CHAPTER 21

The Laplace Transform

DEFINITION

Let $f(x)$ be defined for $0 \le x < \infty$ and let s denote an arbitrary real variable. The *Laplace transform of* $f(x)$, designated by either $\mathscr{L}\{f(x)\}$ or $F(s)$, is

$$\mathscr{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx \qquad (21.1)$$

for all values of s for which the improper integral converges. Convergence occurs when the limit

$$\lim_{R\to\infty} \int_0^R e^{-sx} f(x)\,dx \qquad (21.2)$$

exists. If this limit does not exist, the improper integral diverges and $f(x)$ has no Laplace transform. When evaluating the integral in Eq. (*21.1*), the variable s is treated as a constant because the integration is with respect to x.

The Laplace transforms for a number of elementary functions are calculated in Problems 21.4 through 21.8; additional transforms are given in Appendix A.

PROPERTIES OF LAPLACE TRANSFORMS

Property 21.1. ***(Linearity).*** If $\mathscr{L}\{f(x)\} = F(s)$ and $\mathscr{L}\{g(x)\} = G(s)$, then for any two constants c_1 and c_2

$$\mathscr{L}\{c_1 f(x) + c_2 g(x)\} = c_1\mathscr{L}\{f(x)\} + c_2\mathscr{L}\{g(x)\} = c_1F(s) + c_2G(s) \qquad (21.3)$$

Property 21.2. If $\mathscr{L}\{f(x)\} = F(s)$, then for any constant a

$$\mathscr{L}\{e^{ax} f(x)\} = F(s-a) \qquad (21.4)$$

Property 21.3. If $\mathscr{L}\{f(x)\} = F(s)$, then for any positive integer n

$$\mathscr{L}\{x^n f(x)\} = (-1)^n \frac{d^n}{ds^n}[F(s)] \qquad (21.5)$$

Property 21.4. If $\mathscr{L}\{f(x)\} = F(s)$ and if $\lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{f(x)}{x}$ exists, then

$$\mathscr{L}\left\{\frac{1}{x}f(x)\right\} = \int_s^\infty F(t)\,dt \tag{21.6}$$

Property 21.5. If $\mathscr{L}\{f(x)\} = F(s)$, then

$$\mathscr{L}\left\{\int_0^x f(t)\,dt\right\} = \frac{1}{s}F(s) \tag{21.7}$$

Property 21.6. If $f(x)$ is periodic with period ω, that is, $f(x+\omega) = f(x)$, then

$$\mathscr{L}\{f(x)\} = \frac{\int_0^\omega e^{-sx} f(x)\,dx}{1 - e^{-\omega s}} \tag{21.8}$$

FUNCTIONS OF OTHER INDEPENDENT VARIABLES

For consistency only, the definition of the Laplace transform and its properties, Eqs. (*21.1*) through (*21.8*), are presented for functions of x. They are equally applicable for functions of any independent variable and are generated by replacing the variable x in the above equations by any variable of interest. In particular, the counterpart of Eq. (*21.1*) for the Laplace transform of a function of t is

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Solved Problems

21.1. Determine whether the improper integral $\int_2^\infty \frac{1}{x^2}\,dx$ converges.

Since

$$\lim_{R\to\infty}\int_2^R \frac{1}{x^2}\,dx = \lim_{R\to\infty}\left(-\frac{1}{x}\right)\Bigg|_2^R = \lim_{R\to\infty}\left(-\frac{1}{R}+\frac{1}{2}\right) = \frac{1}{2}$$

the improper integral converges to the value $\frac{1}{2}$.

21.2. Determine whether the improper integral $\int_9^\infty \frac{1}{x}\,dx$ converges.

Since

$$\lim_{R\to\infty}\int_9^R \frac{1}{x}\,dx = \lim_{R\to\infty} \ln|x|\Big|_9^R = \lim_{R\to\infty}(\ln R - \ln 9) = \infty$$

the improper integral diverges.

21.3. Determine those values of s for which the improper integral $\int_0^\infty e^{-sx}\,dx$ converges.

For $s = 0$,

$$\int_0^\infty e^{-sx}\,dx = \int_0^\infty e^{-(0)(x)}\,dx = \lim_{R\to\infty}\int_0^R (1)\,dx = \lim_{R\to\infty} x\Big|_0^R = \lim_{R\to\infty} R = \infty$$

hence the integral diverges. For $s \neq 0$,

$$\int_0^\infty e^{-sx}dx = \lim_{R\to\infty}\int_0^\infty e^{-sx}dx = \lim_{R\to\infty}\left[-\frac{1}{s}e^{-sx}\right]_{x=0}^{x=R}$$

$$= \lim_{R\to\infty}\left(\frac{-1}{s}e^{-sR} + \frac{1}{s}\right)$$

when $s < 0$, $-sR > 0$; hence the limit is ∞ and the integral diverges. When $s > 0$, $-sR < 0$; hence, the limit is $1/s$ and the integral converges.

21.4. Find the Laplace transform of $f(x) = 1$.

Using Eq. (*21.1*) and the results of Problem 21.3, we have

$$F(s) = \mathcal{L}\{1\} = \int_0^\infty e^{-sx}(1)\,dx = \frac{1}{s} \quad (\text{for } s > 0)$$

(See also entry 1 in Appendix A.)

21.5. Find the Laplace transform of $f(x) = x^2$.

Using Eq. (*21.1*) and integration by parts twice, we find that

$$F(s) = \mathcal{L}\{x^2\} = \int_0^\infty e^{-sx}x^2dx = \lim_{R\to\infty}\int_0^R x^2e^{-sx}dx$$

$$= \lim_{R\to\infty}\left[-\frac{x^2}{s}e^{-sx} - \frac{2x}{s^2}e^{-sx} - \frac{2}{s^3}e^{-sx}\right]_{x=0}^{x=R}$$

$$= \lim_{R\to\infty}\left(-\frac{R^2}{s}e^{-sR} - \frac{2R}{s^2}e^{-sR} - \frac{2}{s^3}e^{-sR} + \frac{2}{s^3}\right)$$

For $s < 0$, $\lim_{R\to\infty}[-(R^2/s)e^{-sR}] = \infty$, and the improper integral diverge. For $s > 0$, it follows from repeated use of L'Hôpital's rule that

$$\lim_{R\to\infty}\left(-\frac{R^2}{s}e^{-sR}\right) = \lim_{R\to\infty}\left(\frac{-R^2}{se^{sR}}\right) = \lim_{R\to\infty}\left(\frac{-2R}{s^2e^{sR}}\right)$$

$$= \lim_{R\to\infty}\left(\frac{-2}{s^3e^{sR}}\right) = 0$$

$$\lim_{R\to\infty}\left(-\frac{2R}{s}e^{-sR}\right) = \lim_{R\to\infty}\left(\frac{-2R}{se^{sR}}\right) = \lim_{R\to\infty}\left(\frac{-2}{s^2e^{sR}}\right) = 0$$

Also, $\lim_{R\to\infty}[-(2/s^3)e^{-sR}] = 0$ directly; hence the integral converges, and $F(s) = 2/s^3$. For the special cases $s = 0$, we have

$$\int_0^\infty e^{-sx}x^2dx = \int_0^\infty e^{-s(0)}x^2dx = \lim_{R\to\infty}\int_0^R x^2dx = \lim_{R\to\infty}\frac{R^3}{3} = \infty$$

Finally, combining all cases, we obtain $\mathcal{L}\{x^2\} = 2/s^3$, $s > 0$. (See also entry 3 in Appendix A.)

21.6. Find $\mathcal{L}\{e^{ax}\}$.

Using Eq. (*21.1*), we obtain

$$F(s) = \mathcal{L}\{e^{ax}\} = \int_0^\infty e^{-sx}e^{ax}dx = \lim_{R\to\infty}\int_0^R e^{(a-s)x}dx$$

$$= \lim_{R\to\infty}\left[\frac{e^{(a-s)x}}{a-s}\right]_{x=0}^{x=R} = \lim_{R\to\infty}\left[\frac{e^{(a-s)R}-1}{a-s}\right]$$

$$= \frac{1}{s-a} \quad (\text{for } s > a)$$

Note that when $s \le a$, the improper integral diverges. (See also entry 7 in Appendix A.)

21.7. Find $\mathcal{L}\{\sin ax\}$.

Using Eq. (*21.1*) and integration by parts twice, we obtain

$$\begin{aligned}\mathcal{L}\{\sin ax\} &= \int_0^\infty e^{-sx}\sin ax\,dx = \lim_{R\to\infty}\int_0^R e^{-sx}\sin ax\,dx \\ &= \lim_{R\to\infty}\left[\frac{-se^{-sx}\sin ax}{s^2+a^2} - \frac{ae^{-sx}\cos ax}{s^2+a^2}\right]_{x=0}^{x=R} \\ &= \lim_{R\to\infty}\left[\frac{-se^{-sR}\sin aR}{s^2+a^2} - \frac{ae^{-sR}\cos aR}{s^2+a^2} + \frac{a}{s^2+a^2}\right] \\ &= \frac{a}{x^2+a^2} \quad (\text{for } s>0)\end{aligned}$$

(See also entry 8 in Appendix A.)

21.8. Find the Laplace transform of $f(x) = \begin{cases} e^x & x \le 2 \\ 3 & x > 2 \end{cases}$.

$$\begin{aligned}\mathcal{L}\{f(x)\} &= \int_0^\infty e^{-sx}f(x)\,dx = \int_0^2 e^{-sx}e^x dx + \int_2^\infty e^{-sx}(3)\,dx \\ &= \int_0^2 e^{(1-s)x}dx + 3\lim_{R\to\infty}\int_2^R e^{-sx}dx = \left.\frac{e^{(1-s)x}}{1-s}\right|_{x=0}^{x=2} - \frac{3}{s}\lim_{R\to\infty}\left.e^{-sx}\right|_{x=2}^{x=R} \\ &= \frac{e^{2(1-s)}}{1-s} - \frac{1}{1-s} - \frac{3}{s}\lim_{R\to\infty}[e^{-Rs} - e^{-2s}] = \frac{1-e^{-2(s-1)}}{s-1} + \frac{3}{s}e^{-2x} \quad (\text{for } s>0)\end{aligned}$$

21.9. Find the Laplace transform of the function graphed in Fig. 21-1.

$$f(x) = \begin{cases} -1 & x \le 4 \\ 1 & x > 4 \end{cases}$$

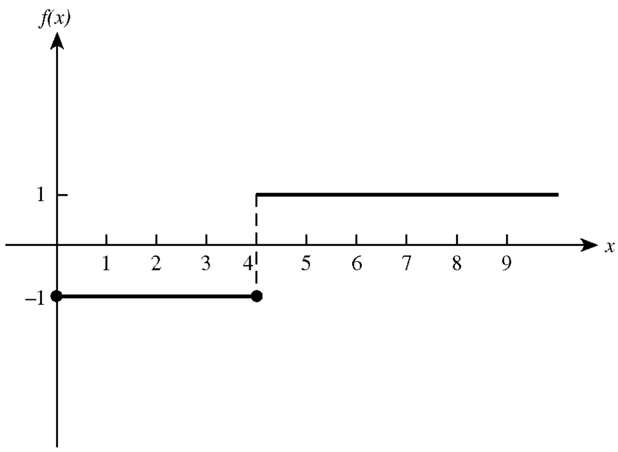

Fig. 21-1

$$\begin{aligned}\mathscr{L}\{f(x)\} &= \int_0^\infty e^{-sx} f(x)\,dx = \int_0^4 e^{-sx}(-1)\,dx + \int_4^\infty e^{-sx}(1)\,dx \\ &= \frac{e^{-sx}}{s}\Bigg|_{x=0}^{x=4} + \lim_{R\to\infty}\int_4^R e^{-sx}\,dx \\ &= \frac{e^{-4s}}{s} - \frac{1}{s} + \lim_{R\to\infty}\left(\frac{-1}{s}e^{-Rs} + \frac{1}{s}e^{-4s}\right) \\ &= \frac{2e^{-4s}}{s} - \frac{1}{s} \quad (\text{for } s > 0)\end{aligned}$$

21.10. Find the Laplace transform of $f(x) = 3 + 2x^2$.

Using Property 21.1 with the results of Problems 21.4 and 21.5, or alternatively, entries 1 and 3 ($n = 3$) of Appendix A, we have

$$\begin{aligned}F(s) &= \mathscr{L}\{3 + 2x^2\} = 3\mathscr{L}\{1\} + 2\mathscr{L}\{x^2\} \\ &= 3\left(\frac{1}{s}\right) + 2\left(\frac{2}{s^3}\right) = \frac{3}{s} + \frac{4}{s^3}\end{aligned}$$

21.11. Find the Laplace transform of $f(x) = 5\sin 3x - 17e^{-2x}$.

Using Property 21.1 with the results of Problems 21.6 ($a = -2$) and 21.7 ($a = 3$), or alternatively, entries 7 and 8 of Appendix A, we have

$$\begin{aligned}F(s) &= \mathscr{L}\{5\sin 3x - 17e^{-2x}\} = 5\mathscr{L}\{\sin 3x\} - 17\mathscr{L}\{e^{-2x}\} \\ &= 5\left(\frac{3}{s^2 + (3)^2}\right) - 17\left(\frac{1}{s - (-2)}\right) = \frac{15}{s^2 + 9} - \frac{17}{s + 2}\end{aligned}$$

21.12. Find the Laplace transform of $f(x) = 2\sin x + 3\cos 2x$.

Using Property 21.1 with entries 8 ($a = 1$) and 9 ($a = 2$) of Appendix A, we have

$$\begin{aligned}F(s) &= \mathscr{L}\{2\sin x + 3\cos 2x\} = 2\mathscr{L}\{\sin x\} + 3\mathscr{L}\{\cos 2x\} \\ &= 2\frac{1}{s^2 + 1} + 3\frac{s}{s^2 + 4} = \frac{2}{s^2 + 1} + \frac{3s}{s^2 + 4}\end{aligned}$$

21.13. Find the Laplace transform of $f(x) = 2x^2 - 3x + 4$.

Using Property 21.1 repeatedly with entries 1, 2 and 3 ($n = 3$) of Appendix A, we have

$$\begin{aligned}F(s) &= \mathscr{L}\{2x^2 - 3x + 4\} = 2\mathscr{L}\{x^2\} - 3\mathscr{L}\{x\} + 4\mathscr{L}\{1\} \\ &= 2\left(\frac{2}{s^3}\right) - 3\left(\frac{1}{s^2}\right) + 4\left(\frac{1}{s}\right) = \frac{4}{s^3} - \frac{3}{s^2} + \frac{4}{s}\end{aligned}$$

21.14. Find $\mathscr{L}\{xe^{4x}\}$.

This problem can be done three ways.

