# Introduction

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# Bibliography


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# What are Differential equations

## Definition

Differential equation A *differential equation* is an equation that relates in a nontrivial manner an unknown function and one or more of the derivatives or differentials of the unknown function with respect to one or more independent variables.

$$F(x, y, y', \dots, y^{(n)}) = 0 \tag{1}$$

# Classification of Differential Equations

Differential equations are classified in several different ways:

- ordinary or partial
- linear or nonlinear
- homogeneous or nonhomogeneous
- autonomous or nonautonomous
- first order, second order, ..., $n$ th-order

**Definition**

The *order* of a differential equation is the order of the highest derivative that appears (non-trivially) in the equation.

- A differential equation is an *ordinary differential equation* if the only derivatives of the unknown function are ordinary derivatives.
- A differential equation is a *partial differential equation* if the only derivatives of the unknown function are partial derivatives.

**Example (Ordinary differential equations)**

- $\frac{dx}{xt} = 1 + x^2$ (first-order, nonlinear)
- $\frac{d^2x}{dt^2} + x = 3\cos(t)$ (second-order, linear, nonhomogeneous)
- $\frac{d^3y}{dx^3} = 3\frac{d^2y}{dx^2} - 5y = 0$ (third-order, linear, homogeneous)

# Ordinary differential equations

## Definition (An Ordinary Differential Equation)

An ordinary differential equation of the $n$-th order is the equation of the form

$$F(x, y(x), y'(c), \ldots, y^{(n)}(c)) = 0 \tag{2}$$

where sought-for function is the function

$$y\colon [a, b] \to \mathbb{R}^d$$

fulfilling condition (2) where

$$F\colon [a, b] \times \underbrace{\mathbb{R}^d \times \ldots \times \mathbb{R}^d}_{n+1} \to \mathbb{R}^k$$

is at least continuous.

If $k = d$ and one can solve $F$ for $y^{(n)}(x)$ then the equation (2) takes the form

$$y^{(n)} = f(x, y(x), \ldots, y^{(n-1)}(x)) \tag{3}$$

where

$$f \colon [a, b] \times \underbrace{\mathbb{R}^d \times \ldots \times \mathbb{R}^d}_{n} \to \mathbb{R}^d.$$

# Solution of Differential Equations

## Definition

To say that $y = g(x)$ is a *solution* of the differential equation

$$F(x, , y, y'(x), \ldots, y^{(n)}(x)) = 0$$

on the interval $[a, b]$ means that

$$F(x, g(x), g'(x), \ldots, g^{(n)}(x)) = 0$$

for every choice of $x$ in an interval $[a, b]$

In other words, a solution, when substituted into the differential equation makes the equation identically true for $x \in [a, b]$

## Definition

A graph of a solution of differential equation is called an *integral curve* of the equation.

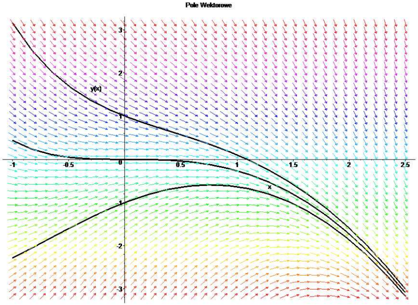


Figure: Vector field

# The Cauchy problem (the initial value problem)

*The Cauchy problem* is a problem of finding the solution $y = y(x)$ of the equation

$$F(x, y(x), y'(x), \ldots, y^{(n)}) = 0$$

satisfying the initial condition

$$\begin{cases} y(x_0) = y_0, \\ y'(x) = y_1, \\ \vdots \\ y^{(n-1)} = y_{n-1}, \end{cases}$$

where $x_0 \in (a, b)$ and $y_0, y_1, \ldots, y_{n-1}$ some constants.

# First order differential equation

The general form of an equation of the first order is

$$F(x, y(x), y'(x)) = 0 \tag{4}$$

If the equation (4) can be solved for $y'(x)$ then one has

$$y'(x) = f(x, y(x))$$

*an equation of the first order solved for the derivative.*

# The Cauchy problem (the initial value problem)

*The Cauchy problem* is a problem of finding the solution $y = y(x)$ of the equation

$$y'(x) = f(x, y(x))$$

satisfying the initial condition

$$y(x_0) = y_0.$$

Geometrically this means that an integral curve passing through a given point $M_0(x_0, y_0)$ in the $xOy$ plane is sought.

