CHAPTER 11

Infinite Series

The early developers of the calculus, including Newton and Leibniz, were well aware of the importance of infinite series. The values of many functions such as sine and cosine were geometrically obtainable only in special cases. Infinite series provided a way of developing extensive tables of values for them.

This chapter begins with a statement of what is meant by infinite series, then the question of when these sums can be assigned values is addressed. Much information can be obtained by exploring infinite sums of constant terms; however, the eventual objective in analysis is to introduce series that depend on variables. This presents the possibility of representing functions by series. Afterward, the question of how continuity, differentiability, and integrability play a role can be examined.

The question of dividing a line segment into infinitesimal parts has stimulated the imaginations of philosophers for a very long time. In a corruption of a paradox introduce by Zeno of Elea (in the fifth century B.C.) a dimensionless frog sits on the end of a one-dimensional log of unit length. The frog jumps halfway, and then halfway and halfway ad infinitum. The question is whether the frog ever reaches the other end. Mathematically, an unending sum,

$$\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}+\cdots$$

is suggested. "Common sense" tells us that the sum must approach one even though that value is never attained. We can form sequences of partial sums

$$S_1=\frac{1}{2},\, S_2=\frac{1}{2}+\frac{1}{4},\ldots,\, S_n=\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}+\cdots$$

and then examine the limit. This returns us to Chapter 2 and the modern manner of thinking about the infinitesimal.

In this chapter consideration of such sums launches us on the road to the theory of infinite series.

DEFINITIONS OF INFINITE SERIES AND THEIR CONVERGENCE AND DIVERGENCE

Definition: The sum

$$S=\sum_{n=1}^{\infty} u_n = u_1+u_2+\cdots+u_n+\cdots \tag{1}$$

is an *infinite series*. Its value, if one exists, is the limit of the sequence of partial sums $\{S_n\}$

$$S = \lim_{n \to \infty} S_n \tag{2}$$

If there is a unique value, the series is said to *converge* to that *sum*, S. If there is not a unique sum the series is said to *diverge*.

Sometimes the character of a series is obvious. For example, the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ generated by the frog on the log surely converges, while $\sum_{n=1}^{\infty} n$ is divergent. On the other hand, the variable series

$$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

raises questions.

This series may be obtained by carrying out the division $\frac{1}{1-x}$. If $-1 < x < 1$, the sums S_n yields an approximations to $\frac{1}{1-x}$ and (2) is the exact value. The indecision arises for $x = -1$. Some very great mathematicians, including Leonard Euler, thought that S should be equal to $\frac{1}{2}$, as is obtained by substituting -1 into $\frac{1}{1-x}$. The problem with this conclusion arises with examination of $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ and observation that appropriate associations can produce values of 1 or 0. Imposition of the condition of uniqueness for convergence put this series in the category of divergent and eliminated such possibility of ambiguity in other cases.

FUNDAMENTAL FACTS CONCERNING INFINITE SERIES

1. If Σu_n converges, then $\lim_{n \to \infty} u_n = 0$ (see Problem 2.26, Chap. 2). The converse, however, is not necessarily true, i.e., if $\lim_{n \to \infty} u_n = 0$, Σu_n may or may not converge. It follows that if the nth term of a series does *not* approach zero the series is divergent.
2. Multiplication of each term of a series by a constant different from zero does not affect the convergence or divergence.
3. Removal (or addition) of a *finite* number of terms from (or to) a series does not affect the convergence or divergence.

SPECIAL SERIES

1. **Geometric series** $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$, where a and r are constants, converges to $S = \frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geqq 1$. The sum of the first n terms is $S_n = \frac{a(1-r^n)}{1-r}$ (see Problem 2.25, Chap. 2).
2. **The *p* series** $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$, where p is a constant, converges for $p > 1$ and diverges for $p \leqq 1$. The series with $p = 1$ is called the *harmonic series*.

TESTS FOR CONVERGENCE AND DIVERGENCE OF SERIES OF CONSTANTS

More often than not, exact values of infinite series cannot be obtained. Thus, the search turns toward information about the series. In particular, its convergence or divergence comes in question. The following tests aid in discovering this information.

1. **Comparison test** for series of non-negative terms.

   (*a*) *Convergence*. Let $v_n \geqq 0$ for all $n > N$ and suppose that Σv_n converges. Then if $0 \leqq u_n \leqq v_n$ for all $n > N$, Σu_n also converges. Note that $n > N$ means *from some term onward*. Often, $N = 1$.

   EXAMPLE. Since $\dfrac{1}{2^n + 1} \leqq \dfrac{1}{2^n}$ and $\sum \dfrac{1}{2^n}$ converges, $\sum \dfrac{1}{2^n + 1}$ also converges.

   (*b*) *Divergence*. Let $v_n \geqq 0$ for all $n > N$ and suppose that Σv_n diverges. Then if $u_n \geqq v_n$ for all $n > N$, Σu_n also diverges.

   EXAMPLE. Since $\dfrac{1}{\ln n} > \dfrac{1}{n}$ and $\sum_{n=2}^{\infty} \dfrac{1}{n}$ diverges, $\sum_{n=2}^{\infty} \dfrac{1}{\ln n}$ also diverges.

2. **The Limit-Comparison or Quotient Test** for series of non-negative terms.

   (*a*) If $u_n \geqq 0$ and $v_n \geqq 0$ and if $\lim_{n \to \infty} \dfrac{u_n}{v_n} = A \neq 0$ or ∞, then Σu_n and Σv_n either both converge or both diverge.

   (*b*) If $A = 0$ in (*a*) and Σv_n converges, then Σu_n converges.

   (*c*) If $A = \infty$ in (*a*) and Σv_n diverges, then Σu_n diverges.

   This test is related to the comparison test and is often a very useful alternative to it. In particular, taking $v_n = 1/n^p$, we have from known facts about the p series the

Theorem 1. Let $\lim_{n \to \infty} n^p u_n = A$. Then

(*i*) Σu_n converges if $p > 1$ and A is finite.

(*ii*) Σu_n diverges if $p \leqq 1$ and $A \neq 0$ (A may be infinite).

EXAMPLES. **1**. $\sum \dfrac{n}{4n^3 - 2}$ converges since $\lim_{n \to \infty} n^2 \cdot \dfrac{n}{4n^3 - 2} = \dfrac{1}{4}$.

2. $\sum \dfrac{\ln n}{\sqrt{n+1}}$ diverges since $\lim_{n \to \infty} n^{1/2} \cdot \dfrac{\ln n}{(n+1)^{1/2}} = \infty$.

3. **Integral test** for series of non-negative terms.

   If $f(x)$ is positive, continuous, and monotonic decreasing for $x \geqq N$ and is such that $f(n) = u_n, n = N, N+1, N+2, \ldots,$ then Σu_n converges or diverges according as $\int_N^{\infty} f(x)\,dx = \lim_{M \to \infty} \int_n^M f(x)\,dx$ converges or diverges. In particular we may have $N = 1$, as is often true in practice.

   This theorem borrows from the next chapter since the integral has an unbounded upper limit. (It is an improper integral. The convergence or divergence of these integrals is defined in much the same way as for infinite series.)

   EXAMPLE: $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges since $\lim_{M \to \infty} \int_1^M \dfrac{dx}{x^2} = \lim_{M \to \infty} \left(1 - \dfrac{1}{M}\right)$ exists.

4. **Alternating series test.** An *alternating series* is one whose successive terms are alternately positive and negative.

   An alternating series converges if the following two conditions are satisfied (see Problem 11.15).

   (*a*) $|u_{n+1}| \leqq |u_n|$ for $n \geqq N$ (Since a fixed number of terms does not affect the convergence or divergence of a series, N may be any positive integer. Frequently it is chosen to be 1.)

   (*b*) $\lim_{n \to \infty} u_n = 0$ $\left(\text{or } \lim_{n \to \infty} |u_n| = 0\right)$

EXAMPLE. For the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, we have $u_n = \frac{(-1)^{n-1}}{n}$, $|u_n| = \frac{1}{n}$, $|u_{n+1}| = \frac{1}{n+1}$. Then for $n \geqq 1$, $|u_{n+1}| \leqq |u_n|$. Also $\lim_{n\to\infty} |u_n| = 0$. Hence, the series converges.

Theorem 2. The numerical error made in stopping at any particular term of a convergent alternating series which satisfies conditions (*a*) and (*b*) is less than the absolute value of the next term.

EXAMPLE. If we stop at the 4th term of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$, the error made is less than $\frac{1}{5} = 0.2$.

5. **Absolute and conditional convergence.** The series Σu_n is called *absolutely convergent* if $\Sigma |u_n|$ converges. If Σu_n converges but $\Sigma |u_n|$ diverges, then Σu_n is called *conditionally convergent*.

Theorem 3. If $\Sigma |u_n|$ converges, then Σu_n converges. In words, an absolutely convergent series is convergent (see Problem 11.17).

EXAMPLE 1. $\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \cdots$ is absolutely convergent and thus convergent, since the series of absolute values $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ converges.

EXAMPLE 2. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges, but $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges. Thus, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent.

Any of the tests used for series with non-negative terms can be used to test for absolute convergence. Also, tests that compare successive terms are common. Tests 6, 8, and 9 are of this type.

6. **Ratio test.** Let $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = L$. Then the series Σu_n
   (*a*) converges (absolutely) if $L < 1$
   (*b*) diverges if $L > 1$.
   If $L = 1$ the test fails.
7. **The *n*th root test.** Let $\lim_{n\to\infty} \sqrt[n]{|u_n|} = L$. Then the series Σu_n
   (*a*) converges (absolutely) if $L < 1$
   (*b*) diverges if $L > 1$.
   If $L = 1$ the test fails.
8. **Raabe's test**. Let $\lim_{n\to\infty} n \left(1 - \left|\frac{u_n + 1}{u_n}\right|\right) = L$. Then the series Σu_n
   (*a*) converges (absolutely) if $L > 1$
   (*b*) diverges or converges conditionally if $L < 1$.

   If $L = 1$ the test fails.
   This test is often used when the ratio tests fails.

9. **Gauss' test.** If $\left|\frac{u_{n+1}}{u_n}\right| = 1 - \frac{L}{n} + \frac{c_n}{n^2}$, where $|c_n| < P$ for all $n > N$, then the series Σu_n
   (*a*) converges (absolutely) if $L > 1$
   (*b*) diverges or converges conditionally if $L \leqq 1$.
   This test is often used when Raabe's test fails.

THEOREMS ON ABSOLUTELY CONVERGENT SERIES

Theorem 4. (Rearrangement of Terms) The terms of an absolutely convergent series can be rearranged in any order, and all such rearranged series will converge to the same sum. However, if the terms of a conditionally convergent series are suitably rearranged, the resulting series may diverge or converge to *any* desired sum (see Problem 11.80).

Theorem 5. (Sums, Differences, and Products) The sum, difference, and product of two absolutely convergent series is absolutely convergent. The operations can be performed as for finite series.

INFINITE SEQUENCES AND SERIES OF FUNCTIONS, UNIFORM CONVERGENCE

We opened this chapter with the thought that functions could be expressed in series form. Such representation is illustrated by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$$

where

$$\sin x = \lim_{n\to\infty} S_n, \quad \text{with} \quad S_1 = x, S_2 = x - \frac{x^3}{3!}, \ldots S_n = \sum_{k=1}^{n} (-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}.$$

Observe that until this section the sequences and series depended on one element, n. Now there is variation with respect to x as well. This complexity requires the introduction of a new concept called *uniform convergence*, which, in turn, is fundamental in exploring the continuity, differentiation, and integrability of series.

Let $\{u_n(x)\}, n = 1, 2, 3, \ldots$ be a sequence of functions defined in $[a, b]$. The sequence is said to converge to $F(x)$, or to have the limit $F(x)$ in $[a, b]$, if for each $\epsilon > 0$ and each x in $[a, b]$ we can find $N > 0$ such that $|u_n(x) - F(x)| < \epsilon$ for all $n > N$. In such case we write $\lim_{n\to\infty} u_n(x) = F(x)$. The number N may depend on x as well as ϵ. If it depends *only* on ϵ and not on x, the sequence is said to converge to $F(x)$ *uniformly* in $[a, b]$ or to be *uniformly convergent* in $[a, b]$.

The infinite series of functions

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + u_3(x) + \cdots \tag{3}$$

is said to be convergent in $[a, b]$ if the sequence of partial sums $\{S_n(x)\}$, $n = 1, 2, 3, \ldots,$ where $S_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x)$, is convergent in $[a, b]$. In such case we write $\lim_{n\to\infty} S_n(x) = S(x)$ and call $S(x)$ the *sum* of the series.

It follows that $\Sigma u_n(x)$ converges to $S(x)$ in $[a, b]$ if for each $\epsilon > 0$ and each x in $[a, b]$ we can find $N > 0$ such that $|S_n(x) - S(x)| < \epsilon$ for all $n > N$. If N depends *only* on ϵ and not on x, the series is called *uniformly convergent* in $[a, b]$.

Since $S(x) - S_n(x) = R_n(x)$, the remainder after n terms, we can equivalently say that $\Sigma u_n(x)$ is uniformly convergent in $[a, b]$ if for each $\epsilon > 0$ we can find N depending on ϵ but not on x such that $|R_n(x)| < \epsilon$ for all $n > N$ and all x in $[a, b]$.

These definitions can be modified to include other intervals besides $a \leqq x \leqq b$, such as $a < x < b$, and so on.

The domain of convergence (absolute or uniform) of a series is the set of values of x for which the series of functions converges (absolutely or uniformly).

EXAMPLE 1. Suppose $u_n = x^n/n$ and $-\frac{1}{2} \leqq x \leqq 1$. Now think of the constant function $F(x) = 0$ on this interval. For any $\epsilon > 0$ and any x in the interval, there is N such that for all $n > N|u_n - F(x)| < \epsilon$, i.e., $|x^n/n| < \epsilon$. Since the limit does not depend on x, the sequence is uniformly convergent.

EXAMPLE 2. If $u_n = x^n$ and $0 \leqq x \leqq 1$, the sequence is not uniformly convergent because (think of the function $F(x) = 0$, $0 \leqq x < 1$, $F(1) = 1$)

$$|x^n - 0| < \epsilon \text{ when } x^n < \epsilon,$$

thus

$$n \ln x < \ln \epsilon.$$

On the interval $0 \leqq x < 1$, and for $0 < \epsilon < 1$, both members of the inequality are negative, therefore, $n > \dfrac{\ln \epsilon}{\ln x}$. Since $\dfrac{\ln \epsilon}{\ln x} = \dfrac{\ln 1 - \ln \epsilon}{\ln 1 - nn\, x} = \dfrac{\ln(/\epsilon)}{\ln(1/x)}$, it follows that we must choose N such that

$$n > N > \frac{\ln 1/\epsilon}{\ln 1/x}$$

From this expression we see that $\epsilon \to 0$ then $\ln \dfrac{1}{\epsilon} \to \infty$ and also as $x \to 1$ from the left $\ln \dfrac{1}{x} \to 0$ from the right; thus, in either case, N must increase without bound. This dependency on both ϵ and x demonstrations that the sequence is not uniformly convergent. For a pictorial view of this example, see Fig. 11-1.

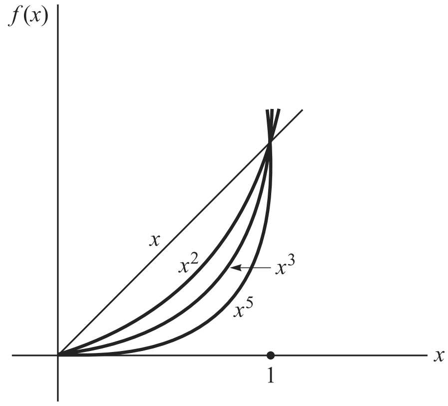

Fig. 11-1

SPECIAL TESTS FOR UNIFORM CONVERGENCE OF SERIES

1. **Weierstrass *M* test.** If sequence of positive constants $M_1, M_2, M_3, \ldots$ can be found such that in some interval

   (a) $|u_n(x)| \leqq M_n \qquad n = 1, 2, 3, \ldots$

   (b) ΣM_n converges

   then $\Sigma u_n(x)$ is uniformly and absolutely convergent in the interval.

