Linear equations of higher order

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1. Linear differential equations of the n-th order
2. Homogeneous linear equations with constant coefficients.
3. Nonhomogeneous linear equations with constant coefficients.
4. Linear differential equations with variable coefficients.

Linear independence of functions

Let $y_1(x), y_2(x), \ldots, y_n(x)$ be a finite system of n functions defined on the interval (a, b). The functions are said to be linearly dependant in the interval (a, b) if there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all equal zero, such that for all values of x in this interval the identity

$$\alpha_1 y_1(x) + \alpha_2 y_2(x) + \ldots + \alpha_n y_n(x) = 0$$

is valid.
If identity holds only for $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ then the functions $y_1(x), y_2(x), \ldots, y_n(x)$ are said to be linearly independent in the interval (a, b)

Examples of linearly independent functions

Example

Show that functions $1, x, x^2, x^3$ are linearly independent in $(-\infty, +\infty)$

Example

Show that the system of functions $e^{k_1 x}$, $e^{k_2 x}$, $e^{k_3 x}$, where k_1, k_2, k_3 are pairwise different, is linearly independent in $(-\infty, +\infty)$.

Example

Show that the system of functions $e^{\alpha x} \sin \beta x$, $e^{\alpha x} \cos \beta x$, where $\beta \neq 0$ is linearly independent in $(-\infty, +\infty)$.

Remark

Functions $\varphi_1(x)$ and $\varphi_2(x)$ are linearly independent in the interval (a, b) if their ratio is not constant in that interval.

Wronskian determinant

Definition

Let n functions have derivatives of the $(n-1)^{th}$ order The determinant

$$W[y_1, y_2, \dots, y_n] = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \ddots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}$$

is called the Wronskian determinant.

It is in general a function of x defined in some interval.

Theorem

If a system of functions $y_1(x), y_2(x), \ldots, y_n(x)$ is linearly dependent in the interval $[a, b]$ then its Wronskian is identically equal to zero in this interval.

Gramian

Let $y_1(x), y_2(x), \ldots, y_n(x)$ be a system of functions given in the interval $[a, b]$.
We set

$$(y_i, y_j) = \int_a^b y_i(x) y_j(x) dx, \quad i, j = 1, \ldots, n$$

Gramian

Definition

The determinant

$$\Gamma(y_1, \ldots, y_n) = \begin{vmatrix} (y_1, y_1) & (y_1, y_2) & \ldots & (y_1, y_n) \\ (y_2, y_1) & (y_2, y_2) & \ldots & (y_2, y_n) \\ \ldots & \ddots & \ldots & \ldots \\ (y_n, y_1) & (y_n, y_2) & \ldots & (y_n, y_n) \end{vmatrix}$$

is called the Gramian of the system of functions $y_1(x), y_2(x), \ldots, y_n(x)$.

Theorem

For a system of functions $y_1(x), y_2(x), \ldots, y_n(x)$ to be linearly dependent it is necessary and sufficient that its Gramian should be zero.

Homogeneous linear equations with constant coefficients.

Definition

The differential equation of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0, \tag{1}$$

where $a_0, a_1, \ldots, a_n$ are real constants and $a_0 \neq 0$ is called homogeneous linear equations with constant coefficients.

Algorithm of finding general solution

1. Set up for equation (1) the characteristic equation

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \qquad (2)$$

2. Find the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation
3. Write out linearly independent particular solutions of the differential equation (1) taking into account that

a) corresponding to each real single root λ of the characteristic equation is a particular solution $y = e^{\lambda x}$

b) corresponding to each single pair of complex conjugate roots $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ of the characteristic equation are two particular solutions $y = e^{\alpha x} \cos \beta x, \quad y = e^{\alpha x} \sin \beta x$

c) corresponding to each real root λ of multiplicity s of the characteristic equation are s linearly independent particular solutions

$$y = e^{\lambda x}, \quad y = xe^{\lambda x}, \quad y = x^2 e^{\lambda x}, \ldots, \quad y = x^{s-1} e^{\lambda x}$$

d) corresponding to each pair of complex conjugate roots $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ of multiplicity s of the characteristic equation are $2s$ particular solutions

$$y = e^{\alpha x} \cos \beta x, \quad y = xe^{\alpha x} \cos \beta x, \ldots, \quad y = x^{s-1} e^{\alpha x} \cos \beta x$$

$$y = e^{\alpha x} \sin \beta x, \quad y = xe^{\alpha x} \sin \beta x, \ldots, \quad y = x^{s-1} e^{\alpha x} \sin \beta x$$

The number of particular solutions of the differential equations (1) thus constructed is equal to the order of the equation.
All the solutions constructed are linearly independent in the aggregate an make up the fundamental system of the differential equation (1)

