, find an expression for the charge on the capacitor at any time t.

14.63. Determine the steady-state charge on the capacitor in the circuit described in Problem 14.62 and write it in the form of Eq. (*14.13*).

14.64. An RCL circuit connected in series has R = 5 ohms, $C = 10^{-2}$ farad, $L = \frac{1}{8}$ henry, and no applied voltage. Find the subsequent steady-state current in the circuit. *Hint*: Initial conditions are not needed.

14.65. An RCL circuit connected in series with R = 5 ohms, $C = 10^{-2}$ farad, and $L = \frac{1}{8}$ henry has applied voltage $E(t) = \sin t$. Find the steady-state current in the circuit. *Hint*: Initial conditions are not needed.

14.66. Determine the equilibrium position of a cylinder of radius 3 in, height 20 in, and weight 5π lb that is floating with its axis vertical in a deep pool of water of weight density 62.5 lb/ft^3.

14.67. Find an expression for the motion of the cylinder described in Problem 14.66 if it is disturbed from its equilibrium position by submerging an additional 2 in of height below the water line and with a velocity of 1 ft/sec in the downward direction.

14.68. Write the harmonic motion of the cylinder described in Problem 14.67 in the form of Eq. (*14.13*).

14.69. Determine the equilibrium position of a cylinder of radius 2 ft, height 4 ft, and weight 600 lb that is floating with its axis vertical in a deep pool of water of weight density 62.5 lb/ft^3.

14.70. Find an expression for the motion of the cylinder described in Problem 14.69 if it is released from rest with 1 ft of its height submerged in water.

14.71. Determine (*a*) the circular frequency, (*b*) the natural frequency, and (*c*) the period for the vibrations described in Problem 14.70.

14.72. Determine (*a*) the circular frequency, (*b*) the natural frequency, and (*c*) the period for the vibrations described in Problem 14.67.

14.73. Determine the equilibrium position of a cylinder of radius 3 cm, height 10 cm, and mass 700 g that is floating with its axis vertical in a deep pool of water of mass density 1 g/cm^3.

14.74. Solve Problem 14.73 if the liquid is not water but another substance with mass density 2 g/cm^3.

14.75. Determine the equilibrium position of a cylinder of radius 30 cm, height 500 cm, and weight 2.5×10^7 dynes that is floating with its axis vertical in a deep pool of water of weight density 980 dynes/cm^3.

14.76. Find an expression for the motion of the cylinder described in Problem 14.75 if it is set in motion from its equilibrium position by striking it to produce an initial velocity of 50 cm/sec in the downward direction.

14.77. Find the general solution to Eq. (*14.10*) and determine its period.

14.78. Determine the radius of a cylinder weighing 5 lb with its axis vertical that oscillates in a pool of deep water ($\rho = 62.5$ lb/ft^3) with a period of 0.75 sec. *Hint:* Use the results of Problem 14.77.

14.79. Determine the weight of a cylinder having a diameter of 1 ft with its axis vertical that oscillates in a pool of deep water ($\rho = 62.5$ lb/ft^3) with a period of 2 sec. *Hint:* Use the results of Problem 14.77.

14.80. A rectangular box of width w, length l, and height h floats in a pool of liquid of weight density ρ with its height parallel to the vertical axis. The box is set into motion by displacing it x_0 units from its equilibrium position and giving it an initial velocity of v_0. Determine the differential equation governing the subsequent motion of the box.

14.81. Determine (*a*) the period of oscillations for the motion described in Problem 14.80 and (*b*) the change in that period if the length of the box is doubled.