Improper integrals. Definite integral - applications

M.W.

Mathematics Teaching and Distance Learning Centre
Gdańsk University of Technology

2012-2017

Improper integral

Definition (Improper integrals - first kind)

Suppose a $\int_a^t f(x)dx$ exists for every $t \in [a, \infty)$. The improper integral is defined by

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

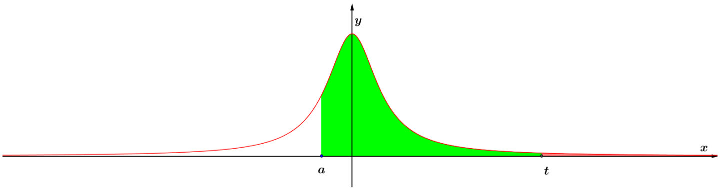
- If the limit exists, we say that $\int_a^{\infty} f(x)dx$ is convergent.
- If the limit does not exist, we say that $\int_a^{\infty} f(x)dx$ is divergent.

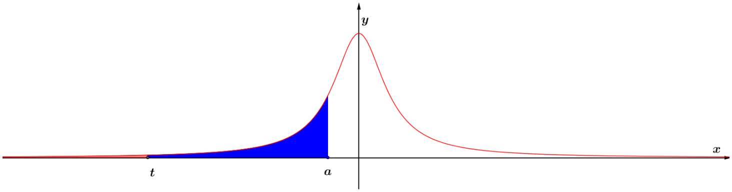
Definition

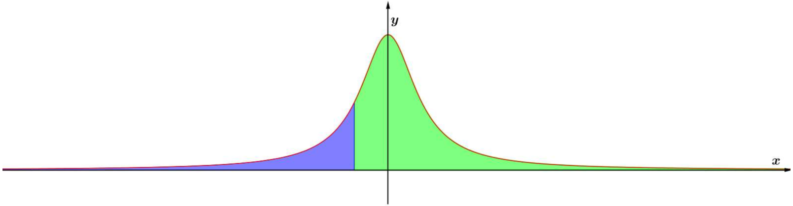
The improper integral $\int\limits_{-\infty}^{b} f(x)dx$ can be defined similarly.

If both $\int\limits_{a}^{\infty} f(x)dx$ and $\int\limits_{-\infty}^{a} f(x)dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$$


y
x
a
t


y
x
t
a


y
x

Improper integral

Example

1. $\int\limits_{1}^{\infty} \frac{dx}{x^4}$
2. $\int\limits_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$
3. $\int\limits_{1}^{\infty} x \sin x dx$

Exercise

1. $$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$$

2. $$\int_{e^2}^{\infty} \frac{1}{x\ln x}\,dx$$

3. $$\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$$

Definition (Improper integrals - second kind)

If f is continuous on $[a, b)$ and is discontinuous at b, then

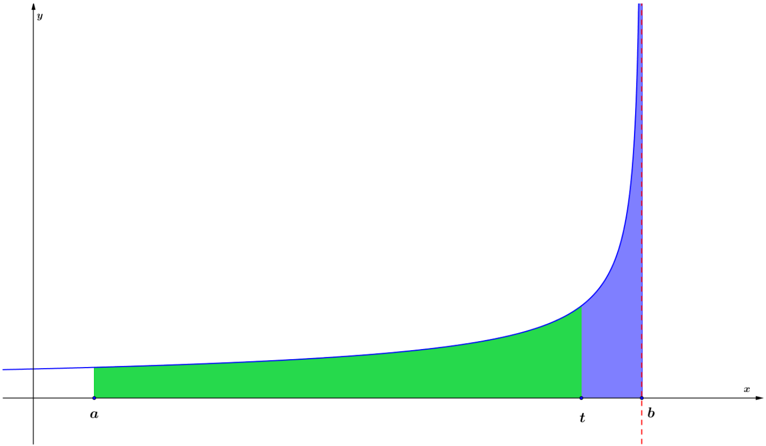
$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

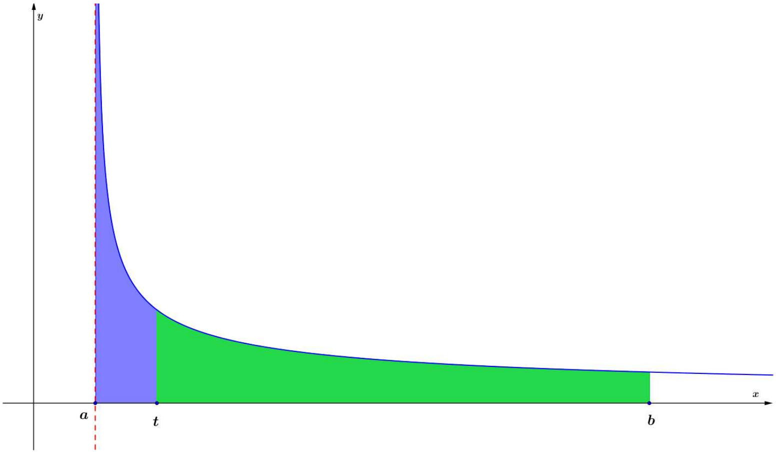
If f is continuous on $(a, b]$ and is discontinuous at a, then