(*a*) Using entry 14 of Appendix A with $n = 2$ and $a = 4$, we have directly that

$$\mathcal{L}\{xe^{4x}\} = \frac{1}{(s-4)^2}$$

(*b*) Set $f(x) = x$. Using Property 21.2 with $a = 4$ and entry 2 of Appendix A, we have

$$F(s) = \mathcal{L}\{f(x)\} = \mathcal{L}\{x\} = \frac{1}{s^2}$$

and
$$\mathcal{L}\{e^{4x}x\} = F(s-4) = \frac{1}{(s-4)^2}$$

(*c*) Set $f(x) = e^{4x}$. Using Property 21.3 with $n = 1$ and the results of Problem 21.6, or alternatively, entry 7 of Appendix A with $a = 4$, we find that

$$F(s) = \mathcal{L}\{f(x)\} = \mathcal{L}\{e^{4x}\} = \frac{1}{s-4}$$

and
$$\mathcal{L}\{xe^{4x}\} = -F'(s) = -\frac{d}{ds}\left(\frac{1}{s-4}\right) = \frac{1}{(s-4)^2}$$

21.15. Find $\mathcal{L}\{e^{-2x}\sin 5x\}$.

This problem can be done two ways.

(*a*) Using entry 15 of Appendix A with $b = -2$ and $a = 5$, we have directly that

$$\mathcal{L}\{e^{-2x}\sin 5x\} = \frac{5}{[s-(-2)]^2 + (5)^2} = \frac{5}{(s+2)^2 + 25}$$

(*b*) Set $f(x) = \sin 5x$. Using Property 21.2 with $a = -2$ and the results of Problem 21.7, or alternatively, entry 8 of Appendix A with $a = 5$, we have

$$F(s) = \mathcal{L}\{f(x)\} = \mathcal{L}\{\sin 5x\} = \frac{5}{s^2 + 25}$$

and
$$\mathcal{L}\{e^{-2x}\sin 5x\} = F(s-(-2)) = F(s+2) = \frac{5}{(s+2)^2 + 25}$$

21.16. Find $\mathcal{L}\{x\cos\sqrt{7}x\}$.

This problem can be done two ways.

(*a*) Using entry 13 of Appendix A with $a = \sqrt{7}$, we have directly that

$$\mathcal{L}\{x\cos\sqrt{7}x\} = \frac{s^2 - (\sqrt{7})^2}{[s^2 + (\sqrt{7})^2]^2} = \frac{s^2 - 7}{(s^2 + 7)^2}$$

(*b*) Set $f(x) = \cos\sqrt{7}x$. Using Property 21.3 with $n = 1$ and entry 9 of Appendix A with $a = \sqrt{7}$, we have

$$F(s) = \mathcal{L}\{\cos\sqrt{7}x\} = \frac{s}{s^2 + (\sqrt{7})^2} = \frac{s}{s^2 + 7}$$

and $$\mathscr{L}\{x\cos\sqrt{7}x\} = -\frac{d}{ds}\left(\frac{s}{s^2+7}\right) = \frac{s^2-7}{(s^2+7)^2}$$

21.17. Find $\mathscr{L}\{e^{-x}\cos 2x\}$.

Let $f(x) = x\cos 2x$. From entry 13 of Appendix A with $a = 2$, we obtain

$$F(s) = \frac{s^2-4}{(s^2+4)^2}$$

Then, from Property 21.2 with $a = -1$,

$$\mathscr{L}\{e^{-x}x\cos 2x\} = F(s+1) = \frac{(s+1)^2-4}{[(s+1)^2+4]^2}$$

21.18. Find $\mathscr{L}\{x^{7/2}\}$.

Define $f(x) = \sqrt{x}$. Then $x^{7/2} = x^3\sqrt{x} = x^3 f(x)$ and, from entry 4 of Appendix A, we obtain

$$F(s) = \mathscr{L}\{f(x)\} = \mathscr{L}\{\sqrt{x}\} = \frac{1}{2}\sqrt{\pi}\,s^{-3/2}$$

It now follows from Property 21.3 with $n = 3$ that

$$\mathscr{L}\{x^3\sqrt{x}\} = (-1)^3\frac{d^3}{ds^3}\left(\frac{1}{2}\sqrt{\pi}s^{-3/2}\right) = \frac{105}{16}\sqrt{\pi}\,s^{-9/2}$$

which agrees with entry 6 of Appendix A for $n = 4$.

21.19. Find $\mathscr{L}\left\{\dfrac{\sin 3x}{x}\right\}$.

Taking $f(x) = \sin 3x$, we find from entry 8 of Appendix A with $a = 3$ that

$$F(s) = \frac{3}{s^2+9} \quad \text{or} \quad F(t) = \frac{3}{t^2+9}$$

Then, using Property 21.4, we obtain

$$\begin{aligned}\mathscr{L}\left\{\frac{\sin 3x}{x}\right\} &= \int_s^\infty \frac{3}{t^2+9}\,dt = \lim_{R\to\infty}\int_s^R \frac{3}{t^2+9}\,dt \\ &= \lim_{R\to\infty}\arctan\frac{t}{3}\bigg|_s^R \\ &= \lim_{R\to\infty}\left(\arctan\frac{R}{3} - \arctan\frac{s}{3}\right) \\ &= \frac{\pi}{2} - \arctan\frac{s}{3}\end{aligned}$$

21.20. Find $\mathscr{L}\left\{\int_0^x \sin 2t\,dt\right\}$.

Taking $f(t) = \sinh 2t$, we have $f(x) = \sinh 2x$. It now follows from entry 10 of Appendix A with $a = 2$ that

$F(s) = 2/(s^2 - 4)$, and then, from Property 21.5 that

$$\mathscr{L}\left\{\int_0^x \sinh 2t\, dt\right\} = \frac{1}{s}\left(\frac{2}{s^2-4}\right) = \frac{2}{s(s^2-4)}$$

21.21. Prove that if $f(x+\omega) = -f(x)$, then

$$\mathscr{L}\{f(x)\} = \frac{\int_0^\omega e^{-sx} f(x)dx}{1+e^{-\omega s}} \tag{1}$$

Since

$$f(x+2\omega) = f[(x+\omega)+\omega] = -f(x+\omega) = -[-f(x)] = f(x)$$

$f(x)$ is periodic with period 2ω. Then, using Property 21.6 with ω replaced by 2ω, we have

$$\mathscr{L}\{f(x)\} = \frac{\int_0^{2\omega} e^{-sx} f(x)dx}{1-e^{-2\omega s}} = \frac{\int_0^\omega e^{-sx} f(x)dx + \int_\omega^{2\omega} e^{-sx} f(x)dx}{1-e^{-2\omega s}}$$

Substituting $y = x - \omega$ into the second integral, we find that

$$\int_\omega^{2\omega} e^{-sx} f(x)dx = \int_0^\omega e^{-s(y+\omega)} f(y+\omega)dy = e^{-\omega s}\int_0^\omega e^{-sy}[-f(y)]dy$$
$$= -e^{-\omega s}\int_0^\omega e^{-sy} f(y)dy$$

The last integral, upon changing the dummy variable of integration back to x, equals

$$-e^{-\omega s}\int_0^\omega e^{-sx} f(x)dx$$

Thus,

$$\mathscr{L}\{f(x)\} = \frac{(1-e^{-\omega s})\int_0^\omega e^{-sx} f(x)dx}{1-e^{-2\omega s}}$$
$$= \frac{(1-e^{-\omega s})\int_0^\omega e^{-sx} f(x)dx}{(1-e^{-\omega s})(1+e^{-\omega s})} = \frac{\int_0^\omega e^{-sx} f(x)dx}{1+e^{-\omega s}}$$

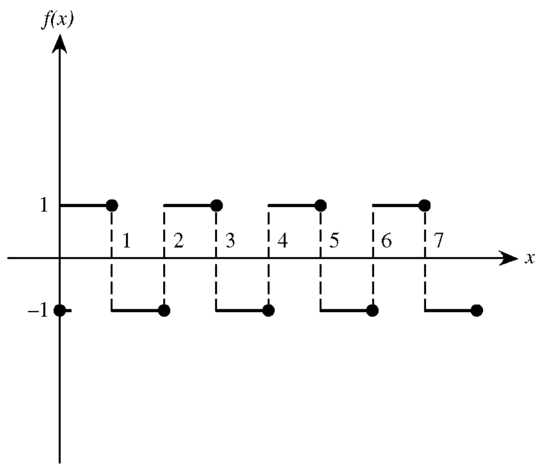

Fig. 21-2

21.22. Find $\mathscr{L}\{f(x)\}$ for the square wave shown in Fig. 21-2.

This problem can be done two ways.

(*a*) Note that $f(x)$ is periodic with period $\omega = 2$, and in the interval $0 < x \le 2$ it can be defined analytically by

$$f(x) = \begin{cases} 1 & 0 < x \le 1 \\ -1 & 1 < x \le 2 \end{cases}$$

From Eq. (*21.8*), we have

$$\mathscr{L}\{f(x)\} = \frac{\int_0^2 e^{-sx} f(x)dx}{1 - e^{-2s}}$$

Since

$$\int_0^2 e^{-sx} f(x)\,dx = \int_0^1 e^{-sx}(1)dx + \int_1^2 e^{-sx}(-1)dx$$

$$= \frac{1}{s}(e^{-2s} - 2e^{-s} + 1) = \frac{1}{s}(e^{-s} - 1)^2$$

it follows that

$$F(s) = \frac{(e^{-s} - 1)^2}{s(1 - e^{-2s})} = \frac{(1 - e^{-s})^2}{s(1 - e^{-s})(1 + e^{-s})} = \frac{1 - e^{-s}}{s(1 + e^{-s})}$$

$$= \left[\frac{e^{s/2}}{e^{s/2}}\right]\left[\frac{1 - e^{-s}}{s(1 + e^{-s})}\right] = \frac{e^{s/2} - e^{-s/2}}{s(e^{s/2} + e^{-s/2})} = \frac{1}{s}\tanh\frac{s}{2}$$

