Derivatives.

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Definition (Notation of the derivative)

Let f be a function defined on an interval I. We say that f is differentiable at a point x_0 in I if the limit

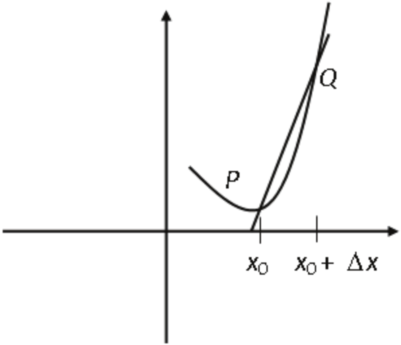
$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

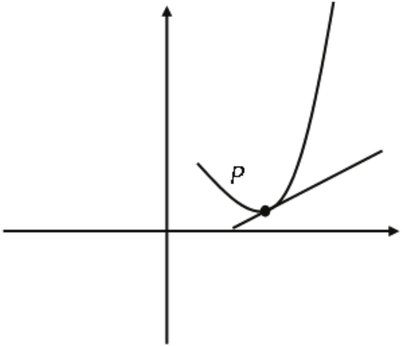
exists. The limit, when it exists, is called the derivative of f at x_0 , and is denoted by $f'(x_0)$.

The expression

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

where $\Delta x \neq 0$, is called a difference quotient of f at x_0.


Q
P
x_0
$x_0 + \Delta x$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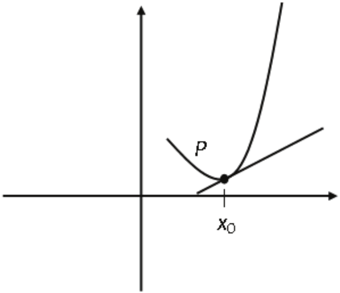
Geometrical interpretation: the derivative $f'(x_0)$, if exists, is the slope of the tangent line to the graph of the function $y = f(x)$ at the point $(x_0, f(x_0))$.

$$y = f(x_0) + f'(x_0)(x - x_0)$$



Physical interpretation: Suppose an object moves along a straight line and its distance from a fixed point on the line at time t is given by $y = f(t)$. Then the derivative $f'(t_0)$ is the instantaneous velocity of the object at the time t_0 .

Suppose $y = f(x)$. It is also common to use the notation

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{df}{dx}\right|_{x=x_0} = f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Definition (Derivative function)

Given a function f. Suppose D is the set of all points at which f is differentiable. The derivative function of f, f', with domain D is defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Theorem (Differentiability $\Rightarrow$ Continuity)

If f is differentiable at x_0, then f is continuous at x_0.

Theorem (Basic rules of differentiation)

Suppose f and g are differentiable at a point x and c is a constant. Then

$$(cf)'(x) = c \cdot f'(x);$$

$$(f+g)'(x) = f'(x) + g'(x);$$

$$(f-g)'(x) = f'(x) - g'(x);$$

$$(fg)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \text{ (Product Rule);}$$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} \text{ (Quotient Rule)}$$

Theorem (Derivatives of some common functions)

$\frac{d}{dx}(C) = 0$ *for any constant* C;

$\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$ *for any nonzero real number* α;

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\frac{1}{\sin^2 x}$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

Theorem (Derivatives of some common functions)

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Theorem (Chain rule)

Suppose h is differentiable at x_0 and g is differentiable at $h(x_0)$. Let $f = g \circ h$. Then f is differentiable at x_0 and

$$f'(x_0) = g'(h(x_0)) \cdot h'(x_0).$$

Theorem (Derivative of inverse function)

Suppose f is differentiable at x_0 with $f(x_0) = y_0$. Then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Theorem (Differentiation of parametric function)

Suppose

$$x = f(t) \text{ and } y = g(t)$$

are differentiable functions of t. *Then*

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

provided that $\frac{dx}{dt} \neq 0$.

Theorem (Differentiation of parametric function)

Suppose

$$x = f(t),\; y = g(t) \;\; and \;\; \frac{dy}{dx}$$

are differentiable functions of t. *Then*

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

provided that $\frac{dx}{dt} \neq 0$.

Definition (Higher derivatives)

Let f be a function.

The n-th derivative of f, if exists, is denoted by $f(n)$ and is defined inductively by

$$f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x),$$

with the convention that $f^{(0)}(x) = f(x)$.