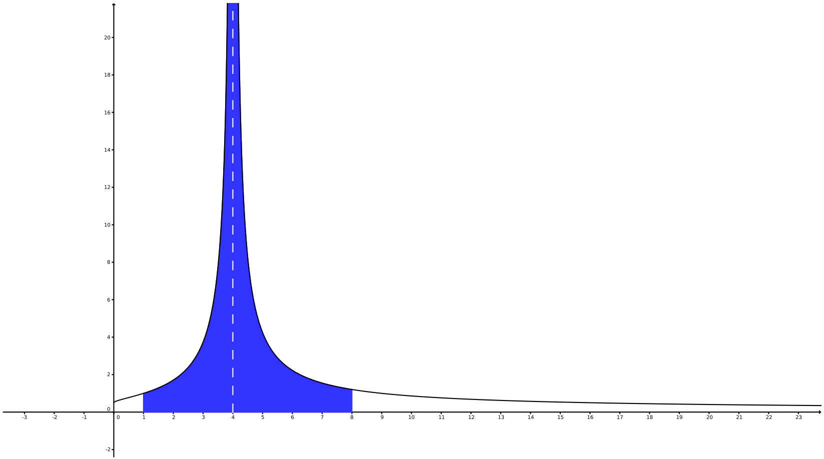
$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

If the limit exists, we say that $\int_a^b f(x)dx$ is convergent.

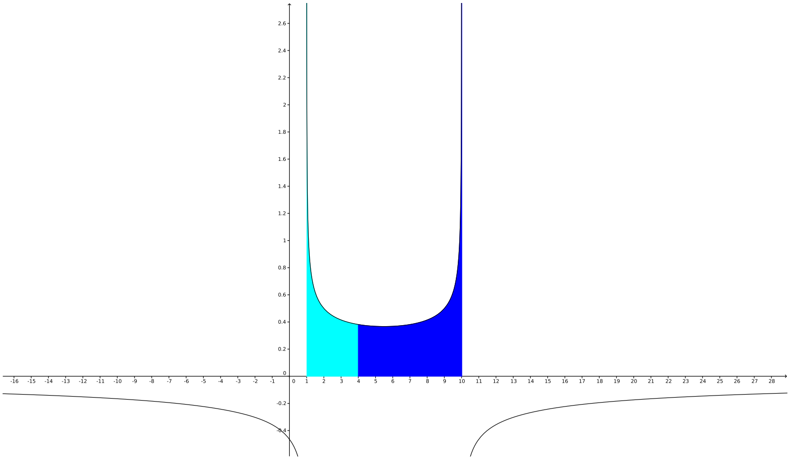
b


y
x
a
t
b


y
x
a
t
b



20
18
16
14
12
10
8
6
4
2
0
-2
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23



2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-16 -15 -14 -13 -12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28

Improper integral

Example

1. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$
2. $\displaystyle\int_1^2 \frac{xdx}{\sqrt{x-1}}$
3. $\displaystyle\int_0^1 \frac{dx}{x \ln x}$
4. $\displaystyle\int_0^2 \frac{dx}{x \ln^2 x}$

Exercise

1. $$\int_0^1 \frac{1}{\sqrt[5]{x^3}}dx$$

2. $$\int_0^{\frac{\pi}{2}} \tan x dx$$

3. $$\int_1^2 \frac{1}{\sqrt{4x - x^2 - 3}}dx$$

4. $$\int_{-1}^1 \frac{1}{x\sqrt[3]{x}}dx$$

Comparison test

Theorem

Let f and g be defined for all $x \geq a$ and integrable on each interval $[a, A]$, $A \geq a$.