(*b*) The square wave $f(x)$ also satisfies the equation $f(x + 1) = -f(x)$. Thus, using (*1*) of Problem 21.21 with $\omega = 1$, we obtain

$$\mathscr{L}\{f(x)\} = \frac{\int_0^1 e^{-sx} f(x)\,dx}{1 + e^{-s}} = \frac{\int_0^1 e^{-sx}(1)\,dx}{1 + e^{-s}}$$

$$= \frac{(1/s)(1 - e^{-s})}{1 + e^{-s}} = \frac{1}{s}\tanh\frac{s}{2}$$

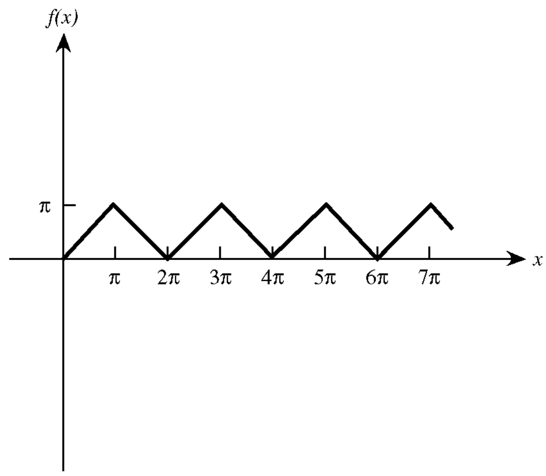

Fig. 21-3

21.23. Find the Laplace transform of the function graphed in Fig. 21-3.

Note that $f(x)$ is periodic with period $\omega = 2\pi$, and in the interval $0 \le x < 2\pi$ it can be defined analytically by

$$f(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x < 2\pi \end{cases}$$

From Eq. (*21.8*), we have

$$\mathscr{L}\{f(x)\} = \frac{\int_0^{2\pi} e^{-sx} f(x)dx}{1 - e^{-2\pi s}}$$

Since

$$\int_0^{2\pi} e^{-sx} f(x)dx = \int_0^{\pi} e^{-sx} x\, dx + \int_0^{2\pi} e^{-sx}(2\pi - x)dx$$
$$= \frac{1}{s^2}(e^{-2\pi s} - 2e^{-\pi s} + 1) = \frac{1}{s^2}(e^{-\pi s} - 1)^2$$

it follows that

$$\mathscr{L}\{f(x)\} = \frac{(1/s^2)(e^{-\pi s} - 1)^2}{1 - e^{-2\pi s}} = \frac{(1/s^2)(e^{-\pi s} - 1)^2}{(1 - e^{-\pi s})(1 + e^{-\pi s})}$$
$$= \frac{1}{s^2}\left(\frac{1 - e^{-\pi s}}{1 + e^{-\pi s}}\right) = \frac{1}{s^2}\tanh\frac{\pi s}{2}$$

21.24. Find $\mathscr{L}\left\{e^{4x} x \int_0^x \frac{1}{t} e^{-4t} \sin 3t\, dt\right\}$.

Using Eq. (*21.4*) with $a = -4$ on the results of Problem 21.19, we obtain

$$\mathscr{L}\left\{\frac{1}{x} e^{-4x} \sin 3x\right\} = \frac{\pi}{2} - \arctan\frac{s+4}{3}$$

It now follows from Eq. (*21.7*) that

$$\mathscr{L}\left\{\int_0^x \frac{1}{t} e^{-4t} \sin 3t\, dt\right\} = \frac{\pi}{2s} - \frac{1}{s}\arctan\frac{s+4}{3}$$

and then from Property 21.3 with $n = 1$,

$$\mathscr{L}\left\{x\int_0^x \frac{1}{t} e^{-4t} \sin 3t\, dt\right\} = \frac{\pi}{2s^2} - \frac{1}{s^2}\arctan\frac{s+4}{3} + \frac{3}{s[9 + (s+4)^2]}$$

Finally, using Eq. (*21.4*) with $a = 4$, we conclude that the required transform is

$$\frac{\pi}{2(s-4)^2} - \frac{1}{(s-4)^2}\arctan\frac{s}{3} + \frac{3}{(s-4)(s^2+9)}$$

21.25. Find the Laplace transforms at (*a*) t, (*b*) e^{at}, and (*c*) $\sin at$, where a denotes a constant.

Using entries 2, 7, and 8 of Appendix A with x replaced by t, we find the Laplace transforms to be, respectively,

$$(a)\quad \mathcal{L}\{t\} = \frac{1}{s^2} \qquad (b)\quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad (c)\quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

21.26. Find the Laplace transforms of (a) θ^2, (b) $\cos a\theta$, (c) $e^{b\theta} \sin a\theta$, where a and b denote constants.

Using entries 3 (with $n = 3$), 9, and 15 of Appendix A with x replaced by θ, we find the Laplace transforms to be, respectively.

$$(a)\quad \mathcal{L}\{\theta^2\} = \frac{2}{s^3} \qquad (b)\quad \mathcal{L}\{\cos a\theta\} = \frac{s}{s^2+a^2} \qquad (c)\quad \mathcal{L}\{e^{b\theta}\sin a\theta\} = \frac{a}{(s-b)^2+a^2}$$

Supplementary Problems

In Problems 21.27 and 21.42, find the Laplace transforms of the given function using Eq. (*21.1*).

21.27. $f(x) = 3$

21.28. $f(x) = \sqrt{5}$

21.29. $f(x) = e^{2x}$

21.30. $f(x) = e^{-6x}$

21.31. $f(x) = x$

21.32. $f(x) = -8x$

21.33. $f(x) = \cos 3x$

21.34. $f(x) = \cos 4x$

21.35. $f(x) = \cos bx$, where b denotes a constant

21.36. $f(x) = xe^{-8x}$

21.37. $f(x) = xe^{bx}$, where b denotes a constant

21.38. $f(x) = x^3$

21.39. $f(x) = \begin{cases} x & 0 \le x \le 2 \\ 2 & x > 2 \end{cases}$

21.40. $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ e^x & 1 < x \le 4 \\ 0 & x > 4 \end{cases}$

21.41. $f(x)$ in Fig. 21-4

21.42. $f(x)$ in Fig. 21-5

In Problems 21.43 and 21.76, use Appendix A and the Properties 21.1 through 21.6, where appropriate, to find the Laplace transforms of the given functions.

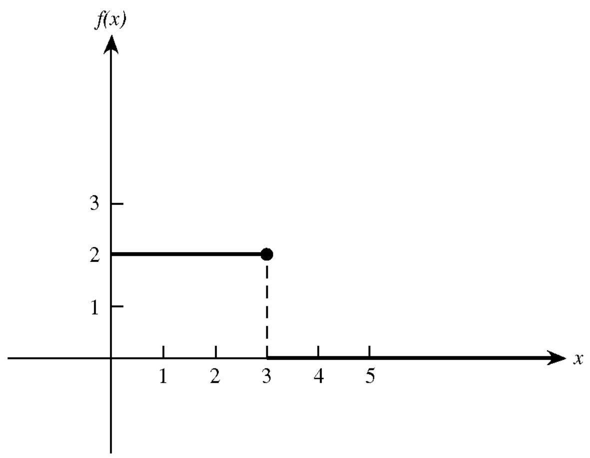

Fig. 21-4

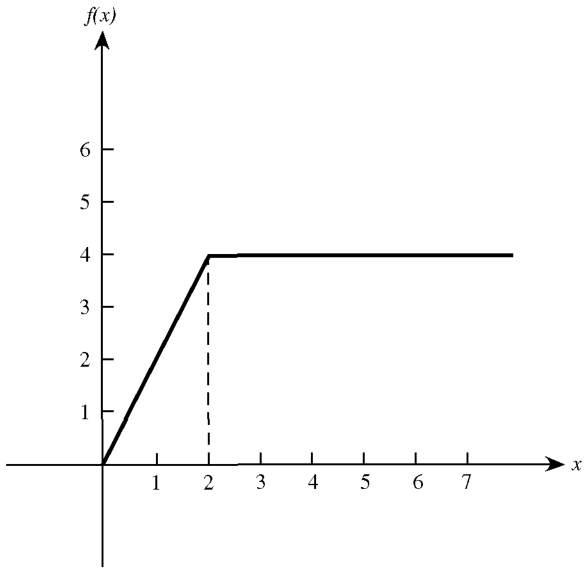

Fig. 21-5

21.43. $f(x) = x^7$

21.44. $f(x) = x \cos 3x$

21.45. $f(x) = x^5 e^{-x}$

21.46. $f(x) = \dfrac{1}{\sqrt{x}}$

21.47. $f(x) = \dfrac{1}{3} e^{-x/3}$

21.48. $f(x) = 5e^{-x/3}$

21.49. $f(x) = 2\sin^2 \sqrt{3}x$

21.50. $f(x) = 8e^{-5x}$

21.51. $f(x) = 3\sin\dfrac{x}{2}$

21.52. $f(x) = -\cos\sqrt{19}x$

21.53. $f(x) = -1.$

21.54. $f(x) = e^{-x} \sin 2x$

21.55. $f(x) = e^{x} \sin 2x$

21.56. $f(x) = e^{-x} \cos 2x$

21.57. $f(x) = e^{3x} \cos 2x$

21.58. $f(x) = e^{3x} \cos 5x$

21.59. $f(x) = e^{5x}\sqrt{x}$

21.60. $f(x) = e^{-5x}\sqrt{x}$

21.61. $f(x) = e^{-2x} \sin^2 x$

21.62. $x^3 + 3 \cos 2x$

21.63. $5e^{2x} + 7e^{-x}$

21.64. $f(x) = 2 + 3x$

21.65. $f(x) = 3 - 4x^2$

21.66. $f(x) = 2x + 5 \sin 3x$

21.67. $f(x) = 2 \cos 3x - \sin 3x$

21.68. $2x^2 \cosh x$

21.69. $2x^2 e^{-x} \cosh x$

21.70. $x^2 \sin 4x$

21.71. $\sqrt{x}\, e^{2x}$

21.72. $\int_0^x t \sinh t \, dt$

21.73. $\int_0^x e^{3t} \cos t \, dt$

21.74. $f(x)$ in Fig. 21-6

21.75. $f(x)$ in Fig. 21-7

21.76. $f(x)$ in Fig. 21-8

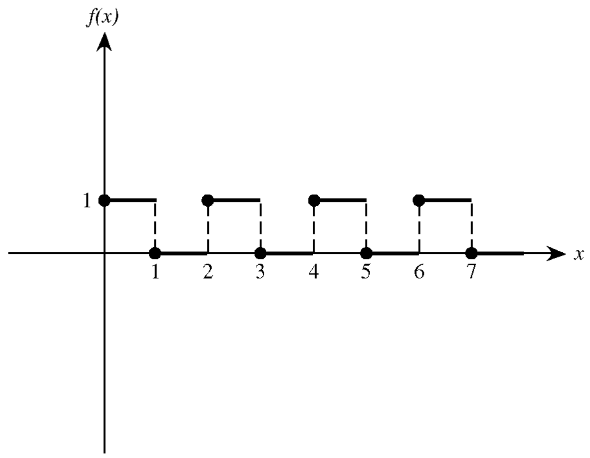

Fig. 21-6

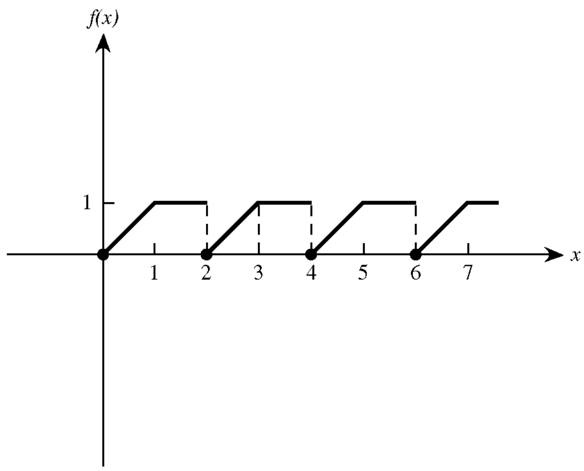

Fig. 21-7

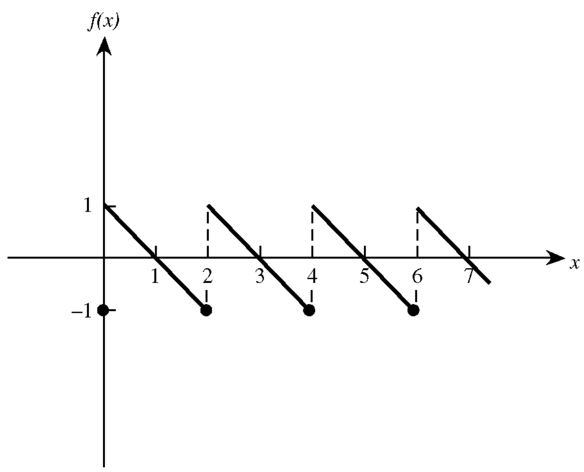

Fig. 21-8

CHAPTER 22

Inverse Laplace Transforms

DEFINITION

An *inverse Laplace transform* of $F(s)$, designated by $\mathscr{L}^{-1}\{F(s)\}$, is another function $f(x)$ having the property that $\mathscr{L}\{f(x)\} = F(s)$. This presumes that the independent variable of interest is x. If the independent variable of interest is t instead, then an inverse Laplace transform of $F(s)$ if $f(t)$ where $\mathscr{L}\{f(t)\} = F(s)$.

The simplest technique for identifying inverse Laplace transforms is to recognize them, either from memory or from a table such as Appendix A (see Problems 22.1 through 22.3). If $F(s)$ is not in a recognizable form, then occasionally it can be transformed into such a form by algebraic manipulation. Observe from Appendix A that almost all Laplace transforms are quotients. The recommended procedure is to first convert the denominator to a form that appears in Appendix A and then the numerator.