   EXAMPLE. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is uniformly and absolutely convergent in $[0, 2\pi]$ since $\left|\dfrac{\cos nx}{n^2}\right| \leqq \dfrac{1}{n^2}$ and $\sum \dfrac{1}{n^2}$ converges.

   This test supplies a sufficient but not a necessary condition for uniform convergence, i.e., a series may be uniformly convergent even when the test cannot be made to apply.

   One may be led because of this test to believe that uniformly convergent series must be absolutely convergent, and conversely. However, the two properties are independent, i.e., a series can be uniformly convergent without being absolutely convergent, and conversely. See Problems 11.30, 11.127.

2. **Dirichlet's test.** Suppose that

   (a) the sequence $\{a_n\}$ is a monotonic decreasing sequence of positive constants having limit zero,

   (b) there exists a constant P such that for $a \leqq x \leqq b$

   $$|u_1(x) + u_2(x) + \cdots + u_n(x)| < P \qquad \text{for all } n > N.$$

   Then the series

   $$a_1 u_1(x) + a_2 u_2(x) + \cdots = \sum_{n=1}^{\infty} a_n u_n(x)$$

   is uniformly convergent in $a \leqq x \leqq b$.

THEOREMS ON UNIFORMLY CONVERGENT SERIES

If an infinite series of functions is uniformly convergent, it has many of the properties possessed by sums of finite series of functions, as indicated in the following theorems.

Theorem 6. If $\{u_n(x)\}, n = 1, 2, 3, \ldots$ are continuous in $[a, b]$ and if $\Sigma u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, then $S(x)$ is continuous in $[a, b]$.

Briefly, this states that a uniformly convergent series of continuous functions is a continuous function. This result is often used to demonstrate that a given series is not uniformly convergent by showing that the sum function $S(x)$ is discontinuous at some point (see Problem 11.30).

In particular if x_0 is in $[a, b]$, then the theorem states that

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} u_n(x) = \sum_{n=1}^{\infty} u_n(x_0)$$

where we use right- or left-hand limits in case x_0 is an endpoint of $[a, b]$.

Theorem 7. If $\{u_n(x)\}, n = 1, 2, 3, \ldots,$ are continuous in $[a, b]$ and if $\Sigma u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, then

$$\int_a^b S(x)\, dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\, dx \tag{4}$$

or

$$\int_a^b \left\{ \sum_{n=1}^{\infty} u_n(x) \right\} dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\, dx \tag{5}$$

Briefly, a uniformly convergent series of continuous functions can be integrated term by term.

Theorem 8. If $\{u_n(x)\}, n = 1, 2, 3, \ldots,$ are continuous and have continuous derivatives in $[a, b]$ and if $\Sigma u_n(x)$ converges to $S(x)$ while $\Sigma u_n'(x)$ is uniformly convergent in $[a, b]$, then in $[a, b]$

$$S'(x) = \sum_{n=1}^{\infty} u_n'(x) \tag{6}$$

or

$$\frac{d}{dx} \left\{ \sum_{n=1}^{\infty} u_n(x) \right\} = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x) \tag{7}$$

This shows conditions under which a series can be differentiated term by term.

Theorems similar to the above can be formulated for sequences. For example, if $\{u_n(x)\}$, $n = 1, 2, 3, \ldots$ is uniformly convergent in $[a, b]$, then

$$\lim_{n \to \infty} \int_a^b u_n(x)\, dx = \int_a^b \lim_{n \to \infty} u_n(x)\, dx \tag{8}$$

which is the analog of Theorem 7.

POWER SERIES

A series having the form

$$a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n \tag{9}$$

where $a_0, a_1, a_2, \ldots$ are constants, is called a *power series* in x. It is often convenient to abbreviate the series (*9*) as $\Sigma a_n x^n$.

In general a power series converges for $|x| < R$ and diverges for $|x| > R$, where the constant R is called the *radius of convergence* of the series. For $|x| = R$, the series may or may not converge.

The interval $|x| < R$ or $-R < x < R$, with possible inclusion of endpoints, is called the *interval of convergence* of the series. Although the ratio test is often successful in obtaining this interval, it may fail and in such cases, other tests may be used (see Problem 11.22).

The two special cases $R = 0$ and $R = \infty$ can arise. In the first case the series converges only for $x = 0$; in the second case it converges for all x, sometimes written $-\infty < x < \infty$ (see Problem 11.25). When we speak of a convergent power series, we shall assume, unless otherwise indicated, that $R > 0$.

Similar remarks hold for a power series of the form (*9*), where x is replaced by $(x - a)$.

THEOREMS ON POWER SERIES

Theorem 9. A power series converges uniformly and absolutely in any interval which lies *entirely within* its interval of convergence.

Theorem 10. A power series can be differentiated or integrated term by term over any interval lying entirely within the interval of convergence. Also, the sum of a convergent power series is continuous in any interval lying entirely within its interval of convergence.

This follows at once from Theorem 9 and the theorems on uniformly convergent series on Pages 270 and 271. The results can be extended to include end points of the interval of convergence by the following theorems.

Theorem 11. *Abel's theorem*. When a power series converges up to and including an endpoint of its interval of convergence, the interval of uniform convergence also extends so far as to include this endpoint. See Problem 11.42.

Theorem 12. *Abel's limit theorem*. If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = x_0$, which may be an interior point or an endpoint of the interval of convergence, then

$$\lim_{x \to x_0} \left\{ \sum_{n=0}^{\infty} a_n x^n \right\} = \sum_{n=0}^{\infty} \left\{ \lim_{x \to x_0} a_n x^n \right\} = \sum_{n=0}^{\infty} a_n x_0^n \tag{10}$$

If x_0 is an end point, we must use $x \to x_0+$ or $x \to x_0-$ in (*10*) according as x_0 is a left- or right-hand end point.

This follows at once from Theorem 11 and Theorem 6 on the continuity of sums of uniformly convergent series.

OPERATIONS WITH POWER SERIES

In the following theorems we assume that all power series are convergent in some interval.

Theorem 13. Two power series can be added or subtracted term by term for each value of x common to their intervals of convergence.

Theorem 14. Two power series, for example, $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, can be multiplied to obtain $\sum_{n=0}^{\infty} c_n x^n$ where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0 \tag{11}$$

the result being valid for each x within the common interval of convergence.

Theorem 15. If the power series $\sum_{n=0}^{\infty} a_n x^n$ is divided by the power series $\Sigma b_n x^n$ where $b_0 \neq 0$, the quotient can be written as a power series which converges for sufficiently small values of x.

Theorem 16. If $y = \sum_{n=0}^{\infty} a_n x^n$, then by substituting $x = \sum_{n=0}^{\infty} b_n y^n$, we can obtain the coefficients b_n in terms of a_n. This process is often called *reversion of series*.

EXPANSION OF FUNCTIONS IN POWER SERIES

This section gets at the heart of the use of infinite series in analysis. Functions are represented through them. Certain forms bear the names of mathematicians of the eighteenth and early nineteenth century who did so much to develop these ideas.

A simple way (and one often used to gain information in mathematics) to explore series representation of functions is to assume such a representation exists and then discover the details. Of course, whatever is found must be confirmed in a rigorous manner. Therefore, assume

$$f(x) = A_0 + A_1(x - c) + A_2(x - c)^2 + \cdots + A_n(x - c)^n + \cdots$$

Notice that the coefficients A_n can be identified with derivatives of f. In particular

$$A_0 = f(c),\ A_1 = f'(c),\ A_2 = \frac{1}{2!} f''(c), \ldots, A_n = \frac{1}{n!} f^{(n)}(c), \ldots$$

This suggests that a series representation of f is

$$f(x) = f(c) + f'(c)(x - c) + \frac{1}{2!} f''(c)(x - c)^2 + \cdots + \frac{1}{n!} f^{(n)}(c)(x - c)^n + \cdots$$

A first step in formalizing series representation of a function, f, for which the first n derivatives exist, is accomplished by introducing *Taylor polynomials* of the function.

$$\begin{aligned} &P_0(x) = f(c) \qquad P_1(x) = f(c) + f'(c)(x - c), \\ &P_2(x) = f(c) + f'(c)(x - c) + \frac{1}{2!} f''(c)(x - c)^2, \\ &P_n(x) = f(c) + f'(c)(x - c) + \cdots + \frac{1}{n!} f^{(n)}(c)(x - c)^n \end{aligned} \tag{12}$$

TAYLOR'S THEOREM

Let f and its derivatives $f', f'', \ldots, f^{(n)}$ exist and be continuous in a closed interval $a \leq x \leq b$ and suppose that $f^{(n+1)}$ exists in the open interval $a < x < b$. Then for c in $[a, b]$,

$$f(x) = P_n(x) + R_n(x),$$

where the remainder $R_n(x)$ may be represented in any of the three following ways.

For each n there exists ξ such that

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-c)^{n+1} \qquad \text{(Lagrange form)} \tag{13}$$

(ξ is between c and x.)

(The theorem with this remainder is a mean value theorem. Also, it is called Taylor's formula.)

For each n there exists ξ such that

$$R_n(x) = \frac{1}{n!} f^{(n+1)}(\xi)(x-\xi)^n(x-c) \qquad \text{(Cauchy form)} \tag{14}$$

$$R_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt \qquad \text{(Integral form)} \tag{15}$$

If all the derivatives of f exist, then

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(c)(x-c)^n \tag{16}$$

This infinite series is called a Taylor series, although when $c = 0$, it can also be referred to as a MacLaurin series or expansion.

One might be tempted to believe that if all derivatives of $f(x)$ exist at $x = c$, the expansion (*16*) would be valid. This, however, is not necessarily the case, for although one can then *formally* obtain the series on the right of (*16*), the resulting series may not converge to $f(x)$. For an example of this see Problem 11.108.

Precise conditions under which the series converges to $f(x)$ are best obtained by means of the theory of functions of a complex variable. See Chapter 16.

The determination of values of functions at desired arguments is conveniently approached through Taylor polynomials.

EXAMPLE. The value of $\sin x$ may be determined geometrically for $0, \frac{\pi}{6}$, and an infinite number of other arguments. To obtain values for other real number arguments, a Taylor series may be expanded about any of these points. For example, let $c = 0$ and evaluate several derivatives there, i.e., $f(0) = \sin 0 = 0$, $f'(0) = \cos 0 = 1$, $f''(0) = -\sin 0 = 0$, $f'''(0) = -\cos 0 = -1$, $f^{1v}(0) = \sin 0 = 0$, $f^{v}(0) = \cos 0 = 1$.

Thus, the MacLaurin expansion to five terms is

$$\sin x = 0 + x - 0 - \frac{1}{3!}x^3 + 0 - \frac{1}{51}x^5 + \cdots$$

Since the fourth term is 0 the Taylor polynomials P_3 and P_4 are equal, i.e.,

$$P_3(x) = P_4(x) = x - \frac{x^3}{3!}$$

and the Lagrange remainder is

$$R_4(x) = \frac{1}{5!}\cos\xi\, x^5$$

Suppose an approximation of the value of $\sin .3$ is required. Then

$$P_4(.3) = .3 - \frac{1}{6}(.3)^3 \approx .2945.$$

The accuracy of this approximation can be determined from examination of the remainder. In particular, (remember $|\cos\xi| \le 1$)

$$|R_4| = \left|\frac{1}{5!}\cos\xi(.3)^5\right| \le \frac{1}{120}\,\frac{243}{10^5} < .000021$$

Thus, the approximation $P_4(.3)$ for $\sin .3$ is correct to four decimal places.

Additional insight to the process of approximation of functional values results by constructing a graph of $P_4(x)$ and comparing it to $y = \sin x$. (See Fig. 11-2.)

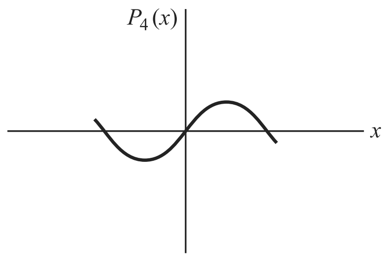

Fig. 11-2

$$P_4(x) = x - \frac{x^3}{6}$$

The roots of the equation are $0, \pm\sqrt{6}$. Examination of the first and second derivatives reveals a relative maximum at $x = \sqrt{2}$ and a relative minimum at $x = -\sqrt{2}$. The graph is a local approximation of the sin curve. The reader can show that $P_6(x)$ produces an even better approximation.

(For an example of series approximation of an integral see the example below.)

SOME IMPORTANT POWER SERIES

The following series, convergent to the given function in the indicated intervals, are frequently employed in practice:

1. $\sin x \quad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots \quad -\infty < x < \infty$
2. $\cos x \quad = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!} + \cdots \quad -\infty < x < \infty$
3. $e^x \quad = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots \quad -\infty < x < \infty$
4. $\ln|1+x| \quad = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots (-1)^{n-1}\frac{x^n}{n} + \cdots \quad -1 < x \leqq 1$
5. $\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| \quad = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots \quad -1 < x < 1$
6. $\tan^{-1} x \quad = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots \quad -1 \leqq x \leqq 1$
7. $(1+x)^p \quad = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\ldots(p-n+1)}{n!}x^n + \cdots$

This is the *binomial series*.

(*a*) If p is a positive integer or zero, the series terminates.

(*b*) If $p > 0$ but is not an integer, the series converges (absolutely) for $-1 \leqq x \leqq 1$.

(*c*) If $-1 < p < 0$, the series converges for $-1 < x \leqq 1$.

(*d*) If $p \leqq -1$, the series converges for $-1 < x < 1$.

For all p the series certainly converges if $-1 < x < 1$.

EXAMPLE. Taylor's Theorem applied to the series for e^x enables us to estimate the value of the integral $\int_0^1 e^{x^2}\,dx$. Substituting x^2 for x, we obtain $\int_0^1 e^{x^2}\,dx = \int_0^1\left(1 + x + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{e^\xi}{5!}x^{10}\right)dx$ where

$$P_4(x) = 1 + x + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8$$

and

$$R_4(x) = \frac{e^\xi}{5!}x^{10}, \qquad 0 < \xi < x$$

Then

$$\int_0^1 P_4(x)\,dx = 1 + \frac{1}{3} + \frac{1}{5(2!)} + \frac{1}{7(3!)} + \frac{1}{9(4!)} \approx 1.4618$$

$$\left|\int_0^1 R_4(x)\,dx\right| \le \int_0^1 \left|\frac{e^{\xi}}{5!}x^{10}\right| dx \le e\int_0^1 \frac{x^{10}}{5!}\,dx = \frac{e}{11.5} < .0021$$

Thus, the maximum error is less than .0021 and the value of the integral is accurate to two decimal places.

SPECIAL TOPICS

1. **Functions defined by series** are often useful in applications and frequently arise as solutions of differential equations. For example, the function defined by

$$J_p(x) = \frac{x^p}{2^p p!}\left\{1 - \frac{x^2}{2(2p+2)} + \frac{x^4}{2\cdot 4(2p+2)(2p+4)} - \cdots\right\}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{p+2n}}{n!(n+p)!} \qquad (16)$$

is a solution of *Bessel's differential equation* $x^2y'' + xy' + (x^2 - p^2)y = 0$ and is thus called a *Bessel function of order p*. See Problems 11.46, 11.110 through 11.113.

Similarly, the *hypergeometric function*

$$F(a, b; c; x) = 1 + \frac{a\cdot B}{1\cdot c}x + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}x^2 + \cdots \qquad (17)$$

is a solution of *Gauss' differential equation* $x(1-x)y'' + \{c - (a+b+1)x\}y' - aby = 0$.

These functions have many important properties.

2. **Infinite series of complex terms**, in particular power series of the form $\sum_{n=0}^{\infty} a_n z^n$, where $z = x + iy$ and a_n may be complex, can be handled in a manner similar to real series.

Such power series converge for $|z| < R$, i.e., interior to a *circle of convergence* $x^2 + y^2 = R^2$, where R is the *radius of convergence* (if the series converges only for $z = 0$, we say that the radius of convergence R is zero; if it converges for all z, we say that the radius of convergence is infinite). On the boundary of this circle, i.e., $|z| = R$, the series may or may not converge, depending on the particular z.