- *If $0 \leq f(x) \leq g(x)$ for all $x \geq x$, then from convergence of the integral $\int\limits_a^{\infty} g(x)dx$ it follows that the integral $\int\limits_a^{\infty} f(x)dx$ is also convergent, and $\int\limits_a^{\infty} f(x)dx \leq \int\limits_a^{\infty} g(x)dx$;*
- *from divergence of the integral $\int\limits_a^{\infty} f(x)dx$ it follows that the integral $\int\limits_a^{\infty} g(x)dx$ is also divergent.*

Comparison test

Theorem

$$\int_1^\infty \frac{1}{x^\alpha}dx = \begin{cases} \text{divergant} & \text{for} \quad \alpha \in (0,1] \\ \text{convergant} & \text{for} \quad \alpha > 1 \end{cases}$$

Theorem

$$\int_0^1 \frac{1}{x^\alpha}dx = \begin{cases} \text{convergant} & \text{for} \quad \alpha \in (0,1) \\ \text{divergant} & \text{for} \quad \alpha \geq 1 \end{cases}$$

Area between curves

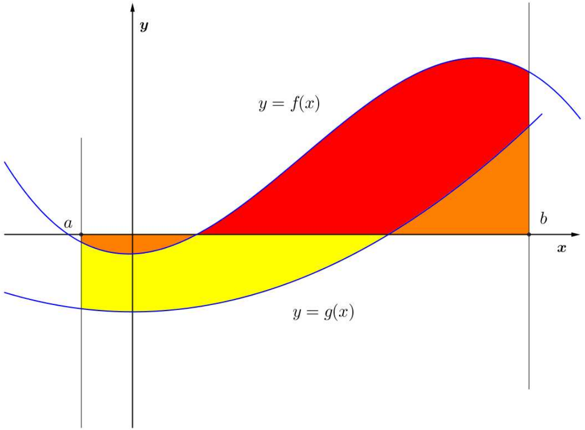
If f and g are integrable on $[a, b]$ and

$$f(x) \geq g(x) \text{ for } x \in [a, b],$$

then the area of the region R between the curves $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ is given as

$$\text{Area of } R = \int_a^b (f(x) - g(x))dx$$

Area between curves



Area between curves

Example

Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curve $y = 2^x$, $y = 2x - x^2$.

Example

Compute the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$

Example

Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curve $y = 2^x$, $y = 2x - x^2$.

Answer: $\frac{3}{\ln 2} - \frac{4}{3}$

Example

Compute the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$

Example

Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curve $y = 2^x$, $y = 2x - x^2$.

Answer: $\frac{3}{\ln 2} - \frac{4}{3}$

Example

Compute the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$

Example

Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curve $y = 2^x$, $y = 2x - x^2$.

Answer: $\frac{3}{\ln 2} - \frac{4}{3}$

Example

Compute the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$

Answer: $\frac{4}{3}$

Area between curves

Example

Find the area of the figure contained between the parabola $x^2 = 4y$ and the witch of Agnesi $y = \frac{8}{x^2+4}$.

Example

Compute the area of the figure which lies in the first quadrant inside the circle $x^2 + y^2 = 3$ and is bounded by the parabolas $x^2 = 2y$ and $y^2 = 2x$

Area between curves

Example

Find the area of the figure contained between the parabola $x^2 = 4y$ and the witch of Agnesi $y = \frac{8}{x^2+4}$.

Answer: $2\pi - \frac{4}{3}$

Example

Compute the area of the figure which lies in the first quadrant inside the circle $x^2 + y^2 = 3$ and is bounded by the parabolas $x^2 = 2y$ and $y^2 = 2x$

Example

Find the area of the figure contained between the parabola $x^2 = 4y$ and the witch of Agnesi $y = \frac{8}{x^2+4}$.

Answer: $2\pi - \frac{4}{3}$

Example

Compute the area of the figure which lies in the first quadrant inside the circle $x^2 + y^2 = 3$ and is bounded by the parabolas $x^2 = 2y$ and $y^2 = 2x$

Area between curves

Example

Find the area of the figure contained between the parabola $x^2 = 4y$ and the witch of Agnesi $y = \frac{8}{x^2+4}$.

Answer: $2\pi - \frac{4}{3}$

Example

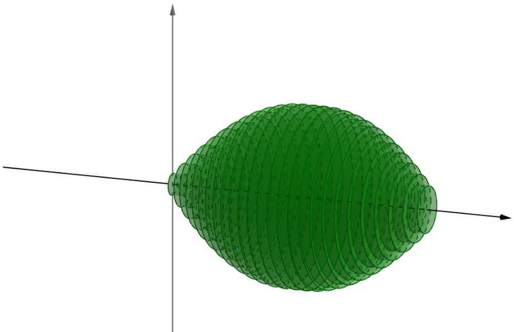
Compute the area of the figure which lies in the first quadrant inside the circle $x^2 + y^2 = 3$ and is bounded by the parabolas $x^2 = 2y$ and $y^2 = 2x$

