The Laplace Transform

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Definition (The Laplace Transform)

Let $f(t)$ be defined for $t \geq 0$. The Laplace transform of $f(t)$ denoted $F(x)$ or $\mathcal{L}(s)$, is an integral transform given by the Laplace integral

$$F(s) = \int_{0}^{\infty} f(t)e^{-ts}dt.$$

Provided that this (improper) integral exists, i.e. that the integral is convergent.
The function $f(t)$ is called original.

Example

Find the Laplace transform of $f(t) = 1$.

Example

Find the Laplace transform of $f(t) = t$.

Example

Find the Laplace transform of $f(t) = e^t$.

Example

Find the Laplace transform of $f(t) = \sin t$.

Example

Find the Laplace transform of $f(t) = \cos t$.

Example

Find the Laplace transform of $f(t) = \sin^2 \omega t$.

Example

Find the Laplace transform of $f(t) = \sin^2(2\omega t)$.

Some properties of the Laplace Transform

1. $\mathcal{L}[af(t) + bf(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[f(t)]$
2. If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$.
3. If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[f(t-a)] = e^{-sa}F(s)$.
4. If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

The derivative of Laplace transforms

1. $\mathcal{L}[(tf(t)] = -\frac{d}{ds}\mathcal{L}[f(t)]$.

Important transforms

Original $f(t)$	Transform $F(s)$
1	$\frac{1}{s}$, $\Re(s) > 0$
t^n	$\frac{n!}{s^{n+1}}$, $\Re(s) > 0$
e^{at}	$\frac{1}{s-a}$, $\Re(s) > a$
$\sin(\omega t)$	$\frac{1}{s^2+\omega^2}$, $\Re(s) > 0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$, $\Re(s) > 0$
$e^{at}f(t)$	$F(s-a)$

Important transforms

Original $f(t)$	Transform $F(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$\int_0^t f(r)dr$	$\frac{1}{s}F(s)$
$f * g$	$F(s)G(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$

convolution of functions: $f * g = \int_0^t f(t-r)g(r)dr$

Example

Find transforms of the given functions:

1. $f(t) = t - 4$,
2. $f(t) = 2t - 4t^2$,
3. $f(t) = t^3 - e^t$,
4. $f(t) = (t-4)^3$,
5. $f(t) = \sin 2t - \cos 4t$,
6. $f(t) = \cos^2 2t - \sin^2 2t$,
7. $f(t) = e^{-3t} \sin 5t$,
8. $f(t) = t^3(\sin 2t + e^{4t})$.

Inverse Laplace transform

Definition (Inverse Laplace transform)

Transformation $\mathcal{L}^{-1}$ is called the Inverse Laplace Transform:

$$\mathcal{L}^{-1}[F(s)] = f(t) \text{ if } \mathcal{L}[f(t)] = F(s).$$

Example

Find the inverse Laplace transform of the given function:

1. $F(s) = -\frac{2}{s}$,
2. $F(s) = \frac{1}{s(s+1)}$,
3. $F(s) = \frac{s+4}{(s-1)(s+2)}$,
4. $F(s) = \frac{5}{s^2+2s+5}$.
5. $F(s) = \frac{3s}{(s+2)^3}$

Example (E AI 2012/13)

Find the original if the Laplace transform of f is given

$$F(s) = \frac{3s^2 - 2s + 9}{s^3 - s^2 + 4s - 4}$$

if $s = 1$ is one of the roots of denominator.

Example (E E)

Find the original if the Laplace transform of f is given

$$F(s) = \frac{3s^2 - 7s + 10}{s^3 - 3s^2 + s + 5}$$

if $s = -1$ is one of the roots of denominator

Example

Find the original if the Laplace transform of f is given

$$F(s) = \frac{s^3 - 2s^2 + 4s + 8}{s^4 + 4s^3 + 8s^2}$$

Example

Find the original if the Laplace transform of f is given

$$F(s) = \frac{4s^2 + 20s + 26}{s^3 + 6s^2 + 13s}$$

Example

Find the original if the Laplace transform of f is given

$$F(s) = \frac{13s + 26}{s^3 + 4s^2 + 13s}$$

Example

Find the original if the Laplace transform of f is given

$$F(s) = \frac{2s^3 + 2s^2 + 3s + 3}{s^4 + s^3 + s^2}$$

Example

Find the original if the Laplace transform of f is given

$$F(s) = \frac{3s^2 + 3s + 6}{s^3 + 2s^2 + 3s}$$

The Laplace Transform Method od Solving an Initial Value Problem

The Laplace Transform of the nth derivative:

$$\mathcal{L}[y^{(n)}(t)] = s^n Y(s) - \sum_{k=1}^{n} s^{n-k} y^{(k-1)}(0).$$

Example

Solve the initial value problem:

1. $y'' + 4y = 8\sin 2t,\ y(0) = 0,\ y'(0) = 2;$
2. $y''' - y'' - y' + y = 6e^{t},\ y(0) = y'(0) = y''(0) = 0.$