MANIPULATING DENOMINATORS

The method of *completing the square* converts a quadratic polynomial into the sum of squares, a form that appears in many of the denominators in Appendix A. In particular, for the quadratic $as^2 + bs + c$, where a, b, and c denote constants,

$$\begin{aligned} as^2 + bs + c &= a\left(s^2 + \frac{b}{a}s\right) + c \\ &= a\left[s^2 + \frac{b}{a}s + \left(\frac{b}{2a}\right)^2\right] + \left[c - \frac{b^2}{4a}\right] \\ &= a\left(s + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \\ &= a(s+k)^2 + h^2 \end{aligned}$$

where $k = b/2a$ and $h = \sqrt{c - (b^2/4a)}$. (See Problems 22.8 through 22.10.)

The method of *partial fractions* transforms a function of the form $a(s)/b(s)$, where both $a(s)$ and $b(s)$ are polynomials in s, into the sum of other fractions such that the denominator of each new fraction is either a first-degree or a quadratic polynomial raised to some power. The method requires only that (1) the degree of $a(s)$ be less than the degree of $b(s)$ (if this is not the case, first perform long division, and consider the remainder term) and (2) $b(s)$ be factored into the product of distinct linear and quadratic polynomials raised to various powers.

The method is carried out as follows. To each factor of $b(s)$ of the form $(s - a)^m$, assign a sum of m fractions, of the form

$$\frac{A_1}{s-a}+\frac{A_2}{(s-a)^2}+\cdots+\frac{A_m}{(s-a)^m}$$

To each factor of $b(s)$ of the form $(s^2 + bs + c)^p$, assign a sum of p fractions, of the form

$$\frac{B_1 s+C_1}{s^2+bs+c}+\frac{B_2 s+C_2}{(s^2+bs+c)^2}+\cdots+\frac{B_p s+C_p}{(s^2+bs+c)^p}$$

Here A_i, B_j, and C_k $(i = 1, 2,\dots, m; j, k = 1, 2,\dots, p)$ are constants which still must be determined.

Set the original fraction $a(s)/b(s)$ equal to the sum of the new fractions just constructed. Clear the resulting equation of fractions and then equate coefficients of like powers of s, thereby obtaining a set of simultaneous linear equations in the unknown constants A_i, B_j, and C_k. Finally, solve these equations for A_i, B_j, and C_k. (See Problems 22.11 through 22.14.)

MANIPULATING NUMERATORS

A factor $s - a$ in the numerators may be written in terms of the factor $s - b$, where both a and b are constants, through the identity $s - a = (s - b) + (b - a)$. The multiplicative constant a in the numerator may be written explicitly in terms of the multiplicative constant b through the identity

$$a=\frac{a}{b}(b)$$

Both identities generate recognizable inverse Laplace transforms when they are combined with:

Property 22.1. (Linearity). If the inverse Laplace transforms of two functions $F(s)$ and $G(s)$ exist, then for any constants c_1 and c_2,

$$\mathscr{L}^{-1}\{c_1F(s) + c_2G(s)\} = c_1\mathscr{L}^{-1}\{F(s)\} + c_2\mathscr{L}^{-1}\{G(s)\}$$

(See Problems 22.4 through 22.7.)

Solved Problems

22.1. Find $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$.

Here $F(s) = 1/s$. From either Problem 21.4 or entry 1 of Appendix A, we have $\mathscr{L}\{1\} = 1/s$. Therefore, $\mathscr{L}^{-1}\{1/s\} = 1$.

22.2. Find $\mathscr{L}^{-1}\left\{\frac{1}{s-8}\right\}$.

From either Problem 21.6 or entry 7 of Appendix A with $a = 8$, we have

$$\mathscr{L}\{e^{8x}\}=\frac{1}{s-8}$$

Therefore,

$$\mathscr{L}^{-1}\left\{\frac{1}{s-8}\right\}=e^{8x}$$

22.3. Find $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+6}\right\}$.

From entry 9 of Appendix A with $a=\sqrt{6}$, we have

$$\mathscr{L}\{\cos\sqrt{6}x\} = \frac{s}{s^2+(\sqrt{6})^2} = \frac{s}{s^2+6}$$

Therefore,

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+6}\right\} = \cos\sqrt{6}x$$

22.4. Find $\mathscr{L}^{-1}\left\{\dfrac{5s}{(s^2+1)^2}\right\}$.

The given function is similar in form to entry 12 of Appendix A. The denominators become identical if we take $a = 1$. Manipulating the numerator of the given function and using Property 22.1, we obtain

$$\mathscr{L}^{-1}\left\{\frac{5s}{(s^2+1)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{\frac{5}{2}(2s)}{(s^2+1)^2}\right\} = \frac{5}{2}\mathscr{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} = \frac{5}{2}x\sin x$$

22.5. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{\sqrt{s}}\right\}$.

The given function is similar in form to entry 5 of Appendix A. Their denominators are identical; manipulating the numerator of the given function and using Property 22.1, we obtain

$$\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi}}\mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{x}}$$

22.6. Find $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2-9}\right\}$.

The denominator of this function is identical to the denominator of entries 10 and 11 of Appendix A with $a = 3$. Using Property 22.1 followed by a simple algebraic manipulation, we obtain

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2-9}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2-9}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2-9}\right\} = \cosh 3x + \mathscr{L}^{-1}\left\{\frac{1}{3}\left(\frac{3}{s^2-(3)^2}\right)\right\}$$

$$= \cosh 3x + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2-(3)^2}\right\} = \cosh 3x + \frac{1}{3}\sinh 3x$$

22.7. Find $\mathscr{L}^{-1}\left\{\dfrac{s}{(s-2)^2+9}\right\}$.

The denominator of this function is identical to the denominators of entries 15 and 16 of Appendix A with $a = 3$ and $b = 2$. Both the given function and entry 16 have the *variable s* in their numerators, so they are the most closely matched. Manipulating the numerator of the given function and using Property 22.1, we obtain

$$\mathscr{L}^{-1}\left\{\frac{s}{(s-2)^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{(s-2)+2}{(s-2)^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s-2}{(s-2)^2+9}\right\} + \mathscr{L}^{-1}\left\{\frac{2}{(s-2)^2+9}\right\}$$

$$= e^{2x}\cos 3x + \mathscr{L}^{-1}\left\{\frac{2}{(s-2)^2+9}\right\} = e^{2x}\cos 3x + \mathscr{L}^{-1}\left\{\frac{2}{3}\left(\frac{3}{(s-2)^2+9}\right)\right\}$$

$$= e^{2x}\cos 3x + \frac{2}{3}\mathscr{L}^{-1}\left\{\frac{3}{(s-2)^2+9}\right\} = e^{2x}\cos 3x + \frac{2}{3}e^{2x}\sin 3x$$

22.8. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2-2s+9}\right\}$.

No function of this form appears in Appendix A. But, by completing the square, we obtain

$$s^2-2s+9=(s^2-2s+1)+(9-1)=(s-1)^2+(\sqrt{8})^2$$

Hence,

$$\frac{1}{s^2-2s+9}=\frac{1}{(s-1)^2+(\sqrt{8})^2}=\left(\frac{1}{\sqrt{8}}\right)\frac{\sqrt{8}}{(s-1)^2+(\sqrt{8})^2}$$

Then, using Property 22.1 and entry 15 of Appendix A with $a=\sqrt{8}$ and $b=1$, we find that

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2-2s+9}\right\}=\frac{1}{\sqrt{8}}\mathscr{L}^{-1}\left\{\frac{\sqrt{8}}{(s-1)^2+(\sqrt{8})^2}\right\}=\frac{1}{\sqrt{8}}e^x\sin\sqrt{8}x$$

22.9. Find $\mathscr{L}^{-1}\left\{\dfrac{s+4}{s^2+4s+8}\right\}$.

No function of this form appears in Appendix A. Completing the square in the denominator, we have

$$s^2+4s+8=(s^2+4s+4)+(8-4)=(s+2)^2+(2)^2$$

Hence,

$$\frac{s+4}{s^2+4s+8}=\frac{s+4}{(s+2)^2+(2)^2}$$

This expression also is not found in Appendix A. However, if we rewrite the numerator as $s+4=(s+2)+2$ and then decompose the fraction, we have

$$\frac{s+4}{s^2+4s+8}=\frac{s+2}{(s+2)^2+(2)^2}+\frac{2}{(s+2)^2+(2)^2}$$

Then, from entries 15 and 16 of Appendix A,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s+4}{s^2+4s+8}\right\}&=\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+(2)^2}\right\}+\mathscr{L}^{-1}\left\{\frac{2}{(s+2)^2+(2)^2}\right\}\\&=e^{-2x}\cos 2x+e^{-2x}\sin 2x\end{aligned}$$

22.10. Find $\mathscr{L}^{-1}\left\{\dfrac{s+2}{s^2-3s+4}\right\}$.

No function of this form appears in Appendix A. Completing the square in the denominator, we obtain

$$s^2-3s+4=\left(s^2-3s+\frac{9}{4}\right)+\left(4-\frac{9}{4}\right)=\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2$$

so that

$$\frac{s+2}{s^2-3s+4}=\frac{s+2}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}$$

We now rewrite the numerator as

$$s+2=s-\frac{3}{2}+\frac{7}{2}=\left(s-\frac{3}{2}\right)+\sqrt{7}\left(\frac{\sqrt{7}}{2}\right)$$

so that

$$\frac{s+2}{s^2-3s+4}=\frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}+\sqrt{7}\frac{\frac{\sqrt{7}}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}$$

Then,

$$\mathscr{L}^{-1}\left\{\frac{s+2}{s^2-3s+4}\right\}=\mathscr{L}^{-1}\left\{\frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\}+\sqrt{7}\mathscr{L}^{-1}\left\{\frac{\frac{\sqrt{7}}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\}$$

$$=e^{(3/2)x}\cos\frac{\sqrt{7}}{2}x+\sqrt{7}e^{(3/2)x}\sin\frac{\sqrt{7}}{2}x$$

22.11. Use partial function to decompose $\dfrac{1}{(s+1)(s^2+1)}$.

To the linear factor $s + 1$, we associate the fraction $A/(s + 1)$; whereas to the quadratic factor $s^2 + 1$, we associate the fraction $(Bs + C)/(s^2 + 1)$. We then set

$$\frac{1}{(s+1)(s^2+1)}\equiv\frac{A}{s+1}+\frac{Bs+C}{s^2+1} \tag{1}$$

Clearing fractions, we obtain

$$1 \equiv A(s^2+1)+(Bs+C)\,(s+1) \tag{2}$$

or

$$s^2(0)+s(0)+1\equiv s^2(A+B)+s(B+C)+(A+C)$$

Equating coefficients of like powers of s, we conclude that $A + B = 0$, $B + C = 0$, and $A + C = 1$. The solution of this set of equations is $A = \frac{1}{2}$, $B = -\frac{1}{2}$, and $C = \frac{1}{2}$. Substituting these values into (*1*), we obtain the partial-fractions decomposition

$$\frac{1}{(s+1)(s^2+1)}\equiv\frac{\frac{1}{2}}{s+1}+\frac{-\frac{1}{2}s+\frac{1}{2}}{s^2+1}$$

The following is an alternative procedure for finding the constants A, B, and C in (*1*). Since (*2*) must hold for all s, it must in particular hold $s = -1$. Substituting this value into (*2*), we immediately find $A = \frac{1}{2}$. Equation (*2*) must also hold for $s = 0$. Substituting this value along with $A = \frac{1}{2}$ into (*2*), we obtain $C = \frac{1}{2}$. Finally, substituting any other value of s into (*2*), we find that $B = -\frac{1}{2}$.