We also use the symbol $\frac{d^n y}{dx^n}$ to denote the n-th derivative of $y = f(x)$.

In particular

$$f^{(1)}(x) = f'(x)$$

$$f^{(2)}(x) = (f')'(x) = f''(x)$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^3 y}{dx^3} = \frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right)$$

Theorem (Product rule)

If f and g are functions with derivatives up to order 2, then

$$(fg)' = fg' + fg'$$

$$(fg)'' = fg'' + 2f'g' + f''g$$

Theorem (Leibniz's formula)

If f and g are functions with derivatives up to order n, then

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} =$$

$$\binom{n}{0} f^{(0)} g^{(n)} + \binom{n}{1} f^{(1)} g^{(n-1)} + \binom{n}{2} f^{(2)} g^{(n-2)} + \ldots + \binom{n}{n} f^{(n)} g^{(0)}$$

Theorem (L'Hôpital's rule)

Suppose

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0, \text{ or}$$

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty.$$

If f and g are differentiable, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that the limit on right hand side exists. Similar result also holds for one-sided limits.

Examples

Example

- $\lim\limits_{x\to 0} \frac{e^x - e^{-x} + 3x}{x}$
- $\lim\limits_{x\to 0^+} \frac{\ln \sin x}{\ln \sin 2x}$
- $\lim\limits_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$
- $\lim\limits_{x\to 1^+} (x-1)\ln(x-1)$
- $\lim\limits_{x\to 1^+} (x-1)^{(x-1)}$
- $\lim\limits_{x\to 2^+} \left(\ln \frac{1}{x-2}\right)^{x-2}$
- $\lim\limits_{x\to \frac{\pi}{4}} (\tan x)^{\tan 2x}$

Definition (Increasing / decreasing functions)

A function f is increasing on an interval I if for any $x, y \in I$,

$$x \leq y \Rightarrow f(x) \leq f(y),$$

and f is decreasing on I if for any $x, y \in I$,

$$x \leq y \Rightarrow f(x) \geq f(y).$$

Theorem (Condition for monotonic functions)

Suppose a function f is differentiable on an interval I.

- *If $f'(x) \geq 0$ for all $x \in I$, then f is increasing on I;*
- *If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I;*
- *If $f'(x) \leq 0$ for all $x \in I$, then f is decreasing on I.*
- *If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.*
- *If $f'(x) = 0$, then f is constant on I.*

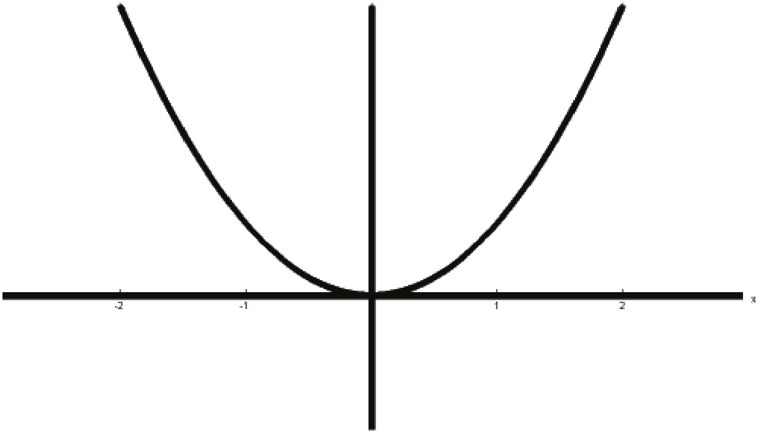
Definition (Absolute Maximum and Minimum)

Suppose f is a function with domain D. f has an absolute maximum on D at a point x_0 if

$$f(x_0) \geq f(x) \text{ for all } x \in D$$

f has an absolute minimum on D at x_0 if

$$f(x_0) \leq f(x) \text{ for all } x \in D$$


-2
-1
1
2
x

Definition (Local Maximum and Minimum)