Note that for $y = 0$ the circle of convergence reduces to the interval of convergence for real power series. Greater insight into the behavior of power series is obtained by use of the theory of functions of a complex variable (see Chapter 16).

3. **Infinite series of functions of two (or more) variables,** such as $\sum_{n=1}^{\infty} u_n(x, y)$ can be treated in a manner analogous to series in one variable. In particular, we can discuss power series in x and y having the form

$$a_{00} + (a_{10}x + a_{01}y) + (a_{20}x^2 + a_{11}xy + a_{02}y^2) + \cdots \qquad (18)$$

using double subscripts for the constants. As for one variable, we can expand suitable functions of x and y in such power series. In particular, the Taylor theroem may be extended as follows.

TAYLOR'S THEOREM (FOR TWO VARIABLES)

Let f be a function of two variables x and y. If all partial derivatives of order n are continuous in a closed region and if all the $(n+1)$ partial derivatives exist in the open region, then

$$f(x_0+h, y_0+k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \cdots$$
$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + R_n \tag{18}$$

where

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k), \qquad 0 < \theta < 1$$

and where the meaning of the operator notation is as follows:

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f = hf_x + kf_y,$$
$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 = h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}$$

and we formally expand $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n$ by the binomial theorem.

Note: In alternate notation $h = \Delta x = x - x_0$, $k = \Delta y = y - y_0$.

If $R_n \to 0$ as $n \to \infty$ then an unending continuation of terms produces the *Taylor series for* $f(x, y)$. Multivariable Taylor series have a similar pattern.

4. **Double Series.** Consider the array of numbers (or functions)

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots \\ u_{21} & u_{22} & u_{23} & \dots \\ u_{31} & u_{32} & u_{33} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Let $S_{mn} = \sum_{p=1}^{m}\sum_{q=1}^{n} u_{pq}$ be the sum of the numbers in the first m rows and first n columns of this array. If there exists a number S such that $\lim_{\substack{m\to\infty \\ n\to\infty}} S_{mn} = S$, we say that the doubles series $\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} u_{pq}$ *converges* to the *sum* S; otherwise, it *diverges*.

Definitions and theorems for double series are very similar to those for series already considered.

5. **Infinite Products.** Let $P_n = (1+u_1)(1+u_2)(1+u_3)\dots(1+u_n)$ denoted by $\prod_{k=1}^{n}(1+u_k)$, where we suppose that $u_k \neq -1, k = 1, 2, 3, \dots$. If there exists a number $P \neq 0$ such that $\lim_{n\to\infty} P_n = P$, we say that the the *infinite product* $((1+u_1)(1+u_2)(1+u_3)\dots = \prod_{k=1}^{\infty}(1+u_k)$, or briefly $\Pi(1+u_k)$, converges to P; otherwise, it diverges.

If $\Pi(1+|u_k|)$ converges, we call the infinite product $\Pi(1+u_k)$ *absolutely convergent*. It can be shown that an absolutely convergent infinite product converges and that factors can in such cases be rearranged without affecting the result.

Theorems about infinite products can (by taking logarithms) often be made to depend on theorems for infinite series. Thus, for example, we have the following theorem.

Theorem. A necessary and sufficient condition that $\Pi(1+u_k)$ converge absolutely is that Σu_k converge absolutely.

6. **Summability.** Let $S_1, S_2, S_3, \ldots$ be the partial sums of a divergent series Σu_n. If the sequence $S_1, \dfrac{S_1, S_2}{2}, \dfrac{S_1 + S_2 + S_3}{3}, \ldots$ (formed by taking arithmetic means of the first n terms of $S_1, S_2, S_3, \ldots$) converges to S, we say that the series Σu_n is *summable* in the *Césaro sense*, or C-1 *summable* to S (see Problem 11.51).

   If Σu_n converges to S, the Césaro method also yields the result S. For this reason the Césaro method is said to be a *regular* method of summability.

   In case the Césaro limit does not exist, we can apply the same technique to the sequence $S_1, \dfrac{S_1 + S_2}{3}, \dfrac{S_1 + S_2 + S_3}{3}, \ldots$. If the C-1 limit for this sequence exists and equals S, we say that Σu_k converges to S in the C-2 sense. The process can be continued indefinitely.

Solved Problems

CONVERGENCE AND DIVERGENCE OF SERIES OF CONSTANTS

11.1. (*a*) Prove that $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ converges and (*b*) find its sum.

$u_n = \dfrac{1}{(2n-1)(2n+1)} = \dfrac{1}{2}\left(\dfrac{1}{2n-1} - \dfrac{1}{2n+1}\right)$. Then

$$\begin{aligned} S_n = u_1 + u_2 + \cdots + u_n &= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \cdots + \frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \end{aligned}$$

Since $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2}$, the series converges and its sum is $\frac{1}{2}$.

The series is sometimes called a *telescoping series*, since the terms of S_n, other than the first and last, cancel out in pairs.

11.2. (*a*) Prove that $\frac{2}{3} + (\frac{2}{3})^2 + (\frac{2}{3})^3 + \cdots = \displaystyle\sum_{n=1}^{\infty} (\tfrac{2}{3})^n$ converges and (*b*) find its sum.

This is a geometric series; therefore, the partial sums are of the form $S_n = \dfrac{a(1-r^n)}{1-r}$. Since $|r| < 1$ $S = \displaystyle\lim_{n\to\infty} S_n = \frac{a}{1-r}$ and in particular with $r = \frac{2}{3}$ and $a = \frac{2}{3}$, we obtain $S = 2$.

11.3. Prove that the series $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

$\displaystyle\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$. Hence by Problem 2.26, Chapter 2, the series is divergent.

11.4. Show that the series whose nth term is $u_n = \sqrt{n+1} - \sqrt{n}$ diverges although $\displaystyle\lim_{n\to\infty} u_n = 0$.

The fact that $\displaystyle\lim_{n\to\infty} u_n = 0$ follows from Problem 2.14(*c*), Chapter 2.

Now $S_n = u_1 + u_2 + \cdots + u_n = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{n+1} - \sqrt{n}) = \sqrt{n+1} - \sqrt{1}$.

Then S_n increases without bound and the series diverges.

This problem shows that $\displaystyle\lim_{n\to\infty} = 0$ is a *necessary* but not *sufficient* condition for the convergence of Σu_n. See also Problem 11.6.

COMPARISON TEST AND QUOTIENT TEST

11.5. If $0 \leqq u_n \leqq v_n$, $n = 1, 2, 3, \ldots$ and if Σv_n converges, prove that Σu_n also converges (i.e., establish the comparison test for convergence).

Let $S_n = u_1 + u_2 + \cdots + u_n$, $T_n = v_1 + v_2 + \cdots + v_n$.

Since Σv_n converges, $\lim\limits_{n\to\infty} T_n$ exists and equals T, say. Also, since $v_n \geqq 0$, $T_n \leqq T$.

Then $S_n = u_1 + u_2 + \cdots + u_n \leqq v_1 + v_2 + \cdots + v_n \leqq T$ or $0 \leqq S_n \leqq T$.

Thus S_n is a bounded monotonic increasing sequence and must have a limit (see Chapter 2), i.e., Σu_n converges.

11.6. Using the comparison test prove that $1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

We have

$$\begin{aligned} 1 &\geqq \tfrac{1}{2} \\ \tfrac{1}{2} + \tfrac{1}{3} &\geqq \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2} \\ \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} &\geqq \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{2} \\ \tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10} + \cdots + \tfrac{1}{15} &\geqq \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \cdots + \tfrac{1}{16} \text{ (8 terms)} = \tfrac{1}{2} \end{aligned}$$

etc. Thus, to any desired number of terms,

$$1 + \left(\tfrac{1}{2} + \tfrac{1}{3}\right) + \left(\tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7}\right) + \cdots \geqq \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdots$$

Since the right-hand side can be made larger than any positive number by choosing enough terms, the given series diverges.

By methods analogous to that used here, we can show that $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$, where p is a constant, diverges if $p \leqq 1$ and converges if $p > 1$. This can also be shown in other ways [see Problem 11.13(*a*)].

11.7. Test for convergence or divergence $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{2n^3 - 1}$.

Since $\ln n < n$ and $\dfrac{1}{2n^3 - 1} \leqq \dfrac{1}{n^3}$, we have $\dfrac{\ln n}{2n^3 - 1} \leqq \dfrac{n}{n^3} = \dfrac{1}{n^2}$.

Then the given series converges, since $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges.

11.8. Let u_n and v_n be positive. If $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \text{constant } A \neq 0$, prove that Σu_n converges or diverges according as Σv_n converges or diverges.

By hypothesis, given $\epsilon > 0$ we can choose an integer N such that $\left|\dfrac{u_n}{v_n} - A\right| < \epsilon$ for all $n > N$. Then for $n = N + 1, N + 2, \ldots$

$$-\epsilon < \frac{u_n}{v_n} - A < \epsilon \qquad \text{or} \qquad (A - \epsilon)v_n < u_n < (A + \epsilon)v_n \tag{1}$$

Summing from $N + 1$ to ∞ (more precisely from $N + 1$ to M and then letting $M \to \infty$),

$$(A - \epsilon)\sum_{N+1}^{\infty} v_n \leqq \sum_{N+1}^{\infty} u_n \leqq (A + \epsilon)\sum_{N+1}^{\infty} v_n \tag{2}$$

There is no loss in generality in assuming $A - \epsilon > 0$. Then from the right-hand inequality of (*2*), Σu_n converges when Σv_n does. From the left-hand inequality of (*2*), Σu_n diverges when Σv_n does. For the cases $A = 0$ or $A = \infty$, see Problem 11.66.

11.9. Test for convergence; (*a*) $\displaystyle\sum_{n=1}^{\infty} \frac{4n^2 - n + 3}{n^3 + 2n}$, (*b*) $\displaystyle\sum_{n=1}^{\infty} \frac{n + \sqrt{n}}{2n^3 - 1}$, (*c*) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2 + 3}$.

(*a*) For large n, $\dfrac{4n^2 - n + 3}{n^3 + 2n}$ is approximately $\dfrac{4n^2}{n^3} = \dfrac{4}{n}$. Taking $u_n = \dfrac{4n^2 - n + 3}{n^3 + 2n}$ and $v_n = \dfrac{4}{n}$, we have $\displaystyle\lim_{n\to\infty} \frac{u_n}{v_n} = 1$.

Since $\Sigma v_n = 4\Sigma 1/n$ diverges, Σu_n also diverges by Problem 11.8.

Note that the purpose of considering the behavior of u_n for large n is to obtain an appropriate comparison series v_n. In the above we could just as well have taken $v_n = 1/n$.

Another method: $\displaystyle\lim_{n\to\infty} n\left(\frac{4n^2 - n + 3}{n^3 + 2n}\right) = 4$. Then by Theorem 1, Page 267, the series converges.

(*b*) For large n, $u_n = \dfrac{n + \sqrt{n}}{2n^3 - 1}$ is approximately $v_n = \dfrac{n}{2n^3} = \dfrac{1}{2n^2}$.

Since $\displaystyle\lim_{n\to\infty} \frac{u_n}{v_n} = 1$ and $\displaystyle\sum v_n = \frac{1}{2}\sum \frac{1}{n^2}$ converges (p series with $p = 2$), the given series converges.

Another method: $\displaystyle\lim_{n\to\infty} n^2\left(\frac{n + \sqrt{n}}{2n^3 - 1}\right) = \frac{1}{2}$. Then by Theorem 1, Page 267, the series converges.

(*c*) $\displaystyle\lim_{n\to\infty} n^{3/2}\left(\frac{\ln n}{n^2 + 3}\right) \leqq \lim_{n\to\infty} n^{3/2}\left(\frac{\ln n}{n^2}\right) = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$ (by L'Hospital's rule or otherwise). Then by Theorem 1 with $p = 3/2$, the series converges.

Note that the method of Problem 11.6(*a*) yields $\dfrac{\ln n}{n^2 + 3} < \dfrac{n}{n^2} = \dfrac{1}{n}$, but nothing can be deduced since $\Sigma 1/n$ diverges.

11.10. Examine for convergence: (*a*) $\displaystyle\sum_{n=1}^{\infty} e^{-n^2}$, (*b*) $\displaystyle\sum_{n=1}^{\infty} \sin^3\left(\frac{1}{n}\right)$.

(*a*) $\displaystyle\lim_{n\to\infty} n^2 e^{-n^2} = 0$ (by L'Hospital's rule or otherwise). Then by Theorem 1 with $p = 2$, the series converges.

(*b*) For large n, $\sin(1/n)$ is approximately $1/n$. This leads to consideration of

$$\lim_{n\to\infty} n^3 \sin^3\left(\frac{1}{n}\right) = \lim_{n\to\infty} \left\{\frac{\sin(1/n)}{1/n}\right\}^3 = 1$$

from which we deduce, by Theorem 1 with $p = 3$, that the given series converges.

INTEGRAL TEST

11.11. Establish the integral test (see Page 267).

We perform the proof taking $N = 1$. Modifications are easily made if $N > 1$.

From the monotonicity of $f(x)$, we have

$$u_{n+1} = f(n+1) \leqq f(x) \leqq f(n) = u_n \qquad n = 1, 2, 3, \ldots$$

Integrating from $x = n$ to $x = n + 1$, using Property 7, Page 92,

$$u_{n+1} \leqq \int_n^{n+1} f(x)\,dx \leqq u_n \qquad n = 1, 2, 3 \ldots$$

Summing from $n = 1$ to $M - 1$,

$$u_2 + u_3 + \cdots + u_M \leqq \int_1^M f(x)\,dx \leqq u_1 + u_2 + \cdots + u_{M-1} \tag{1}$$

If $f(x)$ is strictly decreasing, the equality signs in (*1*) can be omitted.

If $\lim_{M\to\infty} \int_1^M f(x)\,dx$ exists and is equal to S, we see from the left-hand inequality in (*1*) that $u_2 + u_3 + \cdots + u_M$ is monotonic increasing and bounded above by S, so that Σu_n converges.

If $\lim_{M\to\infty} \int_1^M f(x)\,dx$ is unbounded, we see from the right-hand inequality in (*1*) that σu_n diverges.

Thus the proof is complete.

11.12. Illustrate geometrically the proof in Problem 11.11.

Geometrically, $u_2 + u_3 + \cdots + u_M$ is the total area of the rectangles shown shaded in Fig. 11-3, while $u_1 + u_2 + \cdots + u_{M-1}$ is the total area of the rectangles which are shaded and nonshaded.

The area under the curve $y = f(x)$ from $x = 1$ to $x = M$ is intermediate in value between the two areas given above, thus illustrating the result (*1*) of Problem 11.11.

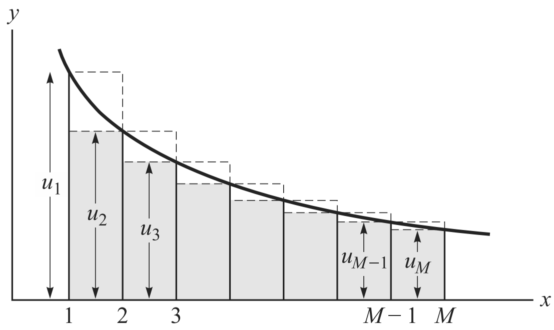

Fig. 11-3

11.13. Test for convergence: (a) $\sum_1^\infty \frac{1}{n^P}$, $p =$ constant; (b) $\sum_1^\infty \frac{n}{n^2+1}$; ($c$) $\sum_2^\infty \frac{1}{n \ln n}$; ($d$) $\sum_1^\infty n e^{-n^2}$.

(a) Consider $\int_1^M \frac{dx}{x^p} = \int_1^M x^{-p}\,dx = \frac{x^{1-p}}{1-p}\bigg|_1^M = \frac{M^{1-p}-1}{1-p}$ where $p \neq 1$.

If $p < 1$, $\lim_{M\to\infty} \frac{M^{1-p}-1}{1-p} = \infty$, so that the integral and thus the series diverges.