22.12. Use partial fractions to decompose $\dfrac{1}{(s^2+1)(s^2+4s+8)}$.

To the quadratic factors $s^2 + 1$ and $s^2 + 4s + 8$, we associate the fractions $(As + B)/(s^2 + 1)$ and $(Cs + D)/(s^2 + 4s + 8)$. We set

$$\frac{1}{(s^2+1)(s^2+4s+8)}\equiv\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4s+8} \tag{1}$$

and clear fractions to obtain

$$1\equiv(As+B)(s^2+4s+8)+(Cs+D)(s^2+1)$$

or

$$s^3(0)+s^2(0)+s(0)+1\equiv s^3(A+C)+s^2(4A+B+D)+s(8A+4B+C)+(8B+D)$$

Equating coefficients of like powers of s, we obtain $A + C = 0$, $4A + B + D = 0$, $8A + 4B + C = 0$, and $8B + D = 1$. The solution of this set of equation is

$$A=-\frac{4}{65}\quad B=\frac{7}{65}\quad C=\frac{4}{65}\quad D=\frac{9}{65}$$

Therefore, $$\frac{1}{(s^2+1)(s^2+4s+8)} \equiv \frac{-\frac{4}{65}s+\frac{7}{65}}{s^2+1}+\frac{\frac{4}{65}s+\frac{9}{65}}{s^2+4s+8}$$

22.13. Use partial fractions to decompose $\dfrac{s+3}{(s-2)(s+1)}$.

To the linear factors $s-2$ and $s+1$, we associate respectively the fractions $A/(s-2)$ and $B/(s+1)$. We set

$$\frac{s+3}{(s-2)(s+1)} \equiv \frac{A}{s-2}+\frac{B}{s+1}$$

and, upon clearing fractions, obtain

$$s+3 \equiv A(s+1)+B(s-2) \tag{1}$$

To find A and B, we use the alternative procedure suggested in Problem 22.11. Substituting $s=-1$ and then $s=2$ into (*1*), we immediately obtain $A=5/3$ and $B=-2/3$. Thus,

$$\frac{s+3}{(s-2)(s+1)} \equiv \frac{5/3}{s-2}-\frac{2/3}{s+1}$$

22.14. Use partial fractions to decompose $\dfrac{8}{s^3(s^2-s-2)}$.

Note that s^2-s-2 factors into $(s-2)(s+1)$. To the factor $s^3=(s-0)^3$, which is a linear polynomial raised to the third power, we associate the sum $A_1/s+A_2/s^2+A_3/s^3$. To the linear factors $(s-2)$ and $(s+1)$, we associate the fractions $B/(s-2)$ and $C/(s+1)$. Then

$$\frac{8}{s^3(s^2-s-2)} \equiv \frac{A_1}{s}+\frac{A_2}{s^2}+\frac{A_3}{s^3}+\frac{B}{s-2}+\frac{C}{s+1}$$

or, clearing fractions,

$$8 \equiv A_1s^2(s-2)(s+1)+A_2s(s-2)(s+1)+A_3(s-2)(s+1)+Bs^3(s+1)+Cs^3(s-2)$$

Letting $s=-1$, 2, and 0, consecutively, we obtain, respectively, $C=8/3$, $B=1/3$, and $A_3=-4$. Then choosing $s=1$ and $s=-2$, and simplifying, we obtain the equations $A_1+A_2=-1$ and $2A_1-A_2=-8$, which have the solutions $A_1=-3$ and $A_2=2$. Note that any other two values for s (not -1, 2, or 0) will also do; the resulting equations may be different, but the solution will be identical. Finally,

$$\frac{2}{s^3(s^2-s-2)} \equiv -\frac{3}{s}+\frac{2}{s^2}-\frac{4}{s^3}+\frac{1/3}{s-2}+\frac{8/3}{s+1}$$

22.15. Find $\mathscr{L}^{-1}\left\{\dfrac{s+3}{(s-2)(s+1)}\right\}$.

No function of this form appears in Appendix A. Using the results of Problem 22.13 and Property 22.1, we obtain

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s+3}{(s-2)(s+1)}\right\} &= \frac{5}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}-\frac{2}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}\\ &= \frac{5}{3}e^{2x}-\frac{2}{3}e^{-x}\end{aligned}$$

22.16. Find $\mathscr{L}^{-1}\left\{\dfrac{8}{s^3(s^2-s-2)}\right\}$.

No function of this form appears in Appendix A. Using the results of Problem 22.14 and Property 22.1, we obtain

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{8}{s^3(s^2-s-2)}\right\} &= -3\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}+2\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}\\ &\quad -2\mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\}+\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{8}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}\\ &= -3+2x-2x^2+\frac{1}{3}e^{2x}+\frac{8}{3}e^{-x}\end{aligned}$$

22.17. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}$.

Using the result of Problem 22.11, and noting that

$$\frac{-\frac{1}{2}s+\frac{1}{2}}{s^2+1}=-\frac{1}{2}\left(\frac{s}{s^2+1}\right)+\frac{1}{2}\left(\frac{1}{s^2+1}\right)$$

we find that

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} &= \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &= \frac{1}{2}e^{-x}-\frac{1}{2}\cos x+\frac{1}{2}\sin x\end{aligned}$$

22.18. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+1)(s^2+4s+8)}\right\}$.

From Problem 22.12, we have

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+8)}\right\}=\mathscr{L}^{-1}\left\{\frac{-\frac{4}{65}s+\frac{7}{65}}{s^2+1}\right\}+\mathscr{L}^{-1}\left\{\frac{\frac{4}{65}s+\frac{9}{65}}{s^2+4s+8}\right\}$$

The first term can be evaluated easily if we note that

$$\frac{-\frac{4}{65}s+\frac{7}{65}}{s^2+1}=\left(-\frac{4}{65}\right)\frac{s}{s^2+1}+\left(\frac{7}{65}\right)\frac{1}{s^2+1}$$

To evaluate the second inverse transforms, we must first complete the square in the denominator, $s^2+4s+8=(s+2)^2+(2)^2$, and then note that

$$\frac{\frac{4}{65}s+\frac{9}{65}}{s^2+4s+8}=\frac{4}{65}\left[\frac{s+2}{(s+2)^2+(2)^2}\right]+\frac{1}{130}\left[\frac{2}{(s+2)^2+(2)^2}\right]$$

Therefore,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+8)}\right\} &= -\frac{4}{65}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{7}{65}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &\quad +\frac{4}{65}\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+(2)^2}\right\}+\frac{1}{130}\mathscr{L}^{-1}\left\{\frac{2}{(s+2)^2+(2)^2}\right\}\\ &= -\frac{4}{65}\cos x+\frac{7}{65}\sin x+\frac{4}{65}e^{-2x}\cos 2x+\frac{1}{130}e^{-2x}\sin 2x\end{aligned}$$

22.19. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\}$.

By the method of partial fractions, we obtain

$$\frac{1}{s(s^2+4)} \equiv \frac{1/4}{s} + \frac{(-1/4)s}{s^2+4}$$

Thus,
$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \frac{1}{4} - \frac{1}{4}\cos 2x$$

Supplementary Problems

Find the inverse Laplace transforms, as a function of x, of the following functions:

22.20. $\dfrac{1}{s^2}$

22.21. $\dfrac{2}{s^2}$

22.22. $\dfrac{2}{s^3}$

22.23. $\dfrac{1}{s^3}$

22.24. $\dfrac{1}{s^4}$

22.25. $\dfrac{1}{s+2}$

22.26. $\dfrac{-2}{s-2}$

22.27. $\dfrac{12}{3s+9}$

22.28. $\dfrac{1}{2s-3}$

22.29. $\dfrac{1}{(s-2)^3}$

22.30. $\dfrac{12}{(s+5)^4}$

22.31. $\dfrac{3s^2}{(s^2+1)^2}$

22.32. $\dfrac{s^2}{(s^2+3)^2}$

22.33. $\dfrac{1}{s^2+4}$

22.34. $\dfrac{2}{(s-2)^2+9}$

22.35. $\dfrac{s}{(s+1)^2+5}$

22.36. $\dfrac{2s+1}{(s-1)^2+7}$

22.37. $\dfrac{1}{2s^2+1}$

22.38. $\dfrac{1}{s^2-2s+2}$

22.39. $\dfrac{s+3}{s^2+2s+5}$

22.40. $\dfrac{s}{s^2-s+17/4}$

22.41. $\dfrac{s+1}{s^2+3s+5}$

22.42. $\dfrac{2s^2}{(s-1)(s^2+1)}$

22.43. $\dfrac{1}{s^2-1}$

22.44. $\dfrac{2}{(s^2+1)(s-1)^2}$

22.45. $\dfrac{s+2}{s^3}$

22.46. $\dfrac{-s+6}{s^3}$

22.47. $\dfrac{s^3+3s}{s^6}$

22.48. $\dfrac{12+15\sqrt{s}}{s^4}$

22.49. $\dfrac{2s-13}{s(s^2-4s+13)}$

22.50. $\dfrac{2(s-1)}{s^2-s+1}$

22.51. $\dfrac{s}{(s^2+9)^2}$

22.52. $\dfrac{1}{2(s-1)(s^2-s-1)} = \dfrac{1/2}{(s-1)(s^2-s-1)}$

22.53. $\dfrac{s}{2s^2+4s+5/2} = \dfrac{(1/2)s}{s^2+2s+5/4}$

CHAPTER 23

Convolutions and the Unit Step Function

CONVOLUTIONS

The *convolution* of two functions $f(x)$ and $g(x)$ is

$$f(x) * g(x) = \int_0^x f(t)g(x-t)\,dt \tag{23.1}$$

Theorem 23.1. $f(x) * g(x) = g(x) * f(x)$.

Theorem 23.2. (Convolution theorem). If $\mathscr{L}\{f(x)\} = F(s)$ and $\mathscr{L}\{g(x)\} = G(s)$, then

$$\mathscr{L}\{f(x) * g(x)\} = \mathscr{L}\{f(x)\}\mathscr{L}\{g(x)\} = F(s)G(s)$$

It follows directly from these two theorems that

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f(x) * g(x) = g(x) * f(x) \tag{23.2}$$

If one of the two convolutions in Eq. (*23.2*) is simpler to calculate, then that convolution is chosen when determining the inverse Laplace transform of a product.

UNIT STEP FUNCTION

The *unit step function* $u(x)$ is defined as

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

As an immediate consequence of the definition, we have for any number c,

$$u(x-c) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$$

The graph of $u(x-c)$ is given if Fig. 23-1.

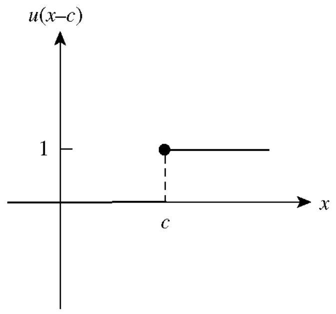

Fig. 23-1

Theorem 23.3. $\mathscr{L}\{u(x-c)\} = \dfrac{1}{s}e^{-cs}.$

TRANSLATIONS

Given a function $f(x)$ defined for $x \geq 0$, the function

$$u(x-c)f(x-c) = \begin{cases} 0 & x < c \\ f(x-c) & x \geq c \end{cases}$$

represents a shift, or translation, of the function $f(x)$ by c units in the positive x-direction. For example, if $f(x)$ is given graphically by Fig. 23-2, then $u(x-c)f(x-c)$ is given graphically by Fig. 23-3.

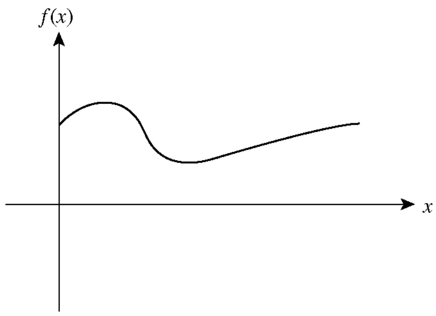

Fig. 23-2

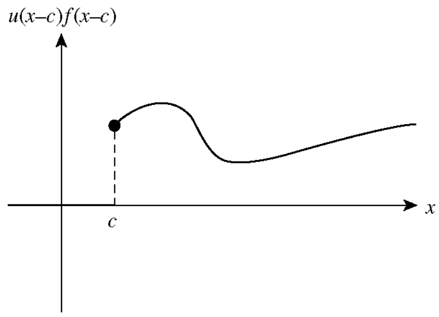

Fig. 23-3

Theorem 23.4. If $F(s) = \mathscr{L}\{f(x)\}$, then

$$\mathscr{L}\{u(x-c)f(x-c)\} = e^{-cs}F(s)$$

Conversely,

$$\mathscr{L}^{-1}\{e^{-cs}F(s)\} = u(x-c)f(x-c) = \begin{cases} 0 & x < c \\ f(x-c) & x \geq c \end{cases}$$

Solved problems

23.1. Find $f(x) * g(x)$ when $f(x) = e^{3x}$ and $g(x) = e^{2x}$.

Here $f(t) = e^{3t}$, $g(x - t) = e^{2(x-t)}$, and

$$f(x) * g(x) = \int_0^x e^{3t} e^{2(x-t)} dt = \int_0^x e^{3t} e^{2x} e^{-2t} dt$$
$$= e^{2x} \int_0^x e^t dt = e^{2x} [e^t]_{t=0}^{t=x} = e^{2x}(e^x - 1) = e^{3x} - e^{2x}$$

23.2. Find $g(x) * f(x)$ for the two functions in problem 23.1 and verify Theorem 23.1.

With $f(x - t) = e^{3(x-t)}$ and $g(t) = e^{2t}$,

$$g(x) * f(x) = \int_0^x g(t) f(x-t)\, dt = \int_0^x e^{2t} e^{3(x-t)} dt$$
$$= e^{3x} \int_0^x e^{-t} dt = e^{3x} [-e^{-t}]_{t=0}^{t=x}$$
$$= e^{3x}(-e^{-x} + 1) = e^{3x} - e^{2x}$$

which, from Problem 23.1 equals $f(x) * g(x)$.