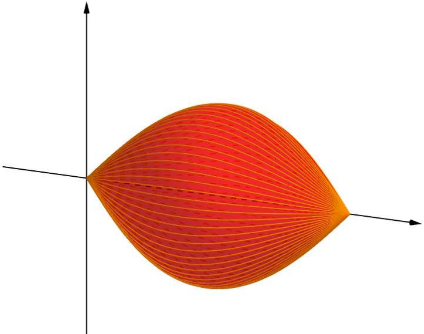
Answer: $\frac{\sqrt{2}}{3} + \frac{3}{2} \arcsin \frac{1}{3}$

Solid of Revolution

Suppose $f(x)$ is a continuous non-negative function for $x \in [a, b]$. If D is the solid obtained by rotating the graph of $y = f(x)$ about the x–axis, then the volume of D is given by

$$\text{Volume of } D = \pi \int_a^b (f(x))^2 \, dx$$





Volume of Solid of Revolution

Example

Compute the volume of the solid generated by revolting about the x-axis the figure bounded by the parabola $y = 0.25x^2 + 2$ and the straight line $5x - 8y + 14 = 0$

Example

Compute the volume of the solid generated by revolting about the y-axis the figure bounded by the parabolas $y = x^2$ and $8x = y^2$

Example

Compute the volume of the solid generated by revolting about the x-axis the figure bounded by the parabola $y = 0.25x^2 + 2$ and the straight line $5x - 8y + 14 = 0$

Answer: $\frac{891}{1280}\pi$

Example

Compute the volume of the solid generated by revolting about the y-axis the figure bounded by the parabolas $y = x^2$ and $8x = y^2$

Volume of Solid of Revolution

Example

Compute the volume of the solid generated by revolting about the x-axis the figure bounded by the parabola $y = 0.25x^2 + 2$ and the straight line $5x - 8y + 14 = 0$

Answer: $\frac{891}{1280}\pi$

Example

Compute the volume of the solid generated by revolting about the y-axis the figure bounded by the parabolas $y = x^2$ and $8x = y^2$

Volume of Solid of Revolution

Example

Compute the volume of the solid generated by revolting about the x-axis the figure bounded by the parabola $y = 0.25x^2 + 2$ and the straight line $5x - 8y + 14 = 0$

Answer: $\frac{891}{1280}\pi$

Example

Compute the volume of the solid generated by revolting about the y-axis the figure bounded by the parabolas $y = x^2$ and $8x = y^2$

Answer: $\frac{24}{5}\pi$

Arc length of a curve

Let f is a continuously differentiable function on $[a, b]$. The arc length of the curve $y = f(x)$ from a to b is given by

$$\text{Arc length } = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Arc length of a curve

Example

Compute the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(0, 0)$ and $(4, 8)$

Example

Compute the arc length of the curve $y = \ln \cos x$ between the points with the abscissas $x = 0$, $x = \frac{\pi}{4}$

Arc length of a curve

Example

Compute the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(0, 0)$ and $(4, 8)$

Answer: $\frac{8}{27}(10\sqrt{10} - 1)$

Example

Compute the arc length of the curve $y = \ln \cos x$ between the points with the abscissas $x = 0$, $x = \frac{\pi}{4}$

Arc length of a curve

Example

Compute the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(0,0)$ and $(4,8)$

Answer: $\frac{8}{27}(10\sqrt{10} - 1)$

Example

Compute the arc length of the curve $y = \ln \cos x$ between the points with the abscissas $x = 0$, $x = \frac{\pi}{4}$

Arc length of a curve

Example

Compute the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(0,0)$ and $(4,8)$

Answer: $\frac{8}{27}(10\sqrt{10} - 1)$

Example

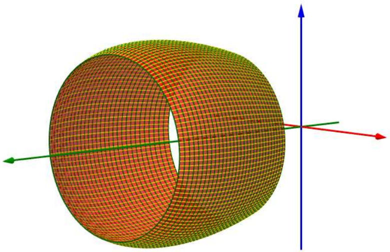
Compute the arc length of the curve $y = \ln \cos x$ between the points with the abscissas $x = 0$, $x = \frac{\pi}{4}$

Answer: $\ln \tan \frac{3\pi}{8}$

Area of a surface of Revolution

Suppose $f(x)$ is a continuous non-negative function for $x \in [a, b]$. If S is the surface obtained by rotating the graph of $y = f(x)$ about the x–axis, then the area of the surface of revolution is given by

$$\text{Area of Surface } S = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx$$



Area of a surface of Revolution

Example

Compute the area of the surface formed by revolting $y = \sin x$, $0 \leq x \leq \pi$ about the x-axis.