If $p > 1$, $\lim_{M\to\infty} \frac{M^{1-p}-1}{1-p} = \frac{1}{p-1}$, so that the integral and thus the series converges.

If $p = 1$, $\int_1^M \frac{dx}{x^p} = \int_1^M \frac{dx}{x} = \ln M$ and $\lim_{M\to\infty} \ln M = \infty$, so that the integral and thus the series diverges.

Thus, the series converges if $p > 1$ and diverges if $p \leqq 1$.

(b) $\lim_{M\to\infty} \int_1^M \frac{x\,dx}{x^2+1} = \lim_{M\to\infty} \frac{1}{2}\ln(x^2+1)|_1^M = \lim_{M\to\infty} \left\{\frac{1}{2}\ln(M^2+1) - \frac{1}{2}\ln 2\right\} = \infty$ and the series diverges.

(c) $\lim_{M\to\infty} \int_2^M \frac{dx}{x \ln x} = \lim_{M\to\infty} \ln(\ln x)|_2^M = \lim_{M\to\infty} \{\ln(\ln M) - \ln(\ln 2)\} = \infty$ and the series diverges.

(d) $\lim_{M\to\infty} \int_1^M x e^{-x^2}\,dx = \lim_{M\to\infty} -\frac{1}{2} e^{-x^2}|_1^M = \lim_{M\to\infty} \left\{\frac{1}{2}e^{-1} - \frac{1}{2}e^{-M^2}\right\} = \frac{1}{2}e^{-1}$ and the series converges.

Note that when the series converges, the value of the corresponding integral is not (in general) the same as the sum of the series. However, the approximate sum of a series can often be obtained quite accurately by using integrals. See Problem 11.74.

11.14. Prove that $\frac{\pi}{4} < \sum_{n=1}^\infty \frac{1}{n^2+1} < \frac{1}{2} + \frac{\pi}{4}$.

From Problem 11.11 it follows that

$$\lim_{M\to\infty}\sum_{n=2}^{M}\frac{1}{n^2+1} < \lim_{M\to\infty}\int_1^M \frac{dx}{x^2+1} < \lim_{M\to\infty}\sum_{n=1}^{M-1}\frac{1}{n^2+1}$$

i.e., $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2+1} < \frac{\pi}{4} < \sum_{n=1}^{\infty}\frac{1}{n^2+1}$, from which $\displaystyle\frac{\pi}{4} < \sum_{n=1}^{\infty}\frac{1}{n^2+1}$ as required.

Since $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2+1} < \frac{\pi}{4}$, we obtain, on adding $\frac{1}{2}$ to each side, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+1} < \frac{1}{2}+\frac{\pi}{4}$.

The required result is therefore proved.

ALTERNATING SERIES

11.15. Given the alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$ where $0 \leqq a_{n+1} \leqq a_n$ and where $\lim_{n\to\infty} a_n = 0$. Prove that (*a*) the series converges, (*b*) the error made in stopping at any term is not greater than the absolute value of the next term.

(*a*) The sum of the series to $2M$ terms is

$$\begin{aligned} S_{2M} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2M-1} - a_{2M}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2M-2} - a_{2M-1}) - a_{2M} \end{aligned}$$

Since the quantities in parentheses are non-negative, we have

$$S_{2M} \geqq 0, \qquad S_2 \leqq S_4 \leqq S_6 \leqq S_8 \leqq \cdots \leqq S_{2M} \leqq a_1$$

Therefore, $\{S_{2M}\}$ is a bounded monotonic increasing sequence and thus has limit S.

Also, $S_{2M+1} = S_{2M} + a_{2M+1}$. Since $\lim_{M\to\infty} S_{2M} = S$ and $\lim_{M\to\infty} a_{2M+1} = 0$ (for, by hypothesis, $\lim_{n\to\infty} a_n = 0$), it follows that $\lim_{M\to\infty} S_{2M+1} = \lim_{M\to\infty} S_{2M} + \lim_{M\to\infty} a_{2M+1} = S + 0 = S$.

Thus, the partial sums of the series approach the limit S and the series converges.

(*b*) The error made in stopping after $2M$ terms is

$$(a_{2M+1} - a_{2M+2}) + (a_{2M+3} - a_{2M+4}) + \cdots = a_{2M+1} - (a_{2M+2} - a_{2M+3}) - \cdots$$

and is thus non-negative and less than or equal to a_{2M+1}, the first term which is omitted.

Similarly, the error made in stopping after $2M+1$ terms is

$$-a_{2M+2} + (a_{2M+3} - a_{2M+4}) + \cdots = -(a_{2M+2} - a_{2M+3}) - (a_{2M+4} - a_{2M+5}) - \cdots$$

which is non-positive and greater than $-a_{2M+2}$.

11.16. (*a*) Prove that the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$ converges. (*b*) Find the maximum error made in approximating the sum by the first 8 terms and the first 9 terms of the series. (*c*) How many terms of the series are needed in order to obtain an error which does not exceed .001 in absolute value?

(*a*) The series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$. If $u_n = \dfrac{(-1)^{n+1}}{2n-1}$, then $a_n = |u_n| = \dfrac{1}{2n-1}$, $a_{n+1} = |u_{n+1}| = \dfrac{1}{2n+1}$. Since $\dfrac{1}{2n+1} \leqq \dfrac{1}{2n-1}$ and since $\displaystyle\lim_{n\to\infty}\frac{1}{2n-1} = 0$, it follows by Problem 11.5(*a*) that the series converges.

(*b*) Use the results of Problem 11.15(*b*). Then the first 8 terms give $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15}$ and the error is positive and does not exceed $\frac{1}{17}$.

Similarly, the first 9 terms are $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17}$ and the error is negative and greater than or equal to $-\frac{1}{19}$, i.e., the error does not exceed $\frac{1}{19}$ in absolute value.

(*c*) The absolute value of the error made in stopping after M terms is less than $1/(2M+1)$. To obtain the desired accuracy, we must have $1/(2M+1) \leqq .001$, from which $M \geqq 499.5$. Thus, at least 500 terms are needed.

ABSOLUTE AND CONDITIONAL CONVERGENCE

11.17. Prove that an absolutely convergent series is convergent.

Given that $\Sigma|u_n|$ converges, we must show that Σu_n converges.

Let $S_M = u_1 + u_2 + \cdots + u_M$ and $T_M = |u_1| + |u_2| + \cdots + |u_M|$. Then

$$\begin{aligned} S_M + T_M &= (u_1 + |u_1|) + (u_2 + |u_2|) + \cdots + (u_M + |u_M|) \\ &\leqq 2|u_1| + 2|u_2| + \cdots + 2|u_M| \end{aligned}$$

Since $\Sigma|u_n|$ converges and since $u_n + |u_n| \geqq 0$, for $n = 1, 2, 3, \ldots$, it follows that $S_M + T_M$ is a bounded monotonic increasing sequence, and so $\lim_{M\to\infty} (S_M + T_M)$ exists.

Also, since $\lim_{M\to\infty} T_M$ exists (since the series is absolutely convergent by hypothesis),

$$\lim_{M\to\infty} S_M = \lim_{M\to\infty} (S_M + T_M - T_M) = \lim_{M\to\infty} (S_M + T_M) - \lim_{M\to\infty} T_M$$

must also exist and the result is proved.

11.18. Investigate the convergence of the series $\dfrac{\sin\sqrt{1}}{1^{3/2}} - \dfrac{\sin\sqrt{2}}{2^{3/2}} + \dfrac{\sin\sqrt{3}}{3^{3/2}} - \cdots$.

Since each term is in absolute value less than or equal to the corresponding term of the series $\dfrac{1}{1^{3/2}} + \dfrac{1}{2^{3/2}} + \dfrac{1}{3^{3/2}} + \cdots$, which converges, it follows that the given series is absolutely convergent and hence convergent by Problem 11.17.

11.19. Examine for convergence and absolute convergence:

(*a*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n^2+1}$, (*b*) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n\ln^2 n}$, (*c*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{n^2}$.

(*a*) The series of absolute values is $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ which is divergent by Problem 11.13(*b*). Hence, the given series is not absolutely convergent.

However, if $a_n = |u_n| = \dfrac{n}{n^2+1}$ and $a_{n+1} = |u_{n+1}| = \dfrac{n+1}{(n+1)^2+1}$, then $a_{n+1} \leqq a_n$ for all $n \geqq 1$, and also $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Hence, by Problem 11.15 the series converges.

Since the series converges but is not absolutely convergent, it is *conditionally convergent*.

(*b*) The series of absolute values is $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln^2 n}$.

By the integral test, this series converges or diverges according as $\displaystyle\lim_{M\to\infty} \int_2^M \frac{dx}{x\ln^2 x}$ exists or does not exist.

If $u = \ln x$, $\displaystyle\int \frac{dx}{x\ln^2 x} = \int \frac{du}{u^2} = -\frac{1}{u} + c = -\frac{1}{\ln x} + c.$

Hence, $\displaystyle\lim_{M\to\infty} \int_2^M \frac{dx}{x\ln^2 x} = \lim_{M\to\infty}\left(\frac{1}{\ln 2} - \frac{1}{\ln M}\right) = \frac{1}{\ln 2}$ and the integral exists. Thus, the series converges.

Then $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n\ln^2 n}$ converges absolutely and thus converges.

Another method:

Since $\dfrac{1}{(n+1)\ln^2(n+1)} \leqq \dfrac{1}{n\ln^2 n}$ and $\lim\limits_{n\to\infty} \dfrac{1}{n\ln^2 n} = 0$, it follows by Problem 11.15(*a*), that the given alternating series converges. To examine its absolute convergence, we must proceed as above.

(*c*) Since $\lim\limits_{n\to\infty} u_n \neq 0$ where $u_n = \dfrac{(-1)^{n-1}2^n}{n^2}$, the given series cannot be convergent. To show that $\lim\limits_{n\to\infty} u_n \neq 0$, it suffices to show that $\lim\limits_{n\to\infty} |u_n| = \lim\limits_{n\to\infty} \dfrac{2^n}{n^2} \neq 0$. This can be accomplished by L'Hospital's rule or other methods [see Problem 11.21(*b*)].

RATIO TEST

11.20. Establish the ratio test for convergence.

Consider first the series $u_1 + u_2 + u_3 + \cdots$ where each term is non-negative. We must prove that if $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = L < 1$, then necessarily Σu_n converges.

By hypothesis, we can choose an integer N so large that for all $n \geqq N$, $(u_{n+1}/u_n) < r$ where $L < r < 1$. Then

$$\begin{aligned} u_{N+1} &< r\,u_N \\ u_{N+2} &< r\,u_{N+1} < r^2\,u_N \\ u_{N+3} &< r\,u_{N+2} < r^3\,u_N \end{aligned}$$

etc. By addition,

$$u_{N+1} + u_{N+2} + \cdots < u_N(r + r^2 + r^3 + \cdots)$$

and so the given series converges by the comparison test, since $0 < r < 1$.

In case the series has terms with mixed signs, we consider $|u_1| + |u_2| + |u_3| + \cdots$. Then by the above proof and Problem 11.17, it follows that if $\lim\limits_{n\to\infty} \left|\dfrac{u_{n+1}}{u_n}\right| = L < 1$, then Σu_n converges (absolutely).

Similarly, we can prove that if $\lim\limits_{n\to\infty} \left|\dfrac{u_{n+1}}{u_n}\right| = L > 1$ the series Σu_n diverges, while if $\lim\limits_{n\to\infty} \left|\dfrac{u_{n+1}}{u_n}\right| = L = 1$ the ratio test fails [see Problem 11.21(*c*)].

11.21. Investigate the convergence of (*a*) $\displaystyle\sum_{n=1}^{\infty} n^4 e^{-n^2}$, (*b*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{n^2}$, (*c*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n^2+1}$.

(*a*) Here $u_n = n^4 e^{-n^2}$. Then

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^4 e^{-(n+1)^2}}{n^4\,e^{-n^2}}\right| = \lim_{n\to\infty} \frac{(n+1)^4\,e^{-(n^2+2n+1)}}{n^4\,e^{-n^2}} \\ &= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^4 e^{-2n-1} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^4 \lim_{n\to\infty} e^{-2n-1} = 1\cdot 0 = 0 \end{aligned}$$

Since $0 < 1$, the series converges.

(*b*) Here $u_n = \dfrac{(-1)^{n-1}2^n}{n^2}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n 2^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-1)^{n-1}2^n}\right| = \lim_{n\to\infty}\frac{2n^2}{(n+1)^2} = 2$$

Since $s > 1$, the series diverges. Compare Problem 11.19(*c*).

(*c*) Here $u_n = \dfrac{(-1)^{n-1}n}{n^2+1}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(n+1)}{(n+1)^2+1}\cdot\frac{n^2+1}{(-1)^{n-1}n}\right| = \lim_{n\to\infty}\frac{(n+1)(n^2+1)}{(n^2+2n+2)n} = 1$$

and the ratio test fails. By using other tests [see Problem 11.19(*a*)], the series is seen to be convergent.

MISCELLANEOUS TESTS

11.22. Test for convergence $1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + \cdots$ where (*a*) $r = 2/3$, (*b*) $r = -2/3$, (*c*) $r = 4/3$.

Here the ratio test is inapplicable, since $\left|\dfrac{u_{n+1}}{u_n}\right| = 2|r|$ or $\frac{1}{2}|r|$ depending on whether n is odd or even.

However, using the nth root test, we have

$$\sqrt[n]{|u_n|} = \begin{cases} \sqrt[n]{2|r^n|} = \sqrt[n]{2}\,|r| & \text{if } n \text{ is odd} \\ \sqrt[n]{|r^n|} = |r| & \text{if } n \text{ is even} \end{cases}$$

Then $\lim_{n\to\infty} \sqrt[n]{|u_n|} = |r|$ (since $\lim_{n\to\infty} 2^{1/n} = 1$).

Thus, if $|r| < 1$ the series converges, and if $|r| > 1$ the series diverges.

Hence, the series converges for cases (*a*) and (*b*), and diverges in case (*c*).

11.23. Test for convergence $\left(\dfrac{1}{3}\right)^2 + \left(\dfrac{1\cdot 4}{3\cdot 6}\right)^2 + \left(\dfrac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\right)^2 + \cdots + \left(\dfrac{1\cdot 4\cdot 7\ldots(3n-2)}{3\cdot 6\cdot 9\ldots(3n)}\right)^2 + \cdots.$

The ratio test fails since $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left(\dfrac{3n+1}{3n+3}\right)^2 = 1$. However, by Raabe's test,

$$\lim_{n\to\infty} n\left(1 - \left|\frac{u_{n+1}}{u_n}\right|\right) = \lim_{n\to\infty} n\left\{1 - \left(\frac{3n+1}{3n+3}\right)^2\right\} = \frac{4}{3} > 1$$

and so the series converges.

11.24. Test for convergence $\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1\cdot 3}{2\cdot 4}\right)^2 + \left(\dfrac{1\cdot 3\cdot 5}{24t}\right)^2 + \cdots + \left(\dfrac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\right)^2 + \cdots.$

The ratio test fails since $\lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left(\dfrac{2n+1}{2n+2}\right)^2 = 1$. Also, Raabe's test fails since

$$\lim_{n\to\infty} n\left(1 - \left|\frac{u_{n+1}}{u_n}\right|\right) = \lim_{n\to\infty} n\left\{1 - \left(\frac{2n+1}{2n+2}\right)^2\right\} = 1$$

However, using long division,

$$\left|\frac{u_{n+1}}{u_n}\right| = \left(\frac{2n+1}{2n+2}\right)^2 = 1 - \frac{1}{n} + \frac{5 - 4/n}{4n^2 + 8n + 4} = 1 - \frac{1}{n} + \frac{c_n}{n^2} \quad \text{where } |c_n| < P$$

so that the series diverges by Gauss' test.