23.3. Find $f(x) * g(x)$ when $f(x) = x$ and $g(x) = x^2$.

Here $f(t) = t$ and $g(x - t) = (x - t)^2 = x^2 - 2xt + t^2$. Thus,

$$f(x) * g(x) = \int_0^x t(x^2 - 2xt + t^2)\, dt$$
$$= x^2 \int_0^x t\, dt - 2x \int_0^x t^2 dt + \int_0^x t^3 dt$$
$$= x^2 \frac{x^2}{2} - 2x \frac{x^3}{3} + \frac{x^4}{4} = \frac{1}{12} x^4$$

23.4. Find $\mathscr{L}^{-1}\left\{\frac{1}{s^2 - 5s + 6}\right\}$ by convolutions.

Note that

$$\frac{1}{s^2 - 5s + 6} = \frac{1}{(s-3)(s-2)} = \frac{1}{s-3} \frac{1}{s-2}$$

Defining $F(s) = 1/(s - 3)$ and $G(s) = 1/(s - 2)$, we have from Appendix A that $f(x) = e^{3x}$ and $g(x) = e^{2x}$. It follows from Eq. (*23.2*) and the results of Problem 23.1 that

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2 - 5s + 6}\right\} = f(x) * g(x) = e^{3x} * e^{2x} = e^{3x} - e^{2x}$$

23.5. Find $\mathscr{L}^{-1}\left\{\frac{6}{s^2 - 1}\right\}$ by convolutions.

Note that

$$\mathscr{L}^{-1}\left\{\frac{6}{s^2 - 1}\right\} = \mathscr{L}^{-1}\left\{\frac{6}{(s-1)(s+1)}\right\} = 6\mathscr{L}^{-1}\left\{\frac{1}{(s-1)} \frac{1}{(s+1)}\right\}$$

Defining $F(s) = 1/(s-1)$ and $G(s) = 1/(s+1)$, we have from Appendix A that $f(x) = e^x$ and $g(x) = e^{-x}$. It follows from Eq. (*23.2*) that

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{6}{s^2-1}\right\} &= 6\mathscr{L}^{-1}\{F(s)G(s)\} = 6e^x * e^{-x}\\ &= 6\int_0^x e^t e^{-(x-t)}\,dt = 6e^{-x}\int_0^x e^{2t}dt\\ &= 6e^{-x}\left[\frac{e^{2x}-1}{2}\right] = 3e^x - 3e^{-x}\end{aligned}$$

23.6. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\}$ by convolutions.

Note that

$$\frac{1}{s(s^2+4)} = \frac{1}{s}\frac{1}{s^2+4}$$

Defining $F(s) = 1/s$ and $G(s) = 1/(s^2+4)$, we have from Appendix A that $f(x) = 1$ and $g(x) = \frac{1}{2}\sin 2x$. It now follows from Eq. (*23.2*) that

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} &= \mathscr{L}^{-1}\{F(s)G(s)\} = g(x) * f(x)\\ &= \int_0^x g(t)f(x-t)\,dt = \int_0^x\left(\frac{1}{2}\sin 2t\right)(1)\,dt\\ &= \frac{1}{4}(1-\cos 2x)\end{aligned}$$

See also Problem 22.19.

23.7. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-1)^2}\right\}$ by convolutions.

If we define $F(s) = G(s) = 1/(s-1)$, then $f(x) = g(x) = e^x$ and

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} &= \mathscr{L}^{-1}\{F(s)G(s)\} = f(x) * g(x)\\ &= \int_0^x f(t)g(x-t)\,dt = \int_0^x e^t e^{x-t}dt\\ &= e^x\int_0^x (1)\,dt = xe^x\end{aligned}$$

23.8. Use the definition of the Laplace transform to find $\mathscr{L}\{u(x-c)\}$ and thereby prove Theorem 23.3.

It follows directly from Eq. (*21.1*) that

$$\begin{aligned}\mathscr{L}\{u(x-c)\} &= \int_0^\infty e^{-sx}u(x-c)\,dx = \int_0^c e^{-sx}(0)\,dx + \int_c^\infty e^{-sx}(1)\,dx\\ &= \int_c^\infty e^{-sx}\,dx = \lim_{R\to\infty}\int_c^R e^{-sx}\,dx = \lim_{R\to\infty}\frac{e^{-sR}-e^{-sc}}{-s}\\ &= \frac{1}{s}e^{-sc} \quad (\text{if } s>0)\end{aligned}$$

23.9. Graph the function $f(x) = u(x - 2) - u(x - 3)$.

Note that

$$u(x-2) = \begin{cases} 0 & x < 2 \\ 1 & x \geq 2 \end{cases} \quad \text{and} \quad u(x-3) = \begin{cases} 0 & x < 3 \\ 1 & x \geq 3 \end{cases}$$

Thus,

$$f(x) = u(x-2) - u(x-3) = \begin{cases} 0 - 0 = 0 & x < 2 \\ 1 - 0 = 1 & 2 \leq x < 3 \\ 1 - 1 = 0 & x \geq 3 \end{cases}$$

the graph of which is given in Fig. 23-4.

23.10. Graph the function $f(x) = 5 - 5u(x - 8)$ for $x \geq 0$.

Note that

$$5u(x-8) = \begin{cases} 0 & x < 8 \\ 5 & x \geq 8 \end{cases}$$

Thus

$$f(x) = 5 - 5u(x-8) = \begin{cases} 5 & x < 8 \\ 0 & x \geq 8 \end{cases}$$

The graph of this function when $x \geq 0$ is given in Fig. 23-5.

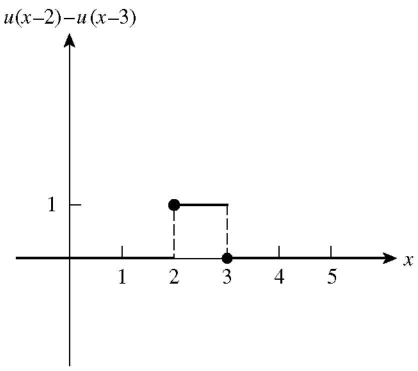

Fig. 23-4

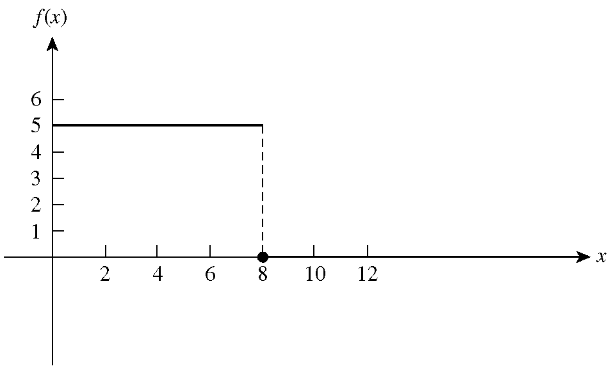

Fig. 23-5

23.11. Use the unit step function to give an analytic representation of the function $f(x)$ graphed in Fig. 23-6.

Note that $f(x)$ is the function $g(x) = x$, $x \geq 0$, translated four units in the positive x-direction. Thus, $f(x) = u(x - 4)g(x - 4) = (x - 4)u(x - 4)$.

23.12. Use the unit step function to give an analytic description of the function $g(x)$ graphed on the interval $(0, \infty)$ in Fig. 23-7. If on the subinterval $(0, a)$ the graph is identical to Fig. 23-2.

Let $f(x)$ represent the function graphed in Fig. 23-2. Then $g(x) = f(x)[1 - u(x - a)]$.

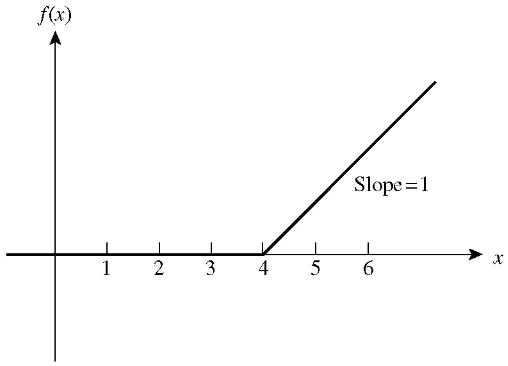

Fig. 23-6

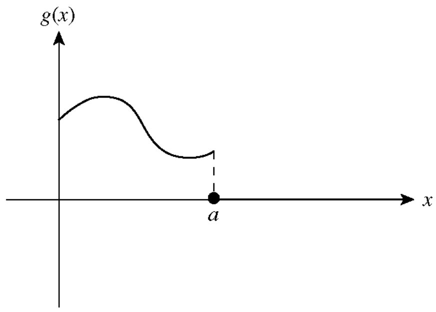

Fig. 23-7

23.13. Find $\mathcal{L}\{g(x)\}$ if $g(x) = \begin{cases} 0 & x < 4 \\ (x-4)^2 & x \geq 4 \end{cases}$.

If we define $f(x) = x^2$, then $g(x)$ can be given compactly as $g(x) = u(x-4)f(x-4) = u(x-4)(x-4)^2$. Then, noting that $\mathcal{L}\{f(x)\} = F(s) = 2/s^3$ and using Theorem 23.4, we conclude that

$$\mathcal{L}\{g(x)\} = \mathcal{L}\{u(x-4)(x-4)^2\} = e^{-4s}\frac{2}{s^3}$$

23.14. Find $\mathcal{L}\{g(x)\}$ if $g(x) = \begin{cases} 0 & x < 4 \\ x^2 & x \geq 4 \end{cases}$.

We first determine a function $f(x)$ such that $f(x-4) = x^2$. Once this has been done, $g(x)$ can be written as $g(x) = u(x-4)f(x-4)$ and Theorem 23.4 can be applied. Now, $f(x-4) = x^2$ only if

$$f(x) = f(x+4-4) = (x+4)^2 = x^2 + 8x + 16$$

Since
$$\mathcal{L}\{f(x)\} = \mathcal{L}\{x^2\} + 8\mathcal{L}\{x\} + 16\mathcal{L}\{1\} = \frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}$$

it follows that

$$\mathcal{L}\{g(x)\} = \mathcal{L}\{u(x-4)f(x-4)\} = e^{-4s}\left(\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right)$$

23.15. Prove Theorem 23.1.

Making the substitution $\tau = x - t$ in the right-hand side of Eq. (*23.1*), we have

$$\begin{aligned} f(x) * g(x) &= \int_0^x f(t)g(x-t)\,dt = \int_x^0 f(x-\tau)g(\tau)(-d\tau) \\ &= -\int_x^0 g(\tau)f(x-\tau)\,d\tau = \int_0^x g(\tau)f(x-\tau)\,d\tau \\ &= g(x) * f(x) \end{aligned}$$

23.16. Prove that $f(x) * [g(x) + h(x)] = f(x) * g(x) + f(x) * h(x)$.

$$\begin{aligned} f(x) * [g(x) + h(x)] &= \int_0^x f(t)[g(x-t) + h(x-t)]\,dt \\ &= \int_0^x [f(t)g(x-t) + f(t)h(x-t)]\,dt \\ &= \int_0^x f(t)g(x-t)\,dt + \int_0^x f(t)h(x-t)\,dt \\ &= f(x) * g(x) + f(x) * h(x) \end{aligned}$$

23.17. The following equation is called an *integral equation of convolution type*.

Assuming that the Laplace Transform for $y(x)$ exists, we solve this equation, and the next two examples, for $y(x)$.

$$y(x) = x + \int_0^x y(t)\sin(x-t)\,dt$$

We see that this integral equation can be written as $y(x) = x + y(x) * \sin x$. Taking the Laplace transform $\mathcal{L}$ of both sides and applying Theorem 23.2, we have

$$\mathcal{L}\{y\} = \mathcal{L}\{x\} + \mathcal{L}\{y\}\mathcal{L}\{\sin x\} = \frac{1}{s^2} + \mathcal{L}\{y\}\frac{1}{s^2+1}.$$

Solving for $\mathcal{L}\{y\}$ yields

$$\mathcal{L}\{y\} = \frac{s^2+1}{s^4}.$$

This implies that $y(x) = x + \frac{x^3}{6}$, which is indeed the solution, as can be verified by direct substitution as follows:

$$x + \int_0^x \left(t + \frac{t^3}{6}\right)\sin(x-t)\,dt = x + \frac{x^3}{6} = y(x)$$

23.18. Use Laplace Transforms to solve the integral equation of convolution type:

$$y(x) = 2 - \int_0^x y(t)e^{x-t}\,dt$$

Here we have $y(x) = 2 - y(x) * e^x$. Continuing as in Problem 23.17, we find that

$$\mathcal{L}\{y\} = \frac{2s-2}{s^2}$$

which gives $y(x) = 2 - 2x$ as the desired solution.

23.19. Use Laplace Transforms to solve the integral equation of convolution type:

$$y(x) = x^3 + \int_0^x 4y(t)\,dt$$

Noting that $y(x) = x^3 + 4 * y(x)$, we find that $\mathcal{L}\{y\} = \frac{6}{s^3(s-4)}$ which gives $y(x) = \frac{3}{32}(-1 + e^{4x} - 4x - 8x^2)$ as the solution.

Supplementary problems

23.20. Find $x * x$.

23.21. Find $2 * x$.

23.22. Find $4x * e^{2x}$.

23.23. Find $e^{4x} * e^{-2x}$.

23.24. Find $x * e^{x}$.

23.25. Find $x * xe^{-x}$.

23.26. Find $3 * \sin 2x$.

23.27. Find $x * \cos x$.

In Problems 23.28 through 23.35, use convolutions to find the inverse Laplace transforms of the given functions.