# Existence and Uniqueness of a Solution

- *Existence* says that there is at least one solution.
- *Uniqueness* says that there is at most one solution

### Theorem

*Let a differential equation*

$$y'(t) = f(x, y) \tag{5}$$

*be given, where the function $f(x, y)$ is determined in some domain $D$ in the $xOy$ plane containing a point $(x_0, y_0)$. If the function $f(x, y)$ is a continuous and bounded function in the domain $D$, then through each internal point $(x_0, y_0)$ of the domain passes at least one integral curve of the equation (5)*

## Definition

A continuous function

$$f : [a, b] \times G \to \mathbb{R}^d,$$

where $G \subset \mathbb{R}$ is said to satisfy a Lipschitz condition in $y$ if there is a constant $L$ (a Lipschitz constant) such that

$$\forall_{x \in [a,b]} \forall_{y,z \in G} ||f(x, y) - f(x, z)|| \leq L||y - z||$$

## Example

Let

$$f : [a, b] \times G \to \mathbb{R}^d$$

be a $C^1$-function ($G$ is a convex set) such that

$$\sup_{x,y} ||\frac{\partial f(x,y)}{\partial y}|| = L < +\infty$$

Then by Lagrange mean theorem one has

$$||f(x,y) - f(x,z)|| \leq ||y - z|| \sup_{0 \leq \theta \leq 1} | \frac{\partial}{\partial y} f(x, y + \theta(y - z))||$$

$$\leq L||x - y||$$

**Theorem**

*Let*

$$f : [a, b] \times G \to \mathbb{R}^d$$

*($G \subset \mathbb{R}^d$) be continuous and satisfy a Lipschitz condition in $y$ with the constant $L$, then the initial value problem $y'(x) = f(x, y)$, $y(x_0) = y_0$, $y_0 \in G$ has a unique solution.*

# Equation not containing sought-for function

It is an equation of the form

$$y' = f(x) \tag{6}$$

where $f$ is defined on an interval $I \subset \mathbb{R}$.
If $f$ is continuous function on $I$, then the form of general solution is

$$y(x) = \int f(x)dx \tag{7}$$

The solution of initial problem

$$\begin{cases} y' = f(x) \\ y(x_0) = y_0 \end{cases} \tag{8}$$

has the form

$$y = \varphi(x) = y_0 + \int_{x_0}^{x} f(t)dt$$

# Example

## Example

Solve the equation

$$y' = 3x^2 + 4x - 1 + e^x$$

$$y = \int (3x^2 + 4x - 1 + e^x)dx = x^3 + 2x^2 - x + e^x + C$$

# Equation not containing independent variable

It is an equation of the form

$$y' = g(y) \tag{9}$$

where $g$ is defined on an interval $J \subset \mathbb{R}$.
Let us note, that if $y_0 \in J$ is a root of a function $g$ i.e. $g(y_0) = 0$ then $y(x) = y_0$ is a specific solution of the equation (9). If the function $g$ is not equal zero in the interval $J$ than

$$\frac{dx}{dy} = \frac{1}{g(y)} \tag{10}$$

is an equation not containing sought-for function $x = x(y)$.
If $g$ is continuous function on $J$, then the form of general solution is

$$x(y) = \int \frac{1}{g(y)} dy \tag{11}$$

# Example

## Example

Solve an initial value problem

$$y' = 1 + y^2, \quad y(\frac{\pi}{2}) = 1$$

# Equations with variables separable

## Definition

A differential equation of the form

$$\varphi(y)dy = f(x)dx \tag{12}$$

is called *an equation with separated variables.*
An equation of the form

$$\varphi_1(x)\psi_1(y)dx = \varphi_2(x)\psi_2(y)dy \tag{13}$$

in which coefficients of the differentials are factors depending on $x$ alone and on $y$ alone is called *an equation with variables separable*.

Dividing the above equation by the product $\psi_1(y)\varphi_2(x)$ reduces it to an equation with separated variables.

$$\frac{\varphi_1(x)}{\varphi_2(x)}dx = \frac{\psi_2(y)}{\psi_1(y)}dy \tag{14}$$

## Remark

Dividing by the product $\psi_1(y)\varphi_2(x)$ may lead to the loss of particular solutions making the product $\psi_1(y)\varphi_2(x)$ zero.

## Proposition

*A differential equation of the form*

$$\frac{dy}{dx} = f(ax + by + c) \tag{15}$$

*where* $a, b \neq 0, c$ *are constants,* $f$ *is continuous function, is converted into an equation with variables separable by the substitution of the variables*

$$u = ax + by + c \tag{16}$$

## Proof.

From equation (16) we get

$$\frac{du}{dx} = a + b\frac{dy}{dx}$$

thus

$$\frac{dy}{d}x = \frac{1}{b}\left(\frac{du}{dx} - a\right)$$

and

$$\frac{du}{dx} = a + bf(u)$$

Eventually

$$\frac{du}{a + bf(u)} = dx$$

□

# Example 1

## Example

Solve the equation

$$3e^x \tan y dx + (2 - e^x)\frac{1}{\cos^2 y} dy = 0$$

# Example 1

## Example

Solve the equation

$$3e^x \tan y dx + (2 - e^x)\frac{1}{\cos^2 y}dy = 0$$

Solution

$$\tan y = C(2 - e^x)^3$$

# Example 2

## Example

Find a particular solution of the equation

$$(1 + e^x)yy' = e^x$$

satisfying the initial condition $y(0) = 1$.