SERIES OF FUNCTIONS

11.25. For what values of x do the following series converge?

$$(a) \quad \sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}, \qquad (b) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}, \qquad (c) \quad \sum_{n=1}^{\infty} n!(x-a)^n, \qquad (d) \quad \sum_{n=1}^{\infty} \frac{n(x-1)^n}{2^n(3n-1)}.$$

(*a*) $u_n = \dfrac{x^{n-1}}{n \cdot 3^n}$. Assuming $x \neq 0$ (if $x = 0$ the series converges), we have

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^n}{(n+1)\cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{x^{n-1}}\right| = \lim_{n\to\infty} \frac{n}{3(n+1)}|x| = \frac{|x|}{3}$$

Then the series converges if $\dfrac{|x|}{3} < 1$, and diverges if $\dfrac{|x|}{3} > 1$. If $\dfrac{|x|}{3} = 1$, i.e., $x = \pm 3$, the test fails.

If $x = 3$ the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

If $x = -3$ the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, which converges.

Then the *interval of convergence* is $-3 \leqq x < 3$. The series diverges outisde this interval.

Note that the series converges absolutely for $-3 < x < 3$. At $x = -3$ the series converges conditionally.

(*b*) Proceed as in part (*a*) with $u_n = \dfrac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$. Then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^n x^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{(-1)^{n-1} x^{2n-1}}\right| = \lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!} x^2$$

$$= \lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} x^2 = \lim_{n\to\infty} \frac{x^2}{(2n+1)(2n)} = 0$$

Then the series converges (absolutely) for all x, i.e., the interval of (absolute) convergence is $-\infty < x < \infty$.

(*c*) $u_n = n!(x-a)^n$, $\displaystyle\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!(x-a)^{n+1}}{n!(x-a)^n}\right| = \lim_{n\to\infty} (n+1)|x-a|$.

This limit is infinite if $x \neq a$. Then the series converges only for $x = a$.

(*d*) $u_n = \dfrac{n(x-1)^n}{2^n(3n-1)}$, $u_{n+1} = \dfrac{(n+1)(x-1)^{n+1}}{2^{n+1}(3n+2)}$. Then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(3n-1)(x-1)}{2n(3n+2)}\right| = \left|\frac{x-1}{2}\right| = \frac{|x-1|}{2}$$

Thus, the series converges for $|x-1| < 2$ and diverges for $|x-1| > 2$.

The test fails for $|x-1| = 2$, i.e., $x - 1 = \pm 2$ or $x = 3$ and $x = -1$.

For $x = 3$ the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3n-1}$, which diverges since the nth term does not approach zero.

For $x = -1$ the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{3n-1}$, which also diverges since the nth term does not approach zero.

Then the series converges only for $|x-1| < 2$, i.e., $-2 < x - 1 < 2$ or $-1 < x < 3$.

11.26. For what values of x does (*a*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{x+2}{x-1}\right)^n$, (*b*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(x+n)(x+n-1)}$ converge?

(*a*) $u_n = \dfrac{1}{2n-1}\left(\dfrac{x+2}{x-1}\right)^n$. Then $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{2n-1}{2n+1}\left|\frac{x+2}{x-1}\right| = \left|\frac{x+2}{x-1}\right|$ if $x \neq 1, -2$.

Then the series converges if $\left|\dfrac{x+2}{x-1}\right| < 1$, diverges if $\left|\dfrac{x+2}{x-1}\right| > 1$, and the test fails if $\left|\dfrac{x+2}{x-1}\right| = 1$, i.e., $x = -\frac{1}{2}$.

If $x = 1$ the series diverges.
If $x = -2$ the series converges.
If $x - \frac{1}{2}$ the series is $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}$ which converges.
Thus, the series converges for $\left|\dfrac{x+2}{x-1}\right| < 1$, $x = -\frac{1}{2}$ and $x = -2$, i.e., for $x \leqq -\frac{1}{2}$.

(*b*) The ratio test fails since $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$, where $u_n = \dfrac{1}{(x+n)(x+n-1)}$. However, noting that

$$\frac{1}{(x+n)(x+n-1)} = \frac{1}{x+n-1} - \frac{1}{x+n}$$

we see that if $x \neq 0, -1, -2, \ldots, -n$,

$$\begin{aligned} S_n = u_1 + u_2 + \cdots + u_n &= \left(\frac{1}{x} - \frac{1}{x+1}\right) + \left(\frac{1}{x+1} - \frac{1}{x+2}\right) + \cdots + \left(\frac{1}{x+n-1} - \frac{1}{x+n}\right) \\ &= \frac{1}{x} - \frac{1}{x+n} \end{aligned}$$

and $\displaystyle\lim_{n\to\infty} S_n = 1/x$, provided $x \neq 0, -1, -2, -3, \ldots$.
Then the series converges for all x except $x = 0, -1, -2, -3, \ldots$, and its sum is $1/x$.

UNIFORM CONVERGENCE

11.27. Find the domain of convergence of $(1-x) + x(1-x) + x^2(1-x) + \cdots$.

Method 1:

$$\begin{aligned} \text{Sum of first } n \text{ terms} = S_n(x) &= (1-x) + x(1-x) + x^2(1-x) + \cdots + x^{n-1}(1-x) \\ &= 1 - x + x - x^2 + x^2 + \cdots + x^{n-1} - x^n \\ &= 1 - x^n \end{aligned}$$

If $|x| < 1$, $\displaystyle\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}(1 - x^n) = 1$.

If $|x| > 1$, $\displaystyle\lim_{n\to\infty} S_n(x)$ does not exist.

If $x = 1$, $S_n(x) = 0$ and $\displaystyle\lim_{n\to\infty} S_n(x) = 0$.

If $x = -1$, $S_n(x) = 1 - (-1)^n$ and $\displaystyle\lim_{n\to\infty} S_n(x)$ does not exist.

Thus, the series converges for $|x| < 1$ and $x = 1$, i.e., for $-1 < x \leqq 1$.

Method 2, using the ratio test.
The series converges if $x = 1$. If $x \neq 1$ and $u_n = x^{n-1}(1-x)$, then $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}|x|$.

Thus, the series converges if $|x| < 1$, diverges if $|x| > 1$. The test fails if $|x| = 1$. If $x = 1$, the series converges; if $x = -1$, the series diverges. Then the series converges for $-1 < x \leqq 1$.

11.28. Investigate the uniform convergence of the series of Problem 11.27 in the interval (*a*) $-\frac{1}{2} < x < \frac{1}{2}$, (*b*) $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$, (*c*) $-.99 \leqq x \leqq .99$, (*d*) $-1 < x < 1$, (*e*) $0 \leqq x < 2$.

(*a*) By Problem 11.27, $S_n(x) = 1 - x^n$, $S(x) = \lim_{n\to\infty} S_n(x) = 1$ if $-\frac{1}{2} < x < \frac{1}{2}$; thus, the series *converges* in this interval. We have

$$\text{Remainder after } n \text{ terms} = R_n(x) = S(x) - S_n(x) = 1 - (1 - x^n) = x^n$$

The series is *uniformly convergent* in the interval if given any $\epsilon > 0$ we can find N dependent on ϵ, *but not on* x, such that $|R_n(x)| < \epsilon$ for all $n > N$. Now

$$|R_n(x)| = |x^n| = |x|^n < \epsilon \quad \text{when} \quad n \ln|x| < \ln\epsilon \quad \text{or} \quad n > \frac{\ln\epsilon}{\ln|x|}$$

since division by $\ln|x|$ (which is negative since $|x| < \frac{1}{2}$) reverses the sense of the inequality.

But if $|x| < \frac{1}{2}$, $\ln|x| < \ln(\frac{1}{2})$, and $n > \dfrac{\ln\epsilon}{\ln|x|} > \dfrac{\ln\epsilon}{\ln(\frac{1}{2})} = N$. Thus, since N is independent of x, the series is uniformly convergent in the interval.

(*b*) In this case $|x| \leqq \frac{1}{2}$, $\ln|x| \leqq \ln(\frac{1}{2})$, and $n > \dfrac{\ln\epsilon}{\ln|x|} \geqq \dfrac{\ln\epsilon}{\ln(\frac{1}{2})} = N$, so that the series is also uniformly convergent in $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$.

(*c*) Reasoning similar to the above, with $\frac{1}{2}$ replaced by .99, shows that the series is uniformly convergent in $-.99 \leqq x \leqq .99$.

(*d*) The arguments used above break down in this case, since $\dfrac{\ln\epsilon}{\ln|x|}$ can be made larger than any positive number by choosing $|x|$ sufficiently close to 1. Thus, no N exists and it follows that the series is *not* uniformly convergent in $-1 < x < 1$.

(*e*) Since the series does not even converge at all points in this interval, it cannot converge uniformly in the interval.

11.29. Discuss the continuity of the sum function $S(x) = \lim_{n\to\infty} S_n(x)$ of Problem 11.27 for the interval $0 \leqq x \leqq 1$.

If $0 \leqq x < 1$, $S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}(1 - x^n) = 1$.

If $x = 1$, $S_n(x) = 0$ and $S(x) = 0$.

Thus, $S(x) = \begin{cases} 1 & \text{if } 0 \leqq x < 1 \\ 0 & \text{if } x = 1 \end{cases}$ and $S(x)$ is discontinuous at $x = 1$ but continuous at all other points in $0 \leqq x < 1$.

In Problem 11.34 it is shown that if a series is uniformly convergent in an interval, the sum function $S(x)$ must be continuous in the interval. It follows that if the sum function is not continuous in an interval, the series cannot be uniformly convergent. This fact is often used to demonstrate the nonuniform convergence of a series (or sequence).

11.30. Investigate the uniform convergence of $x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + \cdots + \dfrac{x^2}{(1+x^2)^n} + \cdots$.

Suppose $x \neq 0$. Then the series is a geometric series with ratio $1/(1+x^2)$ whose sum is (see Problem 2.25, Chap. 2).

$$S(x) = \frac{x^2}{1 - 1/(1+x^2)} = 1 + x^2$$

If $x = 0$ the sum of the first n terms is $S_n(0) = 0$; hence $S(0) = \lim_{n\to\infty} S_n(0) = 0$.

Since $\lim_{x\to 0} S(x) = 1 \neq S(0)$, $S(x)$ is discontinuous at $x = 0$. Then by Problem 11.34, the series cannot be *uniformly convergent* in any interval which includes $x = 0$, although it is (absolutely) *convergent* in any interval. However, it is uniformly convergent in any interval which excludes $x = 0$.

This can also be shown directly (see Problem 11.93).

WEIERSTRASS *M* TEST

11.31. Prove the Weierstrass M test, i.e., if $|u_n(x)| \leqq M_n, n = 1, 2, 3, \ldots,$ where M_n are positive constants such that ΣM_n converges, then $\Sigma u_n(x)$ is uniformly (and absolutely) convergent.

The remainder of the series $\Sigma u_n(x)$ after n terms is $R_n(x) = u_{n+1}(x) + u_{n+2}(x) + \cdots$. Now

$$|R_n(x)| = |u_{n+1}(x) + u_{n+2}(x) + \cdots| \leqq |u_{n+1}(x)| + |u_{n+2}(x)| + \cdots \leqq M_{n+1} + M_{n+2} + \cdots$$

But $M_{n+1} + M_{n+2} + \cdots$ can be made less than ϵ by choosing $n > N$, since ΣM_n converges. Since N is clearly independent of x, we have $|R_n(x)| < \epsilon$ for $n > N$, and the series is uniformly convergent. The absolute convergence follows at once from the comparison test.

11.32. Test for uniform convergence:

$$(a)\ \sum_{n=1}^{\infty} \frac{\cos nx}{n^4}, \quad (b)\ \sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}}, \quad (c)\ \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \quad (d)\ \sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}.$$

(*a*) $\left|\dfrac{\cos nx}{n^4}\right| \leqq \dfrac{1}{n^4} = M_n$. Then since ΣM_n converges (p series with $p = 4 > 1$), the series is uniformly (and absolutely) convergent for all x by the *M test*.

(*b*) By the ratio test, the series converges in the interval $-1 \leqq x \leqq 1$, i.e., $|x| \leqq 1$.

For all x in this interval, $\left|\dfrac{x^n}{n^{3/2}}\right| = \dfrac{|x|^n}{n^{3/2}} \leqq \dfrac{1}{n^{3/2}}$. Choosing $M_n = \dfrac{1}{n^{3/2}}$, we see that ΣM_n converges. Thus, the given series converges uniformly for $-1 \leqq x \leqq 1$ by the M test.

(*c*) $\left|\dfrac{\sin nx}{n}\right| \leqq \dfrac{1}{n}$. However, ΣM_n, where $M_n = \dfrac{1}{n}$, does not converge. The M test cannot be used in this case and we cannot conclude anything about the uniform convergence by this test (see, however, Problem 11.125).

(*d*) $\left|\dfrac{1}{n^2 + x^2}\right| \leqq \dfrac{1}{n^2}$, and $\Sigma \dfrac{1}{n^2}$ converges. Then by the M test the given series converges uniformly for all x.

11.33. If a power series $\Sigma a_n x^n$ converges for $x = x_0$, prove that it converges (*a*) absolutely in the interval $|x| < |x_0|$, (*b*) uniformly in the interval $|x| \leqq |x_1|$, where $|x_1| < |x_0|$.

(*a*) Since $\Sigma a_n x_0^n$ converges, $\lim_{n\to\infty} a_n x_0^n = 0$ and so we can make $|a_n x_0^n| < 1$ by choosing n large enough, i.e., $|a_n| < \dfrac{1}{|x_0|^n}$ for $n > N$. Then

$$\sum_{N+1}^{\infty} |a_n x^n| = \sum_{N+1}^{\infty} |a_n||x|^n < \sum_{N+1}^{\infty} \frac{|x|^n}{|x_0|^n} \tag{1}$$

Since the last series in (*1*) converges for $|x| < |x_0|$, it follows by the comparison test that the first series converges, i.e., the given series is absolutely convergent.

(*b*) Let $M_n = \dfrac{|x_1|^n}{|x_0|^n}$. Then ΣM_n converges since $|x_1| < |x_0|$. As in part (*a*), $|a_n x^n| < M_n$ for $|x| \leqq |x_1|$, so that by the Weierstrass M test, $\Sigma a_n x^n$ is uniformly convergent.

It follows that a power series is uniformly convergent in any interval *within* its interval of convergence.

THEOREMS ON UNIFORM CONVERGENCE

11.34. Prove Theorem 6, Page 271.

We must show that $S(x)$ is continuous in $[a, b]$.

Now $S(x) = S_n(x) + R_n(x)$, so that $S(x+h) = S_n(x+h) + R_n(x+h)$ and thus

$$S(x+h) - S(x) = S_n(x+h) - S_n(x) + R_n(x+h) - R_n(x) \tag{1}$$

where we choose h so that both x and $x+h$ lie in $[a, b]$ (if $x = b$, for example, this will require $h < 0$).

Since $S_n(x)$ is a sum of finite number of continuous functions, it must also be continuous. Then given $\epsilon > 0$, we can find δ so that

$$|S_n(x+h) - S_n(x)| < \epsilon/3 \qquad \text{whenever } |h| < \delta \tag{2}$$

Since the series, by hypothesis, is uniformly convergent, we can choose N so that

$$|R_n(x)| < \epsilon/3 \qquad \text{and} \qquad |R_n(x+h)| < \epsilon/3 \qquad \text{for } n > N \tag{3}$$

Then from (1), (2), and (3),

$$|S(x+h) - S(x)| \leqq |S_n(x+h) - S_n(x)| + |R_n(x+h)| + |R_n(x)| < \epsilon$$

for $|h| < \delta$, and so the continuity is established.