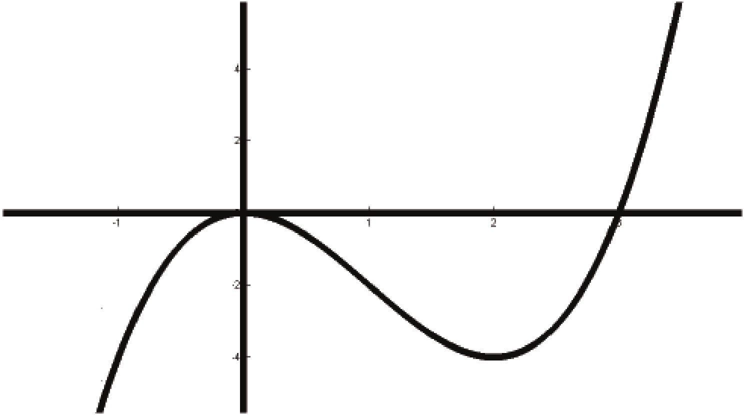
Suppose f is a function with domain D.

f has an local (relative) maximum at x_0 if there is a number $\delta > 0$ such

$$f(x_0) \geq f(x) \text{ for all } x \in (x_0 - \delta, x_0 + \delta) \cap D$$

f has an local (relative) minimum at x_0 if there is a number $\delta > 0$ such

$$f(x_0) \leq f(x) \text{ for all } x \in (x_0 - \delta, x_0 + \delta) \cap D.$$



4
2
-1
1
2
3
-2
-4

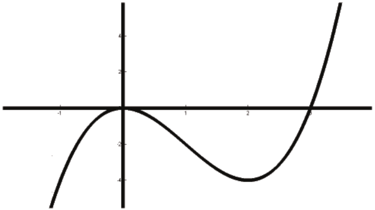
Theorem (Derivative at local minimum / maximum)

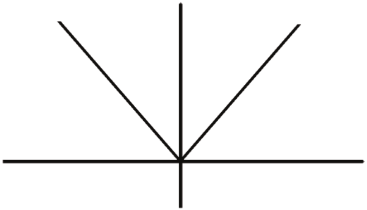
Given a function f with domain D. Suppose

- *f is differentiable at x_0, and*
- *f has a local minimum / maximum at x_0 ,*

then

$$f'(x_0) = 0$$





Example

Find local extrema of the function

$$f(x) = |1 - x^2|.$$

Definition (Critical point)

Given a function f with domain D. An interior point $x_0 \in D$ is a critical point if either

- $f'(x_0) = 0$, or
- f is NOT differentiable at $x = x_0$.

Theorem (First Derivative Test)

Suppose x_0 *is a critical point of a continuous function f If there is* $\delta > 0$ *such that*

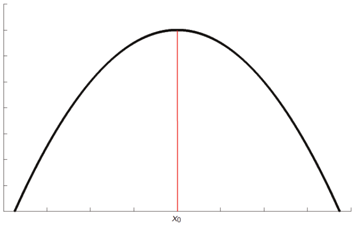
$x \in$	$(x_0 - \delta, x_0)$	$(x_0, x_0 + \delta)$
$f'(x)$	≥ 0	≤ 0

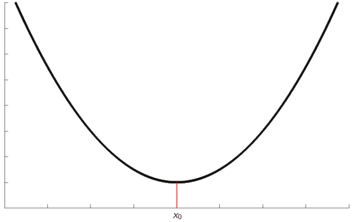
then x_0 *is a local maximum point of* f*;*

If there is $\delta > 0$ *such that*

$x \in$	$(x_0 - \delta, x_0)$	$(x_0, x_0 + \delta)$
$f'(x)$	≤ 0	≥ 0

then x_0 *is a local minimum point of* f*.*





Theorem (Second Derivative Test)

Let f be a differentiable function in some open interval with $f'(x_0) = 0$. Suppose $f''(x_0)$ exists.

- *If $f''(x_0) > 0$, then x_0 is a local minimum of f;*
- *If $f''(x_0) < 0$, then x_0 is a local maximum of f.*

Definition (Concavity)

Let f be a function on an interval I.

The curve $y = f(x)$ is convex on the interval I if for any $a, b \in I$, the line segment joining two points $(a, f(a))$ and $(b, f(b))$ always lies above the curve between that two points;

The curve $y = f(x)$ is concave on the interval I if for any $a, b \in I$, the line segment joining two points $(a, f(a))$ and $(b, f(b))$ always lies below the curve between that two points.