Example

Example

$$y''' - 2y'' - 3y' = 0$$

Example

Example

$$y''' - 2y'' - 3y' = 0$$

Result

$$y = C_1 + C_2 e^{-x} + C_3 e^{3x}$$

Example

Example

$$y^{(5)} - 2y^{(4)} + 2y''' - 4y'' + y' - 2y = 0$$

Example

Example

$$y^{(5)} - 2y^{(4)} + 2y''' - 4y'' + y' - 2y = 0$$

Result

$$y = C_1 e^{2x} + (C_2 + C_3 x)\cos x + (C_4 + C_5 x)\sin x$$

Exercises

1. $3y'' - 2y' - 8y = 0$
2. $3y''' - 3y'' + y' - y = 0, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3$
3. $y'' - 2y' - 2y = 0$
4. $y^{(6)} + 2y^{(5)} + y^{(4)} = 0$
5. $y''' - 8y = 0$
6. $y''' - 2y'' + 2y' = 0$
7. $y^{(4)} - y = 0$

Nonhomogeneous linear equations with constant coefficients

Let

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = f(x), \tag{3}$$

where $a_0, a_1, \ldots, a_n$ are real constants and $a_0 \neq 0$.

Theorem

The general solution of the nonhomogeneous equation (3) is equal to the sum of the general solution of the corresponding homogeneous equation and some particular solution of the nonhomogeneous equation.

In the general case equation (3) can be integrated using the method of variation of arbitrary parameters.

The trial and error method

For the right-hand side of special form the particular solution is easer to find by so-called trial and error method. We use that method when

$$f(x) = e^{\alpha x}[P_l(x)\cos\beta x + Q_m \sin\beta x],$$

where $P_l(x)$ and $Q_m(x)$ being polynomials of degree l and m respectively. The particular solution is of the form

$$y_{ps} = x^s e^{\alpha x}[\widetilde{P}_k(x)\cos\beta x + \widetilde{Q}_k(x)\sin\beta x],$$

where $k = \max(m, l)$, $\widetilde{P}_k(x), \widetilde{Q}_k(x)$ are polynomials of the kth degree, s is multiplicity of th root $\lambda = \alpha + \beta i$ of the characteristic equation. If λ is not the root of characteristic equation then $s = 0$.

Example

Example

$$y''' - y'' + y' - y = x^2 + x$$

Example

Example

$$y''' - y'' + y' - y = x^2 + x$$

Result

$$y = C_1 e^x + C_2 \cos x + C_3 \sin x - x^2 - 3x - 1$$

Example

Example

$$y'' + 3y' + 2y = x \sin x$$

Example

Example

$$y'' + 3y' + 2y = x \sin x$$

Result

$$y = C_1 e^{-x} + C_2 e^{-2x} + (-\frac{3}{10}x + \frac{17}{50}) \cos x + (\frac{1}{10}x + \frac{3}{25}) \sin x$$

Theorem

If $y_k(x)$ is a solution of the equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = f_k(x), \tag{4}$$

$k = 1, 2, \ldots, m$, *then the function*

$$y(x) = \sum_{k=1}^{m} y_k(x)$$

is a solution of the equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = \sum_{k=1}^{m} f_k(x) \tag{5}$$

Exercises

1. $y'' + 3y' = 3$
2. $y'' - 7y' = (x-1)^2$
3. $y'' + 25y = \cos 5x$
4. $y'' + y = \sin x - \cos x$
5. $y'' - y' - 2y = 4x - 2e^x$

Depression of order

If one knows the particular solution $y_1(x)$ of the equations

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x) = 0 \tag{6}$$

then one can depress its order by one (without the loss of linearity of the equation) by substitution

$$u = \left(\frac{y}{y_1}\right)'$$

If one knows k particular linearly independent solutions of equation (6), one can depress the order of the equations of k units.

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x) = f(x) \tag{7}$$

Theorem

The general solution of the nonhomogeneous equation (7) is equal to the sum of the general solution of the corresponding homogeneous equation and some particular solution of the nonhomogeneous equation.

If the fundamental system of corresponding homogeneous equation (6) is known, then it is possible to find the general solution of the nonhomogeneous equation (7) by the method of variation of parameters (*the Lagrange method*)

The general solution of equation (6) is of the form

$$y = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n$$

where $C_1, C_2, \ldots, C_n$ being arbitrary constants.
We shall seek the solution of equation (7) in the form

$$y = C_1(x) y_1 + C_2(x) y_2 + \ldots + C_n(x) y_n$$

where $C_1(x), C_2(x), \ldots, C_n(x)$ some yet unknown functions of x.