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Solution

$$y = \sqrt{1 + \ln\left(\frac{1 + e^x}{2}\right)^2}$$

# Example 3

## Example

Find a particular solution of the equation

$$y' \sin x = y \ln y$$

satisfying the initial condition $y(\frac{\pi}{2}) = e$.

# Example 3

## Example

Find a particular solution of the equation

$$y' \sin x = y \ln y$$

satisfying the initial condition $y(\frac{\pi}{2}) = e$.

Solution

$$y = e^{\tan(\frac{x}{2})}$$

# Example 4

## Example

Find the curve passing through the point $(0, -2)$ such that the slope of the tangent at any of its points is equal to the ordinate of that point increased by 3.

# Example 4

## Example

Find the curve passing through the point $(0, -2)$ such that the slope of the tangent at any of its points is equal to the ordinate of that point increased by 3.

Solution

$$y = e^x - 3$$

# Exercises

Integrate the following equations:

1. $(1+y^2)dx + xydy = 0$
2. $e^{-y}(1+y') = 1$
3. $y' = \sin(x-y)$
4. $y + xy' = a(1+xy),\ y(\frac{1}{a}) = -a$
5. $\tan y' = 0$

## Definition

*A linear equation of the first order* is an equation linear in an unknown function and its derivative. It is of the form

$$\frac{dy}{dx} + p(x)y = q(x) \tag{17}$$

where $p(x)$ and $q(x)$ are given functions of $x$ continuous in the range in which it is required to integrate equation (17)

# Homogeneous linear equation

**Definition**

If $q(x) \equiv 0$ then the equation

$$\frac{dy}{dx} + p(x)y = 0 \tag{18}$$

is called *a homogeneous linear equation*.

The general solution of equation (18) is of the form

$$y = Ce^{-\int p(x)dx} \tag{19}$$

# Homogeneous linear equation

### Theorem

*If the function* $p(x)$ *is continuous in* $(a, b)$*, then* $y = Ce^{-\int p(x)dx}$ *is the general solution of the homogeneous equation* $\frac{dy}{dx} + p(x)y = 0$*. Moreover there exists unique solution satisfying initial condition* $y(x_0) = y_0$*, where* $(x_0, y_0) \in \{(x, y); \quad x \in (a, b) \wedge y \in (-\infty, +\infty)\}$.

# Variation of an arbitrary constant

One of the methods of solving linear equations of the first order is the method of variation of an arbitrary constant which consists in finding the solution of equation (17) in the form

$$y = C(x)e^{-\int p(x)dx} \tag{20}$$

where $C(x)$ is a new unknown function of $x$.

# Algorithm

- If the differential equation is given as

$$a(x)y' + b(x)y = c(x)$$

rewrite it in the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

where $p(x) = \frac{b(x)}{a(x)}$, $q(x) = \frac{c(x)}{a(x)}$

- Find the solution of homogeneous linear equation

$$\frac{dy}{dx} + p(x)y = 0$$

i.e.

$$y = u(x) = Ce^{-\int p(x)dx}$$

# Algorithm

- Evaluate the integral

$$C(x) = \int \frac{q(x)}{e^{-\int p(x)dx}} dx \tag{21}$$

We get the above integral by substituting $y$ and $y'$ in the equation (17) by $C(x)e^{-\int p(x)dx}$ and its derivative.

- Write down the general solution

$$y = u(x) + C(x)e^{-\int p(x)dx} \tag{22}$$

- If you are given an initial condition, use the initial condition to find the constant $C$.

# Example

## Example

Solve the equation

$$y' + xy = 2xe^{-x^2}$$

## Example

Solve the equation

$$y' + xy = 2xe^{-x^2}$$

Solution

$$y(x) = x^2 e^{-x^2} + Ce^{-x^2}$$

# Example

## Remark

It may turn out that the differential equation is linear in $x$ as a function of $y$.

## Example

Solve the equation

$$\frac{dy}{dx} = \frac{1}{x \cos y + \sin 2y}$$

# Example

## Remark

It may turn out that the differential equation is linear in $x$ as a function of $y$.

## Example

Solve the equation

$$\frac{dy}{dx} = \frac{1}{x\cos y + \sin 2y}$$

Solution

$$x = Ce^{\sin y} - 2(1 + \sin y)$$

# Exercises

1. $xy' - 2y = x^3 \cos x$,
2. $y' + y \cos x = \cos x, \; y(0) = 1$,
3. $y' - ye^x = 2xe^{e^x}$,
4. $y' + xe^x y = e^{(1-x)e^x}$.