11.35. Prove Theorem 7, Page 271.

If a function is continuous in $[a, b]$, its integral exists. Then since $S(x)$, $S_n(x)$, and $R_n(x)$ are continuous,

$$\int_a^b S(x) = \int_a^b S_n(x)\,dx + \int_a^b R_n(x)\,dx$$

To prove the theorem we must show that

$$\left|\int_a^b S(x)\,dx - \int_a^b S_n(x)\,dx\right| = \left|\int_a^b R_n(x)\,dx\right|$$

can be made arbitrarily small by choosing n large enough. This, however, follows at once, since by the uniform convergence of the series we can make $|R_n(x)| < \epsilon/(b-a)$ for $n > N$ independent of x in $[a, b]$, and so

$$\left|\int_a^b R_n(x)\,dx\right| \leqq \int_a^b |R_n(x)|\,dx < \int_a^b \frac{\epsilon}{b-a}\,dx = \epsilon$$

This is equivalent to the statements

$$\int_a^b S(x)\,dx = \lim_{n\to\infty} \int_a^b S_n(x)\,dx \qquad \text{or} \qquad \lim_{n\to\infty} \int_a^b S_n(x)\,dx = \int_a^b \left\{\lim_{n\to\infty} S_n(x)\right\} dx$$

11.36. Prove Theorem 8, Page 271.

Let $g(x) = \sum_{n=1}^{\infty} u_n'(x)$. Since, by hypothesis, this series converges uniformly in $[a, b]$, we can integrate term by term (by Problem 11.35) to obtain

$$\begin{aligned}\int_a^x g(x)\,dx &= \sum_{n=1}^{\infty} \int_a^x u_n'(x)\,dx = \sum_{n=1}^{\infty} \{u_n(x) - u_n(a)\} \\ &= \sum_{n=1}^{\infty} u_n(x) - \sum_{n=1}^{\infty} u_n(a) = S(x) - S(a)\end{aligned}$$

because, by hypothesis, $\sum_{n=1}^{\infty} u_n(x)$ converges to $S(x)$ in $[a, b]$.

Differentiating both sides of $\int_a^x g(x)\,dx = S(x) - S(a)$ then shows that $g(x) = S'(x)$, which proves the theorem.

11.37. Let $S_n(x) = nxe^{-nx^2}$, $n = 1, 2, 3, \ldots, 0 \leqq x \leqq 1$.

(a) Determine whether $\lim\limits_{n\to\infty} \int_0^1 S_n(x)\,dx = \int_0^1 \lim\limits_{n\to\infty} S_n(x)\,dx$.

(b) Explain the result in (a).

(a) $\int_0^1 s_n(x)\,dx = \int_0^1 nxe^{-nx^2}\,dx = -\frac{1}{2}e^{-nx^2}|_0^1 = \frac{1}{2}(1 - e^{-n})$. Then

$$\lim_{n\to\infty} \int_0^1 S_n(x)\,dx = \lim_{n\to\infty} \tfrac{1}{2}(1 - e^{-n}) = \tfrac{1}{2}$$

$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} nxe^{-nx^2} = 0, \text{ whether } x = 0 \text{ or } 0 < x \leqq 1. \quad \text{Then,}$$

$$\int_0^1 S(x)\,dx = 0$$

It follows that $\lim\limits_{n\to\infty} \int_0^1 S_n(x)\,dx \neq \int_0^1 \lim\limits_{n\to\infty} S_n(x)\,dx$, i.e., the limit cannot be taken under the integral sign.

(b) The reason for the result in (a) is that although the sequence $S_n(x)$ converges to 0, it does not converge *uniformly* to 0. To show this, observe that the function nxe^{-nx^2} has a maximum at $x = 1/\sqrt{2n}$ (by the usual rules of elementary calculus), the value of this maximum being $\sqrt{\frac{1}{2}n}\,e^{-1/2}$. Hence, as $n \to \infty$, $S_n(x)$ cannot be made arbitrarily small *for all* x and so cannot converge *uniformly* to 0.

11.38. Let $f(x) = \sum\limits_{n=1}^{\infty} \dfrac{\sin nx}{n^3}$. Prove that $\int_0^{\pi} f(x)\,dx = 2\sum\limits_{n=1}^{\infty} \dfrac{1}{(2n-1)^4}$.

We have $\left|\dfrac{\sin nx}{n^3}\right| \leqq \dfrac{1}{n^3}$. Then by the Weierstrass M test the series is uniformly convergent for all x, in particular $0 \leqq x \leqq \pi$, and can be integrated term by term. Thus

$$\begin{aligned}\int_0^{\pi} f(x)\,dx &= \int_0^{\pi}\left(\sum_{n=1}^{\infty}\frac{\sin nx}{n^3}\right)dx = \sum_{n=1}^{\infty}\int_0^{\pi}\frac{\sin nx}{n^3}\,dx \\ &= \sum_{n=1}^{\infty}\frac{1 - \cos n\pi}{n^4} = 2\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) = 2\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\end{aligned}$$

POWER SERIES

11.39. Prove that both the power series $\sum\limits_{n=0}^{\infty} a_n x^n$ and the corresponding series of derivatives $\sum\limits_{n=0}^{\infty} na_n x^{n-1}$ have the same radius of convergence.

Let $R > 0$ be the radius of convergence of $\Sigma a_n x^n$. Let $0 < |x_0| < R$. Then, as in Problem 11.33, we can choose N as that $|a_n| < \dfrac{1}{|x_0|^n}$ for $n > N$.

Thus, the terms of the series $\Sigma|na_n x^{n-1}| = \Sigma\, n|a_n||x|^{n-1}$ can for $n > N$ be made less than corresponding terms of the series $\Sigma\, n\dfrac{|x|^{n-1}}{|x_0|^n}$, which converges, by the ratio test, for $|x| < |x_0| < R$.

Hence, $\Sigma na_n x^{n-1}$ converges absolutely for all points x_0 (no matter how close $|x_0|$ is to R).

If, however, $|x| > R$, $\lim\limits_{n\to\infty} a_n x^n \neq 0$ and thus $\lim\limits_{n\to\infty} na_n x^{n-1} \neq 0$, so that $\Sigma na_n x^{n-1}$ does not converge.

Thus, R is the radius of convergence of $\Sigma n a_n x^{n-1}$.

Note that the series of derivatives may or may not converge for values of x such that $|x| = R$.

11.40. Illustrate Problem 11.39 by using the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2 \cdot 3^n}$.

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2 \cdot 3^{n+1}} \cdot \frac{n^2 \cdot 3^n}{x^n}\right| = \lim_{n\to\infty} \frac{n^2}{3(n+1)^2}|x| = \frac{|x|}{3}$$

so that the series converges for $|x| < 3$. At $x = \pm 3$ the series also converges, so that the interval of convergence is $-3 \leqq x \leqq 3$.

The series of derivatives is

$$\sum_{n=1}^{\infty} \frac{n x^{n-1}}{n^2 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}$$

By Problem 11.25(a) this has the interval of convergence $-3 \leqq x < 3$.

The two series have the same radius of convergence, i.e., $R = 3$, although they do not have the same interval of convergence.

Note that the result of Problem 11.39 can also be proved by the ratio test *if* this test is applicable. The proof given there, however, applies even when the test is not applicable, as in the series of Problem 11.22.

11.41. Prove that in any interval *within* its interval of convergence a power series
(a) represents a continuous function, say, $f(x)$,
(b) can be integrated term by term to yield the integral of $f(x)$,
(c) can be differentiated term by term to yield the derivative of $f(x)$.

We consider the power series $\Sigma a_n x^n$, although analogous results hold for $\Sigma a_n (x-a)^n$.

(a) This follows from Problem 11.33 and 11.34, and the fact that each term $a_n x^n$ of the series is continuous.

(b) This follows from Problems 11.33 and 11.35, and the fact that each term $a_n x^n$ of the series is continuous and thus integrable.

(c) From Problem 11.39, the series of derivatives of a power series always converges within the interval of convergence of the original power series and therefore is uniformly convergent within this interval. Thus, the required result follows from Problems 11.33 and 11.36.

If a power series converges at one (or both) end points of the interval of convergence, it is possible to establish (a) and (b) to include the end point (or end points). See Problem 11.42.

11.42. Prove Abel's theroem that if a power series converges at an end point of its interval of convergence, then the interval of uniform convergence includes this end point.

For simplicity in the proof, we assume the power series to be $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ with the end point of its interval of convergence at $x = 1$, so that the series surely converges for $0 \leqq x \leqq 1$. Then we must show that the series converges uniformly in this interval.

Let

$$R_n(x) = a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \cdots, \qquad R_n = a_n + a_{n+1} + a_{n+2} + \cdots$$

To prove the required result we must show that given any $\epsilon > 0$, we can find N such that $|R_n(x)| < \epsilon$ for all $n > N$, where N is independent of the particular x in $0 \leqq x \leqq 1$.

Now

$$\begin{aligned} R_n(x) &= (R_n - R_{n+1})x^n + (R_{n+1} - R_{n+2})x^{n+1} + (R_{n+2} - R_{n+3})x^{n+2} + \cdots \\ &= R_n x^n + R_{n+1}(x^{n+1} - x^n) + R_{n+2}(x^{n+2} - x^{n+1}) + \cdots \\ &= x^n\{R_n - (1-x)(R_{n+1} + R_{n+2}x + R_{n+3}x^2 + \cdots)\} \end{aligned}$$

Hence, for $0 \leqq x < 1$,

$$|R_n(x)| \leqq |R_n| + (1-x)(|R_{n+1}| + |R_{n+2}|x + |R_{n+3}|x^2 + \cdots) \tag{1}$$

Since Σa_k converges by hypothesis, it follows that given $\epsilon > 0$ we can choose N such that $|R_k| < \epsilon/2$ for all $k \geqq n$. Then for $n > N$ we have from (*1*),

$$|R_n(x)| \leqq \frac{\epsilon}{2} + (1-x)\left(\frac{\epsilon}{2} + \frac{\epsilon}{2}x + \frac{\epsilon}{2}x^2 + \cdots\right) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{2}$$

since $(1-x)(1 + x + x^2 + x^3 + \cdots) = 1$ (if $0 \leqq x < 1$).

Also, for $x = 1$, $|R_n(x)| = |R_n| < \epsilon$ for $n > N$.

Thus, $|R_n(x)| < \epsilon$ for all $n > N$, where N is independent of the value of x in $0 \leqq x \leqq 1$, and the required result follows.

Extensions to other power series are easily made.

11.43. Prove Abel's limit theorem (see Page 272).

As in Problem 11.42, assume the power series to be $\sum_{k=1}^{\infty} a_k x^k$, convergent for $0 \leqq x \leqq 1$.

Then we must show that $\lim_{x \to 1-} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k$.

This follows at once from Problem 11.42, which shows that $\Sigma a_k x^k$ is uniformly convergent for $0 \leqq x \leqq 1$, and from Problem 11.34, which shows that $\Sigma a_k x^k$ is continuous at $x = 1$.

Extensions to other power series are easily made.

11.44. (*a*) Prove that $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$ where the series is uniformly convergent in $-1 \leqq x \leqq 1$.

(*b*) Prove that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$.

(*a*) By Problem 2.25 of Chapter 2, with $r = -x^2$ and $a = 1$, we have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \qquad -1 < x < 1 \tag{1}$$

Integrating from 0 to x, where $-1 < x < 1$, yields

$$\int_0^x \frac{dx}{1+x^2} = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \tag{2}$$

using Problems 11.33 and 11.35.

Since the series on the right of (*2*) converges for $x = \pm 1$, it follows by Problem 11.42 that the series is uniformly convergent in $-1 \leqq x \leqq 1$ and represents $\tan^{-1} x$ in this interval.

(*b*) By Problem 11.43 and part (*a*), we have

$$\lim_{x \to 1-} \tan^{-1} x = \lim_{x \to 1-}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) \qquad \text{or} \qquad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

11.45. Evaluate $\displaystyle\int_0^1 \frac{1 - e^{-x^2}}{x^2}\,dx$ to 3 decimal place accuracy.

We have $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \cdots, \quad -\infty < u < \infty.$

Then if $u = -x^2$, $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{3!} = \frac{x^{10}}{5!} + \cdots, \quad -\infty < x < \infty.$

Thus $\frac{1 - e^{-x^2}}{x^2} = 1 - \frac{x^2}{2!} + \frac{x^4}{3!} - \frac{x^6}{4!} + \frac{x^8}{5!} - \cdots.$

Since the series converges for all x and so, in particular, converges uniformly for $0 \leqq x \leqq 1$, we can integrate term by term to obtain

$$\begin{aligned}
\int_0^1 \frac{1 - e^{-x^2}}{x^2}\, dx &= x - \frac{x^3}{3 \cdot 2!} + \frac{x^5}{5 \cdot 3!} - \frac{x^7}{7 \cdot 4!} + \frac{x^9}{9 \cdot 5!} - \cdots \Big|_0^1 \\
&= 1 - \frac{1}{3 \cdot 2!} + \frac{1}{5 \cdot 3!} - \frac{1}{7 \cdot 4!} + \frac{1}{9 \cdot 5!} - \cdots \\
&= 1 - 0.16666 + 0.03333 - 0.00595 + 0.00092 - \cdots = 0.862
\end{aligned}$$

Note that the error made in adding the first four terms of the alternating series is less than the fifth term, i.e., less than 0.001 (see Problem 11.15).

MISCELLANEOUS PROBLEMS

11.46. Prove that $y = J_p(x)$ defined by (*16*), Page 276, satisfies Bessel's differential equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

The series for $J_p(x)$ converges for all x [see Problem 11.110(*a*)]. Since a power series can be differentiated term by term within its interval of convergence, we have for all x,

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{p+2n}}{2^{p+2n} n!(n+p)!} \\
y' &= \sum_{n=0}^{\infty} \frac{(-1)^n (p+2n) x^{p+2n-1}}{2^{p+2n} n!(n+p)!} \\
y'' &= \sum_{n=0}^{\infty} \frac{(-1)^n (p+2n)(p+2n-1)\, x^{p+2n-2}}{2^{p+2n} n!(n+p)!}
\end{aligned}$$

Then,

$$\begin{aligned}
(x^2 - p^2)y &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{p+2n+2}}{2^{p+2n} n!(n+p)!} - \sum_{n=0}^{\infty} \frac{(-1)^n p^2 x^{p+2n}}{2^{p+2n} n!(n+p)!} \\
xy' &= \sum_{n=0}^{\infty} \frac{(-1)^n (p+2n) x^{p+2n}}{2^{p+2n} n!(n+p)!} \\
x^2 y'' &= \sum_{n=0}^{\infty} \frac{(-1)^n (p+2n)(p+2n-1) x^{p+2n}}{2^{p+2n} n!(n+p)!}
\end{aligned}$$

Adding,

$$\begin{aligned}
x^2y'' + xy' + (x^2 - p^2)y &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{p+2n+2}}{2^{p+2n} n!(n+p)!} \\
&\quad + \sum_{n=0}^{\infty} \frac{(-1)^n[-p^2 + (p+2n) + (p+2n)(p+2n-1)]x^{p+2n}}{2^{p+2n} n!(n+p)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{p+2n+2}}{2^{p+2n} n!(n+p)!} + \sum_{n=0}^{\infty} \frac{(-1)^n[4n(n+p)]x^{p+2n}}{2^{p+2n} n!(n+p)!} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{p+2n}}{2^{p+2n-2}(n-1)!(n-1+p)!} + \sum_{n=1}^{\infty} \frac{(-1)^n 4x^{p+2n}}{2^{p+2n}(n-1)!(n+p-1)!} \\
&= -\sum_{n=1}^{\infty} \frac{(-1)^n 4x^{p+2n}}{2^{p+2n}(n-1)!(n+p-1)!} + \sum_{n=1}^{\infty} \frac{(-1)^n 4x^{p+2n}}{2^{p+2n}(n-1)!(n+p-1)!} \\
&= 0
\end{aligned}$$

11.47. Test for convergence the complex power series $\displaystyle\sum_{n=1}^{\infty} \frac{z^{n-1}}{n^3 \cdot 3^{n-1}}$.

Since $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{z^n}{(n+1)^3 \cdot 3^n} \cdot \frac{n^3 \cdot 3^{n-1}}{z^{n-1}}\right| = \lim_{n\to\infty} \frac{n^3}{3(n+1)^3}|z| = \frac{|z|}{3}$, the series converges for $\dfrac{|z|}{3} < 1$, i.e., $|z| < 3$, and diverges for $|z| > 3$.

For $|z| = 3$, the series of absolute values is $\displaystyle\sum_{n=1}^{\infty} \frac{|z|^{n-1}}{n^3 \cdot 3^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{n^3}$, so that the series is absolutely convergent and thus convergent for $|z| = 3$.