Theorem

$$\begin{cases} y_1 C_1' + y_2 C_2' + \dots y_n C_n' = 0, \\ y_1' C_1' + y_2' C_2' + \dots y_n' C_n' = 0, \\ \vdots \\ y_1^{(n-1)} C_1' + y_2^{(n-1)} C_2' + \dots y_n^{(n-1)} C_n' = f(x). \end{cases} \tag{8}$$

Resolving the above system for $C_i(x)$, $i = 1, 2, \dots, n$ we get

$$\frac{dC_i}{dx} = \varphi_i(x), \quad i = 1, 2, \dots, n$$

whence

$$C_i(x) = \int \varphi_i(x)dx + \widetilde{C}_i, \quad i = 1, 2, \dots, n, \tag{9}$$

where $\widetilde{C}_i$ are arbitrary constants.

Second order equations

In particular, for the second order equations

$$y'' + p_1(x)y' + p_2(x)y = f(x). \tag{10}$$

the system (8) takes form

$$\begin{cases} y_1 C_1' + y_2 C_2' = 0, \\ y_1' C_1' + y_2' C_2' = f(x). \end{cases} \tag{11}$$

Solving (11) for C_1' and C_2' we get

$$C_1' = -\frac{y_2 f(x)}{W[y_1, y_2]}, \quad C_2' = \frac{y_1 f(x)}{W[y_1, y_2]},$$

where $W[y_1, y_2] = y_1 y_2' - y_2 y_1'$. Finally

$$C_1(x) = -\int \frac{y_2 f(x)}{W[y_1, y_2]} dx + \widetilde{C}_1, \tag{12}$$

$$C_2(x) = \int \frac{y_1 f(x)}{W[y_1, y_2]} dx + \widetilde{C}_2. \tag{13}$$

Remark

For the equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x),$$

where $a_0(x) \not\equiv 1$, $a_0(x) \neq 0$ system (11) will look thus

$$\begin{cases} y_1 C_1' + y_2 C_2' = 0, \\ y_1' C_1' + y_2' C_2' = \frac{f(x)}{a_0(x)}. \end{cases}$$

Example 1

Example

Find the general solution of the equation

$$xy'' + 2y' + xy = 0$$

if

$$y_1 = \frac{\sin x}{x}$$

is its particular solution.

Example 1

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Find the general solution of the equation

$$xy'' + 2y' + xy = 0$$

if

$$y_1 = \frac{\sin x}{x}$$

is its particular solution.

Result: $y = C_1 \frac{\cos x}{x} + C_2 \frac{\sin x}{x}$

Example 2

Example

Find the general solution of the equation

$$y'' + \frac{2}{x}y' + y = \frac{1}{x}, \quad x \neq 0$$

Example

Find the general solution of the equation

$$y'' + \frac{2}{x}y' + y = \frac{1}{x}, \quad x \neq 0$$

Result: $y = C_1 \frac{\sin x}{x} + C_2 \frac{\cos x}{x} + \frac{1}{x}$

Example 3

Example

Find the general solution of the equation

$$y'' + y = \frac{1}{\cos x}$$

Example 3

Example

Find the general solution of the equation

$$y'' + y = \frac{1}{\cos x}$$

Result: $y = C_1 \sin x + C_2 \cos x + \cos x \ln|\cos x| + x \sin x$

Example 4

Example

Given the fundamental system of solutions

$$y_1 = \ln x, \quad y_2 = x,$$

of the corresponding homogeneous equation find the particular solution of the equation

$$x^2(1-\ln x)y'' + xy' - y = \frac{(1-\ln x)^2}{x}, \quad x \neq 0$$

satisfying the condition

$$\lim_{x\to+\infty} y = 0$$

Example 4

Example

Given the fundamental system of solutions

$$y_1 = \ln x, \quad y_2 = x,$$

of the corresponding homogeneous equation find the particular solution of the equation

$$x^2(1 - \ln x)y'' + xy' - y = \frac{(1 - \ln x)^2}{x}, \quad x \neq 0$$

satisfying the condition

$$\lim_{x \to +\infty} y = 0$$

Result: g.s. $y = C_1 \ln x + C_2 x + \frac{1 - 2\ln x}{4x}$, p.s. $y = \frac{1 - 2\ln x}{4x}$

Exercises

1. $(2x+1)y'' + (4x-2)y' - 8y = 0, \quad y_1 = e^{mx}$,
2. $(3x+2x^3)y'' + (1+x)y' + 6y = 6$, y_1 is a polynomial,
3. $y'' + y' + e^{-2x}y = e^{-3x}, \quad y_1 = \cos e^{-x}$,
4. $y'' + y = \frac{1}{\sin x}$
5. $y'' - 2y' + y = \frac{e^x}{x^2 \sin x}$,
6. $xy'' - \frac{1}{1+2x^2}y' = 4x^3 e^{x^3}$.