Definite integral

M.W.

Mathematics Teaching and Distance Learning Centre
Gdańsk University of Technology

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Definition (Riemann sum)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. For any positive integer n, take $n + 1$ points in $[a, b]$ so that

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

Let

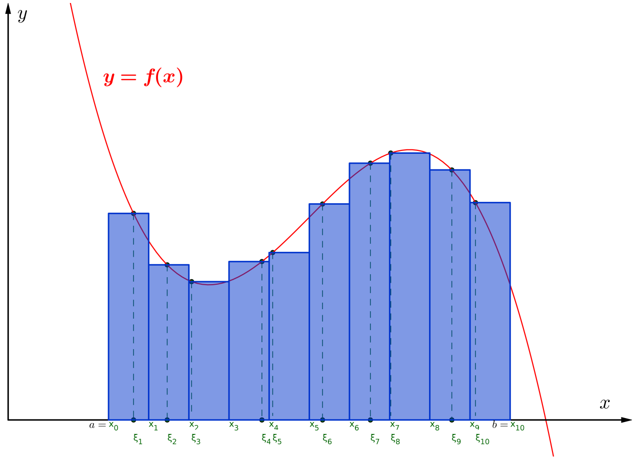
$$\Delta x_j = x_j - x_{j-1}$$

and take values

$$\xi_j \in [x_{j-1}, x_j] \text{ for } j = 1, \ldots, n$$

Then a Riemann sum of f is defined as

$$R_n(f) = \sum_{j=1}^{n} f(\xi_j)\Delta x_j$$



y
y = f(x)
x
a = x0
x1
x2
x3
x4
x5
x6
x7
x8
x9
b = x10
ξ1
ξ2
ξ3
ξ4
ξ5
ξ6
ξ7
ξ8
ξ9
ξ10

Definition (Definite integral)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. The definite integral of f from a to b, written $\int\limits_a^b f(x)dx$, is the limit of the Riemann sum $R_n(f)$ as n tends to infinity, if the limit exists. In the other words,

$$\int\limits_a^b f(x)dx = \lim_{n\to\infty} R_n(f) = \lim_{n\to\infty} \sum_{j=1}^{n} f(\xi_j)\Delta x_j$$

The function f is called the integrand, and a and b are called the limits of integration. In this case, we also say that $f(x)$ is integrable.

Theorem (Continuity $\Rightarrow$ Integrability)

If f is a continuous function on $[a, b]$, then f is integrable.

Example

Proceeding from the definition, compute the integral

$$\int_0^1 x dx$$

Example

Evaluate the integral

$$I = \int_0^5 \sqrt{25 - x^2} dx$$

proceeding from its geometric meaning.

Example

Proceeding from the geometric meaning of the definite integral, show that

- $$\int_{-\pi}^{\pi} \sin x dx = 0$$
- $$\int_{-1}^{1} |x| dx = 1$$

Properties of definite integrals

Theorem (Properties of definite integrals)

Suppose f and g are integrable functions and a, b, and c are real numbers. Then

- $\int\limits_a^b f(x)dx = -\int\limits_b^a f(x)dx;$
- $\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx;$
- $\int\limits_a^b cf(x)dx = c\int\limits_a^b f(x)dx;$
- $\int\limits_a^b (f(x) \pm g(x))dx = \int\limits_a^b f(x)dx \pm \int\limits_a^b g(x)dx.$

The fundamental theorem of calculus - I

Theorem (The fundamental theorem of calculus - I)

If f is continuous on the interval $[a, b]$ and

$$f(x) = F'(x),$$

then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example

Evaluate the integral

$$I = \int_{1}^{\sqrt{3}} \frac{1}{1+x^2} dx$$

Example

Compute the integrals

1. $$\int_0^{\pi} \sin 2x \, dx$$

2. $$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^3 x} \, dx$$

3. $$\int_0^{2} \frac{1}{\sqrt{16 - x^2}} \, dx$$

Example

Compute the integrals

1. $$\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2}\, dx$$

2. $$\int_{0}^{1} \frac{1}{\sqrt{x}+1}\, dx$$

Example

Compute the integral

$$\int_0^1 xe^x \, dx$$

Average of integral

Theorem (Average of integral)

Suppose f is a continuous function on $[a, b]$, then there is a number $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Theorem (The fundamental theorem of calculus - II)

Given a continuous function f on $[a, b]$, we define a function G by

$$G(x) = \int_a^x f(t)dt \text{ for all } x \in [a, b].$$

Then G is differentiable and $G'(x) = f(x)$ for all $x \in (a, b)$.