Thus, the series converges within and on the circle $|z| = 3$.

11.48. Assuming the power series for e^x holds for complex numbers, show that

$$e^{ix} = \cos x + i \sin x$$

Letting $z = ix$ in $e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots$, we have

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \cdots = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= \cos x + i \sin x
\end{aligned}$$

Similarly, $e^{-ix} = \cos x - i \sin x$. The results are called *Euler's identities*.

11.49. Prove that $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n\right)$ exists.

Letting $f(x) = 1/x$ in (*1*), Problem 11.11, we find

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{M} \leqq \ln M \leqq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{M-1}$$

from which we have on replacing M by n,

$$\frac{1}{n} \leqq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n \leqq 1$$

Thus, the sequence $S_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n} - \ln n$ is bounded by 0 and 1.

Consider $S_{n+1} - S_n = \dfrac{1}{n+1} - \ln\left(\dfrac{n+1}{n}\right)$. By integrating the inequality $\dfrac{1}{n+1} \leqq \dfrac{1}{x} \leqq \dfrac{1}{n}$ with respect to x from n to $n+1$, we have

$$\frac{1}{n+1} \leqq \ln\left(\frac{n+1}{n}\right) \leqq \frac{1}{n} \qquad \text{or} \qquad \frac{1}{n+1} - \frac{1}{n} \leqq \frac{1}{n+1} - \ln\left(\frac{n+1}{n}\right) \leqq 0$$

i.e., $S_{n+1} - S_n \leqq 0$, so that S_n is monotonic decreasing.

Since S_n is bounded and monotonic decreasing, it has a limit. This limit, denoted by γ, is equal to $0.577215\ldots$ and is called *Euler's constant*. It is not yet known whether γ is rational or not.

11.50. Prove that the infinite product $\prod_{k=1}^{\infty}(1+u_k)$, where $u_k > 0$, converges if $\sum_{k=1}^{\infty} u_k$ converges.

According to the Taylor series for e^x (Page 275), $1 + x \leqq e^x$ for $x > 0$, so that

$$P_n = \prod_{k=1}^{n}(1+u_k) = (1+u_1)(1+u_2)\cdots(1+u_n) \leqq e^{u_1} \cdot e^{u_2} \cdots e^{u_n} = e^{u_1+u_2+\cdots+u_n}$$

Since $u_1 + u_2 + \cdots$ converges, it follows that P_n is a bounded monotonic increasing sequence and so has a limit, thus proving the required result.

11.51. Prove that the series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ is $C - 1$ summable to $1/2$.

The sequence of partial sums is $1, 0, 1, 0, 1, 0, \ldots$.

Then $S_1 = 1, \dfrac{S_1 + S_2}{2} = \dfrac{1+0}{2} = \dfrac{1}{2}, \dfrac{S_1 + S_2 + S_3}{3} = \dfrac{1+0+1}{3} = \dfrac{2}{3}, \ldots$.

Continuing in this manner, we obtain the sequence $1, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{3}{5}, \frac{1}{2}, \ldots$, the nth term being $T_n = \begin{cases} 1/2 & \text{if } n \text{ is even} \\ n/(2n-1) & \text{if } n \text{ is odd} \end{cases}$. Thus, $\lim_{n\to\infty} T_n = \frac{1}{2}$ and the required result follows.

11.52. (*a*) If $f^{(n+1)}(x)$ is continuous in $[a, b]$ prove that for c in $[a, b]$, $f(x) = f(c) + f'(c)(x-c) + \dfrac{1}{2!}f''(c)(x-c)^2 + \cdots + \dfrac{1}{n!}f^{(n)}(c)(x-c)^n + \dfrac{1}{n!}\displaystyle\int_c^x (x-t)^n f^{(n+1)}(t)\,dt$.

(*b*) Obtain the Lagrange and Cauchy forms of the remainder in Taylor's Formula. (See Page 274.)

The proof of (*a*) is made using mathematical induction. (See Chapter 1.) The result holds for $n = 0$ since

$$f(x) = f(c) + \int_C^x f'(t)\,dt = f(c) + f(x) - f(c)$$

We make the induction assumption that it holds for $n = k$ and then use integration by parts with

$$dv = \frac{(x-t)^k}{k!}\,dt \text{ and } u = f^{k+1}(t)$$

Then

$$v = -\frac{(x-t)^{k+1}}{(k+1)!} \qquad \text{and} \qquad du = f^{k+2}(t)\,dt$$

Thus,

$$\begin{aligned}\frac{1}{k!}\int_C^x (x-t)^k f^{(k+1)}(t)\,dt &= -\frac{f^{k+1}(t)(x-t)^{k+1}}{(k+1)!}\bigg|_C^x + \frac{1}{(k+1)!}\int_C^x (x-t)^{k+1} f^{(k+2)}(t)\,dt \\ &= \frac{f^{k+1}(c)(x-c)^{k+1}}{(k+1)!} + \frac{1}{(k+1)!}\int_C^x (x-t)^{k+1} f^{(k+2)}(t)\,dt\end{aligned}$$

Having demonstrated that the result holds for $k+1$, we conclude that it holds for all positive integers.

To obtain the Lagrange form of the remainder R_n, consider the form

$$f(x) = f(c) + f'(c)(x-c) + \frac{1}{2!}f''(c)(x-c)^2 + \cdots + \frac{K}{n!}(x-c)^n$$

This is the Taylor polynomial $P_{n-1}(x)$ plus $\frac{K}{n!}(x-c)^n$. Also, it could be looked upon as P_n except that in the last term, $f^{(n)}(c)$ is replaced by a number K such that for fixed c and x the representation of $f(x)$ is exact. Now define a new function

$$\Phi(t) = f(t) - f(x) + \sum_{j=1}^{n-1} f^{(j)}(t)\frac{(x-t)^j}{j!} + \frac{K(x-t)^n}{n!}$$

The function Φ satisfies the hypothesis of Rolle's Theorem in that $\Phi(c) = \Phi(x) = 0$, the function is continuous on the interval bound by c and x, and Φ' exists at each point of the interval. Therefore, there exists ξ in the interval such that $\Phi'(\xi) = 0$. We proceed to compute Φ' and set it equal to zero.

$$\Phi'(t) = f'(t) + \sum_{j=1}^{n-1} f^{(j+1)}(t)\frac{(x-t)^j}{j!} - \sum_{j=1}^{n-1} f^{(j)}(t)\frac{(x-t)^{j-1}}{(j-1)!} - \frac{K(x-t)^{n-1}}{(n-1)!}$$

This reduces to

$$\Phi'(t) = \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{K}{(n-1)!}(x-t)^{n-1}$$

According to hypothesis: for each n there is ξ_n such that

$$\Phi(\xi_n) = 0$$

Thus

$$K = f^{(n)}(\xi_n)$$

and the Lagrange remainder is

$$R_{n-1} = \frac{f^{(n)}(\xi_n)}{n!}(x-c)^n$$

or equivalently

$$R_n = \frac{1}{(n+1)!}f^{(n+1)}(\xi_{n+1})(x-c)^{n+1}$$

The Cauchy form of the remainder follows immediately by applying the mean value theorem for integrals. (See Page 274.)

11.53. Extend Taylor's theorem to functions of two variables x and y.

Define $F(t) = f(x_0 + ht, y_0 + kt)$, then applying Taylor's theorem for one variable (about $t = 0$)

$$F(t) = F(0) + F'(0) + \frac{1}{2!}F''(0)t^2 + \cdots + \frac{1}{n!}F^{(n)}(0)t^n + \frac{1}{(n+1)!}F^{(n+1)}(\theta)t^{n+1}, \qquad 0 < \theta < t$$

Now let $t = 1$

$$F(1) = f(x_0 + h, y_0 + k) = F(0) + F'(0) + \frac{1}{2!}F''(0) + \cdots + \frac{1}{n!}F^{(n)}(0) + \frac{1}{(n+1)!}F^{(n+1)}(\theta)$$

When the derivatives $F'(t), \ldots, F^{(n)}(t), F^{(n+1)}(\theta)$ are computed and substituted into the previous expression, the two variable version of Taylor's formula results. (See Page 277, where this form and notational details can be found.)

11.54. Expand $x^2 + 3y - 2$ in powers of $x - 1$ and $y + 2$. Use Taylor's formula with $h = x - x_0$, $k = y - y_0$, where $x_0 = 1$ and $y_0 = -2$.

$$x^2 + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$$

(Check this algebraically.)

11.55. Prove that $\ln \dfrac{x+y}{2} = \dfrac{x+y-2}{2+\theta(x+y-2)}$, $0 < \theta < 1, x > 0, y > 0$. Hint: Use the Taylor formula with the linear term as the remainder.

11.56. Expand $f(x,y) = \sin xy$ in powers of $x-1$ and $y - \dfrac{\pi}{2}$ to second-degree terms.

$$1 - \frac{1}{8}\pi^2(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \left(y - \frac{\pi}{2}\right)^2$$

Supplementary Problems

CONVERGENCE AND DIVERGENCE OF SERIES OF CONSTANTS

11.57. (*a*) Prove that the series $\dfrac{1}{3 \cdot 7} + \dfrac{1}{7 \cdot 11} + \dfrac{1}{11 \cdot 15} + \cdots = \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{(4n-1)(4n+3)}$ converges and (*b*) find its sum.
Ans. (*b*) 1/12

11.58. Prove that the convergence or divergence of a series is not affected by (*a*) multiplying each term by the same non-zero constant, (*b*) removing (or adding) a finite number of terms.

11.59. If Σu_n and Σv_n converge to A and B, respectively, prove that $\Sigma(u_n + v_n)$ converges to $A + B$.

11.60. Prove that the series $\frac{3}{2} + (\frac{3}{2})^2 + (\frac{3}{2})^3 + \cdots = \Sigma(\frac{3}{2})^n$ diverges.

11.61. Find the fallacy: Let $S = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$. Then $S = 1 - (1-1) - (1-1) - \cdots = 1$ and $S = (1-1) + (1-1) + (1-1) + \cdots = 0$. Hence, $1 = 0$.

COMPARISON TEST AND QUOTIENT TEST

11.62. Test for convergence:

(*a*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, (*b*) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n^2-3}$, (*c*) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{(n+1)\sqrt{n+3}}$, (*d*) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n \cdot 5^n}$, (*e*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5n-3}$,

(*f*) $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{(3n+2)n^{4/3}}$.

Ans. (*a*) conv., (*b*) div., (*c*) div., (*d*) conv., (*e*) div., (*f*) conv.

11.63. Investigate the convergence of (*a*) $\displaystyle\sum_{n=1}^{\infty} \frac{4n^2+5n-2}{n(n^2+1)^{3/2}}$, (*b*) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{n - \ln n}{n^2 + 10n^3}}$. *Ans.* (*a*) conv., (*b*) div.

11.64. Establish the comparison test for divergence (see Page 267).

11.65. Use the comparison test to prove that

(*a*) $\sum_{n=1}^{\infty} \leqq \dfrac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leqq 1$, (*b*) $\sum_{n=1}^{\infty} \dfrac{\tan^{-1} n}{n}$ diverges, (*c*) $\sum_{n=1}^{\infty} \dfrac{n^2}{2^n}$ converges.

11.66. Establish the results (*b*) and (*c*) of the quotient test, Page 267.

11.67. Test for convergence:

(*a*) $\sum_{n=1}^{\infty} \dfrac{(\ln n)^2}{n^2}$, (*b*) $\sum_{n=1}^{\infty} \sqrt{n \tan^{-1}(1/n^3)}$, (*c*) $\sum_{n=1}^{\infty} \dfrac{3 + \sin n}{n(1 + e^{-n})}$, (*d*) $\sum_{n=1}^{\infty} n \sin^2(1/n)$.

Ans. (*a*) conv., (*b*) div., (*c*) div., (*d*) div.

11.68. If Σu_n converges, where $u_n \geqq 0$ for $n > N$, and if $\lim_{n\to\infty} nu_n$ exists, prove that $\lim_{n\to\infty} nu_n = 0$.

11.69. (*a*) Test for convergence $\sum_{n=1}^{\infty} \dfrac{1}{n^{1+1/n}}$. (*b*) Does your answer to (*a*) contradict the statement about the p series made on Page 266 that $\Sigma 1/n^p$ converges for $p > 1$?

Ans. (*a*) div.

INTEGRAL TEST

11.70. Test for convergence: (*a*) $\sum_{n=1}^{\infty} \dfrac{n^2}{2n^3 - 1}$, (*b*) $\sum_{n=2}^{\infty} \dfrac{1}{n(\ln n)^3}$, (*c*) $\sum_{n=1}^{\infty} \dfrac{n}{2^n}$, (*d*) $\sum_{n=1}^{\infty} \dfrac{e^{-\sqrt{n}}}{\sqrt{n}}$ (*e*) $\sum_{n=2}^{\infty} \dfrac{\ln n}{n}$,

(*f*) $\sum_{n=10}^{\infty} \dfrac{2^{\ln(\ln n)}}{n \ln n}$.

Ans. (*a*) div., (*b*) conv., (*c*) conv., (*d*) conv., (*e*) div., (*f*) div.

11.71. Prove that $\sum_{n=2}^{\infty} \dfrac{1}{n(\ln n)^p}$, where p is a constant, (*a*) converges if $p > 1$ and (*b*) diverges if $p \leqq 1$.

11.72. Prove that $\dfrac{9}{8} < \sum_{n=1}^{\infty} \dfrac{1}{n^3} < \dfrac{5}{4}$.

11.73. Investigate the convergence of $\sum_{n=1}^{\infty} \dfrac{e^{\tan^{-1} n}}{n^2 + 1}$.

Ans. conv.

11.74. (*a*) Prove that $\frac{2}{3}n^{3/2} + \frac{1}{3} \leqq \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} \leqq \frac{2}{3}n^{3/2} + n^{1/2} - \frac{2}{3}$.

(*b*) Use (*a*) to estimate the value of $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{100}$, giving the maximum error.

(*c*) Show how the accuracy in (*b*) can be improved by estimating, for example, $\sqrt{10} + \sqrt{11} + \cdots + \sqrt{100}$ and adding on the value of $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{9}$ computed to some desired degree of accuracy.

Ans. (*b*) 671.5 ± 4.5

ALTERNATING SERIES

11.75. Test for convergence: (*a*) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{2^n}$, (*b*) $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2 + 2n + 2}$, (*c*) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} n}{3n - 1}$,

(*d*) $\sum_{n=1}^{\infty} (-1)^n \sin^{-1} \dfrac{1}{n}$, (*e*) $\sum_{n=2}^{\infty} \dfrac{(-1)^n \sqrt{n}}{\ln n}$.

Ans. (*a*) conv., (*b*) conv., (*c*) div., (*d*) conv., (*e*) div.

11.76. (*a*) What is the largest absolute error made in approximating the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n(n+1)}$ by the sum of the first 5 terms?
Ans. 1/192
(*b*) What is the least number of terms which must be taken in order that 3 decimal place accuracy will result?
Ans. 8 terms

11.77. (*a*) Prove that $S = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = \frac{4}{3}\left(\frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{3^3} - \cdots\right)$.
(*b*) How many terms of the series on the right are needed in order to calculate S to six decimal place accuracy?
Ans. (*b*) at least 100 terms

ABSOLUTE AND CONDITIONAL CONVERGENCE

11.78. Test for absolute or conditional convergence:

(*a*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+1}$ (*c*) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ (*e*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \sin \frac{1}{\sqrt{n}}$

(*b*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2+1}$ (*d*) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{(n^2+1)^{4/3}}$ (*f*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^3}{2^n - 1}$

Ans. (*a*) abs. conv., (*b*) cond. conv., (*c*) cond. conv., (*d*) div., (*e*) abs. conv., (*f*) abs. conv.

11.79. Prove that $\sum_{n=1}^{\infty} \frac{\cos n\pi a}{x^2+n^2}$ converges absolutely for all real x and a.

11.80. If $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to S, prove that the rearranged series $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2}S$. Explain.
[Hint: Take 1/2 of the first series and write it as $0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + \cdots$; then add term by term to the first series. Note that $S = \ln 2$, as shown in Problem 11.100.]