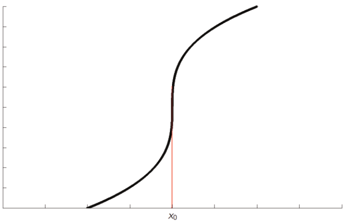
Theorem (Condition for Concavity)

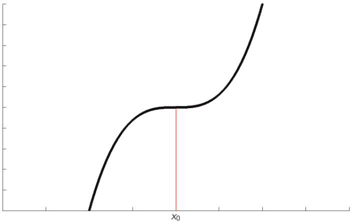
Let f be a function defined on an open interval I and $f''(x)$ exists for all $x \in I$.

- *If $f''(x) \geq 0$ for all $x \in I$, then the curve $y = f(x)$ is convex;*
- *If $f''(x) \leq 0$ for all $x \in I$, then the curve $y = f(x)$ is concave.*

Definition (Point of inflection)

A point x_0 where the graph $y = f(x)$ changes its behaviour from convex to concave or concave to convex is called a point of inflection of the curve.





Theorem

Let $x_0 \in \mathbb{R}$ *and let* f *be defined at least in the neighbourhood* $U(x_0)$ *of* x_0*. If there exists positive number* δ*, such that*

$x \in$	$(x_0 - \delta, x_0)$	$(x_0, x_0 + \delta)$
$f''(x)$	≤ 0	≥ 0

or

$x \in$	$(x_0 - \delta, x_0)$	$(x_0, x_0 + \delta)$
$f''(x)$	≥ 0	≤ 0

then $(x_0, f(x_0))$ *is a point of inflection on the graph of function* f*.*

Theorem

Let $x_0 \in \mathbb{R}$ *and let* f *be defined at lest in the neighbourhood* $U(x_0)$ *of* x_0. *Then, if*

1. $f''(x_0) = f'''(x_0) = \ldots = f^{(n-1)}(x_0) = 0$,
2. $f^{(n)}(x_0) \neq 0$,
3. n *is an odd number, such that* $n \geq 3$,

then $(x_0, f(x_0))$ *is a point of inflection on the graph of function* f.

Examples

Example

Find points of inflection

1. $f(x) = x^4 - 24x^2 + 6x + 5$
2. $f(x) = \ln(x) + 3x - 1$
3. $f(x) = xe^x$

Definition (Asymptote)

- A straight line $x = a$ is called a vertical asymptote to the curve $y = f(x)$ if

$$\lim_{x \to a^-} f(x) \to \pm\infty \text{ and } /\text{or}$$

$$\lim_{x \to a^+} f(x) \to \pm\infty.$$

- A straight line $y = mx + b$ is called an inclined asymptote to the curve $y = f(x)$ if

$$\lim_{x \to \infty} (f(x) - (mx + b)) = 0 \text{ and } /\text{or}$$

$$\lim_{x \to -\infty} (f(x) - (mx + b)) = 0$$

- If $m = 0$, then $y = b$ is called a horizontal asymptote.

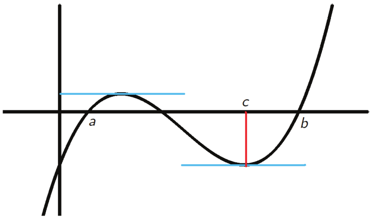
Theorem (Rolle's Theorem)

Let f be a function on $[a, b]$. Suppose

- f *is continuous on* $[a, b]$*, and*
- f *is differentiable in* (a, b)

If $f(a) = f(b) = 0$*, then there is a point* $c \in (a, b)$ *such that*

$$f'(c) = 0.$$



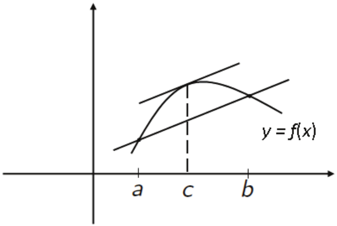
Theorem (Mean Value Theorem)

Let f be a function on $[a, b]$. Suppose

- *f is continuous on $[a, b]$, and*
- *f is differentiable in (a, b).*

Then there is a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Examples

Example

Investigate functions

1. $f(x) = (x-1)^2(x+1)$
2. $f(x) = \frac{x^3}{2(4-x^2)}$
3. $f(x) = x^2 e^{\frac{1}{x}}$
4. $f(x) = (x-1)\ln x$