23.28. $\dfrac{1}{(s-1)(s-2)}$

23.29. $\dfrac{1}{(s)(s)}$

23.30. $\dfrac{2}{s(s+1)}$

23.31. $\dfrac{1}{s^2+3s-40}$

23.32. $\dfrac{3}{s^2(s^2+3)}$

23.33. $\dfrac{1}{s(s^2+4)}$ with $F(s) = 1/s^2$ and $G(s) = s/(s^2+4)$. Compare with Problem 23.6.

23.34. $\dfrac{9}{s(s^2+9)}$

23.35. $\dfrac{9}{s^2(s^2+9)}$

23.36. Graph $f(x) = 2u(x-2) - u(x-4)$.

23.37. Graph $f(x) = u(x-2) - 2u(x-3) + u(x-4)$.

23.38. Use the unit step function to give an analytic representation for the function graphed in Fig. 23-8.

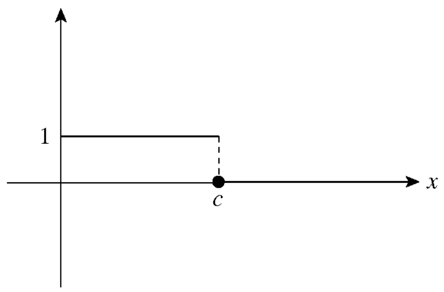

Fig. 23-8

23.39. Graph $f(x) = u(x-\pi)\cos 2(x-\pi)$.

23.40. Graph $f(x) = \dfrac{1}{2}(x-1)^2 u(x-1)$.

In Problems 23.41 through 23.48, find $\mathscr{L}\{g(x)\}$ for the given functions.

23.41. $g(x) = \begin{cases} 0 & x < 1 \\ \sin(x-1) & x \geq 1 \end{cases}$

23.42. $g(x) = \begin{cases} 0 & x < 3 \\ x-3 & x \geq 3 \end{cases}$

23.43. $g(x) = \begin{cases} 0 & x < 3 \\ x & x \geq 3 \end{cases}$

23.44. $g(x) = \begin{cases} 0 & x < 3 \\ x+1 & x \geq 3 \end{cases}$

23.45. $g(x) = \begin{cases} 0 & x < 5 \\ e^{x-5} & x \geq 5 \end{cases}$

23.46. $g(x) = \begin{cases} 0 & x < 5 \\ e^{x} & x \geq 5 \end{cases}$

23.47. $g(x) = \begin{cases} 0 & x < 2 \\ e^{x-5} & x \geq 2 \end{cases}$

23.48. $g(x) = \begin{cases} 0 & x < 2 \\ x^3 + 1 & x \geq 2 \end{cases}$

In Problems 23.49 through 23.55, determine the inverse Laplace transforms of the given functions.

23.49. $\dfrac{s}{s^2+4}e^{-3s}$

23.50. $\dfrac{1}{s^2+4}e^{-5s}$

23.51. $\dfrac{1}{s^2+4}e^{-\pi s}$

23.52. $\dfrac{2}{s-3}e^{-2s}$

23.53. $\dfrac{8}{s+3}e^{-s}$

23.54. $\dfrac{1}{s^3}e^{-2s}$

23.55. $\dfrac{1}{s^2}e^{-\pi s}$

23.56. Prove that for any constant k, $[kf(x)] * g(x) = k[f(x) * g(x)]$.

In Problems 23.57 through 23.60, assume that the Laplace Transform for $y(x)$ exists. Solve for $y(x)$.

23.57. $y(x) = x^3 + \int_0^x (x-t)\,y(t)\,dt$

23.58. $y(x) = e^x + \int_0^x y(t)\,dt$

23.59. $y(x) = 1 + \int_0^x (t-x)\,y(t)\,dt$

23.60. $y(x) = \int_0^x (t-x)\,y(t)\,dt$

CHAPTER 24

Solutions of Linear Differential Equations with Constant Coefficients by Laplace Transforms

LAPLACE TRANSFORMS OF DERIVATIVES

Denote $\mathscr{L}\{y(x)\}$ by $Y(s)$. Then under broad conditions, the Laplace transform of the nth-derivative ($n = 1, 2, 3, \dots$) of $y(x)$ is

$$\mathscr{L}\left\{\frac{d^n y}{dx^n}\right\} = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots - s y^{(n-2)}(0) - y^{(n-1)}(0) \tag{24.1}$$

If the initial conditions on $y(x)$ at $x = 0$ are given by

$$y(0) = c_0, \qquad y'(0) = c_1, \; \dots \;, y^{(n-1)}(0) = c_{n-1} \tag{24.2}$$

then (*24.1*) can be rewritten as

$$\mathscr{L}\left\{\frac{d^n y}{dx^n}\right\} = s^n Y(s) - c_0 s^{n-1} - c_1 s^{n-2} - \cdots - c_{n-2} s - c_{n-1} \tag{24.3}$$

For the special cases of $n = 1$ and $n = 2$, Eq. (*24.3*) simplifies to

$$\mathscr{L}\{y'(x)\} = sY(s) - c_0 \tag{24.4}$$

$$\mathscr{L}\{y''(x)\} = s^2 Y(s) - c_0 s - c_1 \tag{24.5}$$

SOLUTIONS OF DIFFERENTIAL EQUATIONS

Laplace transforms are used to solve initial-value problems given by the nth-order linear differential equation with constant coefficients

$$b_n \frac{d^n y}{dx^n} + b_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + b_1 \frac{dy}{dx} + b_0 y = g(x) \tag{24.6}$$

together with the initial conditions specified in Eq. (*24.2*). First, take the Laplace transform of both sides of Eq. (*24.6*), thereby obtaining an algebraic equation for $Y(s)$. Then solve for $Y(s)$ *algebraically*, and finally take inverse Laplace transforms to obtain $y(x) = \mathscr{L}^{-1}\{Y(s)\}$.

Unlike previous methods, where first the differential equation is solved and then the initial conditions are applied to evaluate the arbitrary constants, the Laplace transform method solves the entire initial-value problem in one step. There are two exceptions: when no initial conditions are specified and when the initial conditions are not at $x = 0$. In these situations, c_0 through c_n in Eqs. (*24.2*) and (*24.3*) remain arbitrary and the solution to differential Eq. (*24.6*) is found in terms of these constants. They are then evaluated separately when appropriate subsidiary conditions are provided. (See Problems 24.11 through 24.13.)

Solved problems

24.1. Solve $y' - 5y = 0$; $y(0) = 2$.

Taking the Laplace transform of both sides of this differential equation and using Property 24.4, we obtain $\mathscr{L}\{y'\} - 5\mathscr{L}\{y\} = \mathscr{L}\{0\}$. Then, using Eq. (*24.4*) with $c_0 = 2$, we find

$$[sY(s) - 2] - 5Y(s) = 0 \quad \text{from which} \quad Y(s) = \frac{2}{s-5}$$

Finally, taking the inverse Laplace transform of $Y(s)$, we obtain

$$y(x) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{2}{s-5}\right\} = 2\mathscr{L}^{-1}\left\{\frac{1}{s-5}\right\} = 2e^{5x}$$

24.2. Solve $y' - 5y = e^{5x}$; $y(0) = 0$.

Taking the Laplace transform of both sides of this differential equation and using Property 24.4, we find that $\mathscr{L}\{y'\} - 5\mathscr{L}\{y\} = \mathscr{L}\{e^{5x}\}$. Then, using Appendix A and Eq. (*24.4*) with $c_0 = 0$, we obtain

$$[sY(s) - 0] - 5Y(s) = \frac{1}{s-5} \quad \text{from which} \quad Y(s) = \frac{1}{(s-5)^2}$$

Finally, taking the inverse transform of $Y(s)$, we obtain

$$y(x) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{(s-5)^2}\right\} = xe^{5x}$$

(see Appendix A, entry 14).

24.3. Solve $y' + y = \sin x$; $y(0) = 1$.

Taking the Laplace transform of both sides of the differential equation, we obtain

$$\mathscr{L}\{y'\} + \mathscr{L}\{y\} = \mathscr{L}\{\sin x\} \quad \text{or} \quad [sY(s) - 1] + Y(s) = \frac{1}{s^2+1}$$

Solving for $Y(s)$, we find

$$Y(s) = \frac{1}{(s+1)(s^2+1)} + \frac{1}{s+1}$$

Taking the inverse Laplace transform, and using the result of Problem 22.17, we obtain

$$\begin{aligned} y(x) &= \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= \left(\frac{1}{2}e^{-x} - \frac{1}{2}\cos x + \frac{1}{2}\sin x\right) + e^{-x} = \frac{3}{2}e^{-x} - \frac{1}{2}\cos x + \frac{1}{2}\sin x \end{aligned}$$

24.4. Solve $y'' + 4y = 0$; $y(0) = 2$, $y'(0) = 2$.

Taking Laplace transforms, we have $\mathscr{L}\{y''\} + 4\mathscr{L}\{y\} = \mathscr{L}\{0\}$. Then, using Eq. (*24.5*) with $c_0 = 2$ and $c_1 = 2$, we obtain

$$[s^2Y(s) - 2s - 2] + 4Y(s) = 0$$

or

$$Y(s) = \frac{2s+2}{s^2+4} = \frac{2s}{s^2+4} + \frac{2}{s^2+4}$$

Finally, taking the inverse Laplace transform, we obtain

$$y(x) = \mathscr{L}^{-1}\{Y(s)\} = 2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = 2\cos 2x + \sin 2x$$

24.5. Solve $y'' - 3y' + 4y = 0$; $y(0) = 1$, $y'(0) = 5$.

Taking Laplace transforms, we obtain $\mathscr{L}\{y''\} - 3\mathscr{L}\{y'\} + 4\mathscr{L}\{y\} = \mathscr{L}\{0\}$. Then, using *both* Eqs. (*24.4*) *and* (*24.5*) with $c_0 = 1$ and $c_1 = 5$, we have

$$[s^2Y(s) - s - 5] - 3[sY(s) - 1] + 4Y(s) = 0$$

or

$$Y(s) = \frac{s+2}{s^2 - 3s + 4}$$

Finally, taking the inverse Laplace transform and using the result of Problem 22.10, we obtain

$$y(x) = e^{(3/2)x}\cos\frac{\sqrt{7}}{2}x + \sqrt{7}e^{(3/2)x}\sin\frac{\sqrt{7}}{2}x$$

24.6. Solve $y'' - y' - 2y = 4x^2$; $y(0) = 1$, $y'(0) = 4$.

Taking Laplace transforms, we have $\mathscr{L}\{y''\} - \mathscr{L}\{y'\} - 2\mathscr{L}\{y\} = 4\mathscr{L}\{x^2\}$. Then, using *both* Eqs. (*24.4*) *and* (*24.5*) with $c_0 = 1$ and $c_1 = 4$, we obtain

$$[s^2Y(s) - s - 4] - [sY(s) - 1] - 2Y(s) = \frac{8}{s^3}$$

or, upon solving for $Y(s)$,

$$Y(s) = \frac{s+3}{s^2 - s - 2} + \frac{8}{s^3(s^2 - s - 2)}$$

Finally, taking the inverse Laplace transform and using the results of Problems 22.15 and 22.16, we obtain

$$\begin{aligned} y(x) &= \left(\frac{5}{3}e^{2x} - \frac{2}{3}e^{-x}\right) + \left(-3 + 2x - 2x^2 + \frac{1}{3}e^{2x} + \frac{8}{3}e^{-x}\right) \\ &= 2e^{2x} + 2e^{-x} - 2x^2 + 2x - 3 \end{aligned}$$

(See Problem 13.1.)

24.7. Solve $y'' + 4y' + 8y = \sin x$; $y(0) = 1$, $y'(0) = 0$.

Taking Laplace transforms, we obtain $\mathscr{L}\{y''\} + 4\mathscr{L}\{y'\} + 8\mathscr{L}\{y\} = \mathscr{L}\{\sin x\}$. Since $c_0 = 1$ and $c_1 = 0$, this becomes

$$[s^2Y(s) - s - 0] + 4[sY(s) - 1] + 8Y(s) = \frac{1}{s^2+1}$$

Thus,
$$Y(s) = \frac{s+4}{s^2+4s+8} + \frac{1}{(s^2+1)(s^2+4s+8)}$$

Finally, taking the inverse Laplace transform and using the results of Problems 22.9 and 22.18, we obtain

$$\begin{aligned} y(x) &= (e^{-2x}\cos 2x + e^{-2x}\sin 2x) \\ &\quad + \left(-\frac{4}{65}\cos x + \frac{7}{65}\sin x + \frac{4}{65}e^{-2x}\cos 2x + \frac{1}{130}e^{-2x}\sin 2x\right) \\ &= e^{-2x}\left(\frac{69}{65}\cos 2x + \frac{131}{130}\sin 2x\right) + \frac{7}{65}\sin x - \frac{4}{65}\cos x \end{aligned}$$

(See Problem 13.3.)

24.8. Solve $y'' - 2y' + y = f(x)$; $y(0) = 0$, $y'(0) = 0$.

In this equation $f(x)$ is unspecified. Taking Laplace transforms and designating $\mathscr{L}\{f(x)\}$ by $F(s)$, we obtain

$$[s^2Y(s) - (0)s - 0] - 2[sY(s) - 0] + Y(s) = F(s) \quad \text{or} \quad Y(s) = \frac{F(s)}{(s-1)^2}$$

From Appendix A, entry 14, $\mathscr{L}^{-1}\{1/(s-1)^2\} = xe^x$. Thus, taking the inverse transform of $Y(s)$ and using convolutions, we conclude that

$$y(x) = xe^x * f(x) = \int_0^x te^t f(x-t)\,dt$$

24.9. Solve $y'' + y = f(x)$; $y(0) = 0$, $y'(0) = 0$ if $f(x) = \begin{cases} 0 & x < 1 \\ 2 & x \ge 1 \end{cases}$

Note that $f(x) = 2u(x-1)$. Taking Laplace transforms, we obtain

$$[s^2Y(s) - (0)s - 0] + Y(s) = \mathscr{L}\{f(x)\} = 2\mathscr{L}\{u(x-1)\} = 2e^{-s}/s$$

or
$$Y(s) = e^{-s}\frac{2}{s(s^2+1)}$$

Since
$$\mathscr{L}^{-1}\left\{\frac{2}{s(s^2+1)}\right\} = 2\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = 2 - 2\cos x$$

it follows from Theorem 23.4 that

$$y(x) = \mathscr{L}^{-1}\left\{e^{-s}\frac{2}{s(s^2+1)}\right\} = [2 - 2\cos(x-1)]u(x-1)$$

24.10. Solve $y''' + y' = e^x$; $y(0) = y'(0) = y''(0) = 0$.

Taking Laplace transforms, we obtain $\mathscr{L}\{y'''\} + \mathscr{L}\{y'\} = \mathscr{L}(e^x)$. Then, using Eq. (*24.3*) with $n = 3$ and Eq. (*24.4*), we have

$$[s^3Y(s) - (0)s^2 - (0)s - 0] + [sY(s) - 0] = \frac{1}{s-1} \quad \text{or} \quad Y(s) = \frac{1}{(s-1)(s^3+s)}$$

Finally, using the method of partial fractions and taking the inverse transform, we obtain

$$y(x) = \mathscr{L}^{-1}\left\{-\frac{1}{s} + \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}s - \frac{1}{2}}{s^2+1}\right\} = -1 + \frac{1}{2}e^x + \frac{1}{2}\cos x - \frac{1}{2}\sin x$$

24.11. Solve $y' - 5y = 0$.

No initial conditions are specified. Taking the Laplace transform of both sides of the differential equation, we obtain

$$\mathscr{L}\{y'\} - 5\mathscr{L}\{y\} = \mathscr{L}\{0\}$$

Then, using Eq. (*24.4*) with $c_0 = y(0)$ kept arbitrary, we have

$$[sY(s) - c_0] - 5Y(s) = 0 \quad \text{or} \quad Y(s) = \frac{c_0}{s-5}$$

Taking the inverse Laplace transform, we find that

$$y(x) = \mathscr{L}^{-1}\{Y(s)\} = c_0\mathscr{L}^{-1}\left\{\frac{1}{s-5}\right\} = c_0e^{5x}$$

24.12. Solve $y'' - 3y' + 2y = e^{-x}$.

No initial conditions are specified. Taking Laplace transforms, we have $\mathscr{L}\{y''\} - 3\mathscr{L}\{y'\} + 2\mathscr{L}\{y\} = \mathscr{L}(e^{-x})$, or

$$[s^2Y(s) - sc_0 - c_1] - 3[sY(s) - c_0] + 2[Y(s)] = 1/(s+1)$$

Here c_0 and c_1 must remain arbitrary, since they represent $y(0)$ and $y'(0)$, respectively, which are unknown. Thus,

$$Y(s) = c_0\frac{s-3}{s^2-3s+2} + c_1\frac{1}{s^2-3s+2} + \frac{1}{(s+1)(s^2-3s+2)}$$

Using the method of partial fractions and noting that $s^2 - 3s + 2 = (s-1)(s-2)$, we obtain

$$\begin{aligned}
y(x) &= c_0\mathscr{L}^{-1}\left\{\frac{2}{s-1} + \frac{-1}{s-2}\right\} + c_1\mathscr{L}^{-1}\left\{\frac{-1}{s-1} + \frac{1}{s-2}\right\} + \mathscr{L}^{-1}\left\{\frac{1/6}{s+1} + \frac{-1/2}{s-1} + \frac{1/3}{s-2}\right\} \\
&= c_0(2e^x - e^{2x}) + c_1(-e^x + e^{2x}) + \left(\frac{1}{6}e^{-x} - \frac{1}{2}e^x + \frac{1}{3}e^{2x}\right) \\
&= \left(2c_0 - c_1 - \frac{1}{2}\right)e^x + \left(-c_0 + c_1 + \frac{1}{3}\right)e^{2x} + \frac{1}{6}e^{-x} \\
&= d_0e^x + d_1e^{2x} + \frac{1}{6}e^{-x}
\end{aligned}$$

where $d_0 = 2c_0 - c_1 - \frac{1}{2}$ and $d_1 = -c_0 + c_1 + \frac{1}{3}$.

24.13. Solve $y'' - 3y' + 2y = e^{-x}$; $y(1) = 0$, $y'(1) = 0$.

The initial conditions are given at $x = 1$, not $x = 0$. Using the results of Problem 24.12, we have as the solution to just the differential equation

$$y = d_0e^x + d_1e^{2x} + \frac{1}{6}e^{-x}$$

Applying the initial conditions to this last equation, we find that $d_0 = -\frac{1}{2}e^{-2}$ and $d_1 = \frac{1}{3}e^{-3}$; hence,

$$y(x) = -\frac{1}{2}e^{x-2} + \frac{1}{3}e^{2x-3} + \frac{1}{6}e^{-x}$$

24.14. Solve $\dfrac{dN}{dt} = 0.05N;\ N(0) = 20{,}000.$

This is a differential equation for the unknown function $N(t)$ in the independent variable t. We set $N(s) = \mathscr{L}\{N(t)\}$. Taking Laplace transforms of the given differential equation and using (*24.4*) with N replacing y, we have

$$[sN(s) - N(0)] = 0.05N(s)$$

$$[sN(s) - 20{,}000] = 0.05N(s)$$

or, upon solving for $N(s)$,

$$N(s) = \frac{20{,}000}{s - 0.05}$$

Then from Appendix A, entry 7 with $a = 0.05$ and t replacing x, we obtain

$$N(t) = \mathscr{L}^{-1}\{N(s)\} = \mathscr{L}^{-1}\left\{\frac{20{,}000}{s - 0.05}\right\} = 20{,}000\mathscr{L}^{-1}\left\{\frac{1}{s - 0.05}\right\} = 20{,}000e^{0.05t}$$

Compare with (*2*) of Problem 7.1.

24.15. Solve $\dfrac{dI}{dt} + 50I = 5; I(0) = 0.$

This is a differential equation for the unknown function $I(t)$ in the independent variable t. We set $I(s) = \mathscr{L}\{I(t)\}$. Taking Laplace transforms of the given differential equation and using Eq. (*24.4*) with I replacing y, we have

$$[sI(s) - I(0)] + 50I(s) = 5\left(\frac{1}{s}\right)$$

$$[sI(s) - 0] + 50I(s) = 5\left(\frac{1}{s}\right)$$

or, upon solving for $I(s)$,

$$I(s) = \frac{5}{s(s + 50)}$$

Then using the method of partial fractions and Appendix A, with t replacing x, we obtain

$$I(t) = \mathscr{L}^{-1}\{I(s)\} = \mathscr{L}^{-1}\left\{\frac{5}{s(s + 50)}\right\} = \mathscr{L}^{-1}\left\{\frac{1/10}{s} - \frac{1/10}{s + 50}\right\}$$

$$= \frac{1}{10}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{10}\mathscr{L}^{-1}\left\{\frac{1}{s + 50}\right\} = \frac{1}{10} - \frac{1}{10}e^{-50t}$$

Compare with (*1*) of Problem 7.19.

24.16. Solve $\ddot{x} + 16x = 2\sin 4t; x(0) = -\tfrac{1}{2}, \dot{x}(0) = 0.$

This is a differential equation for the unknown function $x(t)$ in the independent variable t. We set $X(s) = \mathscr{L}\{x(t)\}$. Taking Laplace transforms of the given differential equation and using Eq. (*24.5*) with x replacing y, we have

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 16X(s) = 2\left(\frac{4}{s^2 + 16}\right)$$

$$\left[s^2X(s) - s\left(-\frac{1}{2}\right) - 0\right] + 16X(s) = \frac{8}{s^2 + 16}$$

$$(s^2 + 16)X(s) = \frac{8}{s^2 + 16} - \frac{s}{2}$$

or
$$X(s) = \frac{8}{(s^2+16)^2} - \frac{1}{2}\left(\frac{s}{s^2+16}\right)$$

Then using Appendix A, entries 17 and 9 with $a = 4$ and t replacing x, we obtain

$$\begin{aligned}x(t) = \mathscr{L}^{-1}\{X(s)\} &= \mathscr{L}^{-1}\left\{\frac{8}{(s^2+16)^2} - \frac{1}{2}\left(\frac{s}{s^2+16}\right)\right\} \\ &= \frac{1}{16}\mathscr{L}^{-1}\left\{\frac{128}{(s^2+16)^2}\right\} - \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s}{s^2+16}\right\} \\ &= \frac{1}{16}(\sin 4t - 4t\cos 4t) - \frac{1}{2}\cos 4t\end{aligned}$$

Compare with the results of Problem 14.10.

Supplementary Problems

Use Laplace transforms to solve the following problems.

24.17. $y' + 2y = 0;\ y(0) = 1$

24.18. $y' + 2y = 2;\ y(0) = 1$

24.19. $y' + 2y = e^x;\ y(0) = 1$

24.20. $y' + 2y = 0;\ y(1) = 1$

24.21. $y' + 5y = 0;\ y(1) = 0$

24.22. $y' - 5y = e^{5x};\ y(0) = 2$

24.23. $y' + y = xe^{-x};\ y(0) = -2$

24.24. $y' + y = \sin x$

24.25. $y' + 20y = 6\sin 2x;\ y(0) = 6$

24.26. $y'' - y = 0;\ y(0) = 1,\ y'(0) = 1$

24.27. $y'' - y = \sin x;\ y(0) = 0,\ y'(0) = 1$

24.28. $y'' - y = e^x;\ y(0) = 1,\ y'(0) = 0$

24.29. $y'' + 2y' - 3y = \sin 2x;\ y(0) = y'(0) = 0$

24.30. $y'' + y = \sin x;\ y(0) = 0,\ y'(0) = 2$

24.31. $y'' + y' + y = 0;\ y(0) = 4,\ y'(0) = -3$

24.32. $y'' + 2y' + 5y = 3e^{-2x};\ y(0) = 1,\ y'(0) = 1$

24.33. $y'' + 5y' - 3y = u(x-4);\ y(0) = 0,\ y'(0) = 0$

24.34. $y'' + y = 0;\ y(\pi) = 0,\ y'(\pi) = -1$

24.35. $y''' - y = 5;\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 0$

24.36. $y^{(4)} - y = 0;\ y(0) = 1,\ y'(0) = 0,\ y''(0) = 0,\ y'''(0) = 0$

24.37. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = x^2e^x;\ y(0) = 1, y'(0) = 2, y''(0) = 3$

24.38. $\dfrac{dN}{dt} - 0.085N = 0;\ N(0) = 5000$

24.39. $\dfrac{dT}{dt} = 3T;\ T(0) = 100$

24.40. $\dfrac{dT}{dt} + 3T = 90;\ T(0) = 100$

24.41. $\dfrac{dv}{dt} + 2v = 32$

24.42. $\dfrac{dq}{dt} + q = 4\cos 2t;\ q(0) = 0$

24.43. $\ddot{x} + 9\dot{x} + 14x = 0;\ x(0) = 0,\ \dot{x}(0) = -1$

24.44. $\ddot{x} + 4\dot{x} + 4x = 0;\ x(0) = 2,\ \dot{x}(0) = -2$

24.45. $\dfrac{d^2x}{dt^2} + 8\dfrac{dx}{dt} + 25x = 0;\ x(\pi) = 0,\ \dot{x}(\pi) = 6$

24.46. $\dfrac{d^2q}{dt^2} + 9\dfrac{dq}{dt} + 14q = \dfrac{1}{2}\sin t;\ q(0)=0,\ \dot{q}(0) = 1$