11.81. Prove that the terms of an absolutely convergent series can always be rearranged without altering the sum.

RATIO TEST

11.82. Test for convergence:

(*a*) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)e^n}$, (*b*) $\sum_{n=1}^{\infty} \frac{10^{2n}}{(2n-1)!}$, (*c*) $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$, (*d*) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^{3n}}{3^{2n}}$, (*e*) $\sum_{n=1}^{\infty} \frac{(\sqrt{5}-1)^n}{n^2+1}$.

Ans. (*a*) conv. (abs.), (*b*) conv., (*c*) div., (*d*) conv. (abs.), (*e*) div.

11.83. Show that the ratio test cannot be used to establish the conditional convergence of a series.

11.84. Prove that (*a*) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges and (*b*) $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

MISCELLANEOUS TESTS

11.85. Establish the validity of the nth root test on Page 268.

11.86. Apply the nth root test to work Problems 11.82(*a*), (*c*), (*d*), and (*e*).

11.87. Prove that $\frac{1}{3} + (\frac{2}{3})^2 + (\frac{1}{3})^3 + (\frac{2}{3})^4 + (\frac{1}{3})^5 + (\frac{2}{3})^6 + \cdots$ converges.

11.88. Test for convergence: (*a*) $\frac{1}{3}+\frac{1\cdot 4}{3\cdot 6}+\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}+\cdots$, (*b*) $\frac{2}{9}+\frac{2\cdot 5}{9\cdot 12}+\frac{2\cdot 5\cdot 8}{9\cdot 12\cdot 15}+\cdots$.

Ans. (*a*) div., (*b*) conv.

11.89. If a, b, and d are positive numbers and $b > a$, prove that

$$\frac{a}{b}+\frac{a(a+d)}{b(b+d)}+\frac{a(a+d)(a+2d)}{b(b+d)(b+2d)}+\cdots$$

converges if $b - a > d$, and diverges if $b - a \leqq d$.

SERIES OF FUNCTIONS

11.90. Find the domain of convergence of the series:

(*a*) $\sum_{n=1}^{\infty}\frac{x^n}{n^3}$, (*b*) $\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^n}{2^n(3n-1)}$, (*c*) $\sum_{n=1}^{\infty}\frac{1}{n(1+x^2)^n}$, (*d*) $\sum_{n=1}^{\infty}n^2\left(\frac{1-x}{1+x}\right)^n$, (*e*) $\sum_{n=1}^{\infty}\frac{e^{nx}}{n^2-n+1}$

Ans. (*a*) $-1 \leqq x \leqq 1$, (*b*) $-1 < x \leqq 3$, (*c*) all $x \neq 0$, (*d*) $x > 0$, (*e*) $x \leqq 0$

11.91. Prove that $\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^n$ converges for $-1 \leqq x < 1$.

UNIFORM CONVERGENCE

11.92. By use of the definition, investigate the uniform convergence of the series

$$\sum_{n=1}^{\infty}\frac{x}{[1+(n-1)x][1+nx]}$$

$\left[\text{Hint: Resolve the } n\text{th term into partial fractions and show that the } n\text{th partial sum is } S_n(x) = 1 - \frac{1}{1+nx}.\right]$

Ans. Not uniformly convergent in any interval which includes $x = 0$; uniformly convergent in any other interval.

11.93. Work Problem 11.30 directly by first obtaining $S_n(x)$.

11.94. Investigate by any method the convergence and uniform convergence of the series:

(*a*) $\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n$, (*b*) $\sum_{n=1}^{\infty}\frac{\sin^2 nx}{2^n-1}$, (*c*) $\sum_{n=1}^{\infty}\frac{x}{(1+x)^n}, x \geqq 0$.

Ans. (*a*) conv. for $|x| < 3$; unif. conv. for $|x| \leqq r < 3$. (*b*) unif. conv. for all x. (*c*) conv. for $x \geqq 0$; not unif. conv. for $x \geqq 0$, but unif. conv. for $x \geqq r > 0$.

11.95. If $F(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n^3}$, prove that:

(*a*) $F(x)$ is continuous for all x, (*b*) $\lim_{x\to 0} F(x) = 0$, (*c*) $F'(x) = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ is continous everywhere.

11.96. Prove that $\int_0^{\pi}\left(\frac{\cos 2x}{1\cdot 3}+\frac{\cos 4x}{3\cdot 5}+\frac{\cos 6x}{5\cdot 7}+\cdots\right)dx = 0$.

11.97. Prove that $F(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{\sinh n\pi}$ has derivatives of all orders for any real x.

11.98. Examine the sequence $u_n(x) = \frac{1}{1+x^{2n}}, n = 1, 2, 3, \ldots$, for uniform convergence.

11.99. Prove that $\lim_{n\to\infty}\int_0^1\frac{dx}{(1+x/n)^n} = 1 - e^{-1}$.

POWER SERIES

11.100. (*a*) Prove that $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$.

(*b*) Prove that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

$\left[\text{Hint: Use the fact that } \dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \text{ and integrate.}\right]$

11.101. Prove that $\sin^{-1} x = x + \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1 \cdot 3}{2 \cdot 4}\dfrac{x^5}{5} + \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\dfrac{x^7}{7} + \cdots$, $-1 \leqq x \leqq 1$.

11.102. Evaluate (*a*) $\displaystyle\int_0^{1/2} e^{-x^2}\,dx$, (*d*) $\displaystyle\int_0^1 \frac{1-\cos x}{x}\,dx$ to 3 decimal places, justifying all steps.
Ans. (*a*) 0.461, (*b*) 0.486

11.103. Evaluate (*a*) $\sin 40^\circ$, (*b*) $\cos 65^\circ$, (*c*) $\tan 12^\circ$ correct to 3 decimal places.
Ans. (*a*) 0.643, (*b*) 0.423, (*c*) 0.213

11.104. Verify the expansions 4, 5, and 6 on Page 275.

11.105. By multiplying the series for $\sin x$ and $\cos x$, verify that $2 \sin x \cos x = \sin 2x$.

11.106. Show that $e^{\cos x} = e\left(1 - \dfrac{x^2}{2!} + \dfrac{4x^4}{4!} - \dfrac{31x^6}{6!} + \cdots\right)$, $-\infty < x < \infty$.

11.107. Obtain the expansions

(*a*) $\tanh^{-1} x = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \dfrac{x^7}{7} + \cdots \qquad -1 < x < 1$

(*b*) $\ln(x + \sqrt{x^2+1}) = x - \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1 \cdot 3}{2 \cdot 4}\dfrac{x^5}{5} - \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\dfrac{x^7}{7} + \cdots \qquad -1 \leqq x \leqq 1$

11.108. Let $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Prove that the formal Taylor series about $x = 0$ corresponding to $f(x)$ exists but that it does not converge to the given function for any $x \neq 0$.

11.109. Prove that

(*a*) $\dfrac{\ln(1+x)}{1+x} = x - \left(1 + \dfrac{1}{2}\right)x^2 + \left(1 + \dfrac{1}{2} + \dfrac{1}{3}\right)x^3 - \cdots$ for $-1 < x < 1$

(*b*) $\{\ln(1+x)\}^2 = x^2 - \left(1 + \dfrac{1}{2}\right)\dfrac{2x^3}{3} + \left(1 + \dfrac{1}{2} + \dfrac{1}{3}\right)\dfrac{2x^4}{4} - \cdots$ for $-1 < x \leqq 1$

MISCELLANEOUS PROBLEMS

11.110. Prove that the series for $J_p(x)$ converges (*a*) for all x, (*b*) absolutely and uniformly in any finite interval.

11.111. Prove that (*a*) $\dfrac{d}{dx}\{J_0(x)\} = -J_1(x)$, (*b*) $\dfrac{d}{dx}\{x^p J_p(x)\} = x^p J_{p-1}(x)$, (*c*) $J_{p+1}(x) = \dfrac{2p}{x} J_p(x) - J_{p-1}(x)$.

11.112. Assuming that the result of Problem 11.111(*c*) holds for $p = 0, -1, -2, \ldots$, prove that
(*a*) $J_{-1}(x) = -J_1(x)$, (*b*) $J_{-2}(x) = J_2(x)$, (*c*) $J_{-n}(x) = (-1)^n J_n(x)$, $n = 1, 2, 3, \ldots$.

11.113. Prove that $e^{1/2x(t-1/t)} = \displaystyle\sum_{p=-\infty}^{\infty} J_p(x)\, t^p$.
[Hint: Write the left side as $e^{xt/2}e^{-x/2t}$, expand and use Problem 11.112.]

11.114. Prove that $\sum_{n=1}^{\infty} \frac{(n+1)z^n}{n(n+2)^2}$ is absolutely and uniformly convergent at all points within and on the circle $|z| = 1$.

11.115. (*a*) If $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} b_n x^n$ for all x in the common interval of convergence $|x| < R$ where $R > 0$, prove that $a_n = b_n$ for $n = 0, 1, 2, \ldots$. (*b*) Use (*a*) to show that the Taylor expansion of a function exists, the expansion is unique.

11.116. Suppose that $\overline{\lim}\sqrt[n]{|u_n|} = L$. Prove that Σu_n converges or diverges according as $L < 1$ or $L > 1$. If $L = 1$ the test fails.

11.117. Prove that the radius of convergence of the series $\Sigma a_n x^n$ can be determined by the following limits, when they exist, and give examples: (*a*) $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$, (*b*) $\lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$, (*c*) $\overline{\lim_{n\to\infty}} \frac{1}{\sqrt[n]{|a_n|}}$.

11.118. Use Problem 11.117 to find the radius of convergence of the series in Problem 11.22.

11.119. (*a*) Prove that a necessary and sufficient condition that the series Σu_n converge is that, given any $\epsilon > 0$, we can find $N > 0$ depending on ϵ such that $|S_p - S_q| < \epsilon$ whenever $p > N$ and $q > N$, where $S_k = u_1 + u_2 + \cdots + u_k$.

(*b*) Use (*a*) to prove that the series $\sum_{n=1}^{\infty} \frac{n}{(n+1)3^n}$ converges.

(*c*) How could you use (*a*) to prove that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges?

[Hint: Use the Cauchy convergence criterion, Page 25.]

11.120. Prove that the hypergeometric series (Page 276) (*a*) is absolutely convergent for $|x| < 1$, (*b*) is divergent for $|x| > 1$, (*c*) is absolutely divergent for $|x| = 1$ if $a + b - c < 0$, (*d*) satisfies the differential equation $x(1-x)y'' + \{c - (a+b+1)x\}y' - aby = 0$.

11.121. If $F(a, b; c; x)$ is the hypergeometric function defined by the series on Page 276, prove that (*a*) $F(-p, 1; 1; -x) = (1+x)^p$, (*b*) $xF(1, 1; 2; -x) = \ln(1+x)$, (*c*) $F(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2) = (\sin^{-1} x)/x$.

11.122. Find the sum of the series $S(x) = x + \frac{x^3}{1\cdot 3} + \frac{x^5}{1\cdot 3\cdot 5} + \cdots$.
[Hint: Show that $S'(x) - 1 + xS(x)$ and solve.]

Ans. $e^{x^2/2} \int_0^x e^{-x^2/2}\,dx$

11.123. Prove that

$$1 + \frac{1}{1\cdot 3} + \frac{1}{1\cdot 3\cdot 5} + \frac{1}{1\cdot 3\cdot 5\cdot 7} + \cdots = \sqrt{e}\left(1 - \frac{1}{2\cdot 3} + \frac{1}{2^2\cdot 2!\cdot 5} - \frac{1}{2^3\cdot 3!\cdot 7} + \frac{1}{2^4\cdot 4!\cdot 9} - \cdots\right)$$

11.124. Establish the Dirichlet test on Page 270.

11.125. Prove that $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is uniformly convergent in any interval which does not include $0, \pm\pi, \pm 2\pi, \ldots$.

[Hint: use the Dirichlet test, Page 270, and Problem 1.94, Chapter 1.]

11.126. Establish the results on Page 275 concerning the binomial series.
[Hint: Examine the Lagrange and Cauchy forms of the remainder in Taylor's theorem.]

11.127. Prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+x^2}$ converges uniformly for all x, but not absolutely.

11.128. Prove that $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots = \frac{\pi}{3\sqrt{3}} + \frac{1}{3} \ln 2$

11.129. If $x = ye^y$, prove that $y = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{n-1}}{n!} x^n$ for $-1/e < x \leqq 1/e$.

11.130. Prove that the equation $e^{-\lambda} = \lambda - 1$ has only one real root and show that it is given by

$$\lambda = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{n-1} e^{-n}}{n!}$$

11.131. Let $\frac{x}{e^x - 1} = 1 + B_1 x + \frac{B_2 x^2}{2!} + \frac{B_3 x^3}{3!} + \cdots$. (*a*) Show that the numbers B_n, called the *Bernoulli numbers*, satisfy the recursion formula $(B+1)^n - B^n = 0$ where B^k is formally replaced by B_k after expanding. (*b*) Using (*a*) or otherwise, determine $B_1, \ldots, B_6$.

Ans. (*b*) $B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}$.

11.132. (*a*) Prove that $\frac{x}{e^x - 1} = \frac{x}{2}\left(\coth\frac{x}{2} - 1\right)$. (*b*) Use Problem 11.127 and part (*a*) to show that $B_{2k+1} = 0$ if $k = 1, 2, 3, \ldots$.

11.133. Derive the series expansions:

(*a*) $\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \cdots + \frac{B_{2n}(2x)^{2n}}{(2n)!x} + \cdots$

(*b*) $\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} + \cdots (-1)^n \frac{B_{2n}(2x)^{2n}}{(2n)!x} + \cdots$

(*c*) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots (-1)^{n-1} \frac{2(2^{2n} - 1)B_{2n}(2x)^{2n-1}}{(2n)!} + \cdots$

(*d*) $\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7}{360}x^3 + \cdots (-1)^{n-1} \frac{2(2^{2n-1} - 1)B_{2n}x^{2n-1}}{(2n)!} + \cdots$

[Hint: For (*a*) use Problem 11.132; for (*b*) replace x by ix in (*a*); for (*c*) use $\tan x = \cot x - 2\cot 2x$; for (*d*) use $\csc x = \cot x + \tan x/2$.]

11.134. Prove that $\prod_{n=1}^{\infty}\left(1 + \frac{1}{n^3}\right)$ converges.

11.135. Use the definition to prove that $\prod_{n=1}^{\infty}\left(1 + \frac{1}{n}\right)$ diverges.

11.136. Prove that $\prod_{n=1}^{\infty}(1 - u_n)$, where $0 < u_n < 1$, converges if and only if Σu_n converges.

11.137. (*a*) Prove that $\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right)$ converges to $\frac{1}{2}$. (*b*) Evaluate the infinite product in (*a*) to 2 decimal places and compare with the true value.

11.138. Prove that the series $1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + \cdots$ is the $C - 1$ summable to zero.

11.139. Prove that the Césaro method of summability is regular. [Hint: See Page 278.]

11.140. Prove that the series $1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$ converges to $1/(1-x)^2$ for $|x| < 1$.

11.141. A series $\sum_{n=0}^{\infty} a_n$ is called *Abel summable* to S if $S = \lim_{x \to 1-} \sum_{n=0}^{\infty} a_n x^n$ exists. Prove that

(a) $\sum_{n=0}^{\infty} (-1)^n (n+1)$ is Abel summable to 1/4 and

(b) $\sum_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2)}{2}$ is Abel summable to 1/8.

11.142. Prove that the double series $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \dfrac{1}{(m^2+n^2)^p}$, where p is a constant, converges or diverges according as $p > 1$ or $p \leqq 1$, respectively.

11.143. (a) Prove that $\displaystyle\int_x^{\infty} \frac{e^{x-u}}{u}\, du = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots \frac{(-1)^{n-1}(n-1)!}{x^n} + (-1)^n n! \int_x^{\infty} \frac{e^{x-u}}{u^{n+1}}\, du.$

(b) Use (a) to prove that $\displaystyle\int_x^{\infty} \frac{e^{x-u}}{u}\, du \sim \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots$