Chapter 62

Partial Derivatives

FUNCTIONS OF SEVERAL VARIABLES. If a real number z is assigned to each point (x, y) of a part (region) of the xy plane, then z is said to be given as a function, $z = f(x, y)$, of the independent variables x and y. The locus of all points (x, y, z) satisfying $z = f(x, y)$ is a surface in ordinary space. In a similar manner, functions $w = f(x, y, z, \ldots)$ of several independent variables may be defined, but no geometric picture is available.

There are a number of differences between the calculus of one and of two variables. Fortunately, the calculus of functions of three or more variables differs only slightly from that of functions of two variables. The study here will be limited largely to functions of two variables.

LIMITS AND CONTINUITY. We say that a function $f(x, y)$ has a limit A as $x \to x_0$ and $y \to y_0$, and we write $\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = A$, if, for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that, for all (x, y) satisfying

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \qquad (62.1)$$

we have $|f(x, y) - A| < \epsilon$. Here, *(62.1)* defines a deleted neighborhood of (x_0, y_0), namely, all points except (x_0, y_0) lying within a circle of radius δ and center (x_0, y_0).

A function $f(x, y)$ is said to be continuous at (x_0, y_0) provided $f(x_0, y_0)$ is defined and $\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(x_0, y_0)$. (See Problems 1 and 2.)

PARTIAL DERIVATIVES. Let $z = f(x, y)$ be a function of the independent variables x and y. Since x and y are independent, we may (1) allow x to vary while y is held fixed, (2) allow y to vary while x is held fixed, or (3) permit x and y to vary simultaneously. In the first two cases, z is in effect a function of a single variable and can be differentiated in accordance with the usual rules.

If x varies while y is held fixed, then z is a function of x; its derivative with respect to x,

$$f_x(x, y) = \frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

is called *the (first) partial derivative of* $z = f(x, y)$ *with respect to* x.

If y varies while x is held fixed, z is a function of y; its derivative with respect to y,

$$f_y(x, y) = \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

is called *the (first) partial derivative of* $z = f(x, y)$ *with respect to* y. (See Problems 3 to 8.)

If z is defined implicitly as a function of x and y by the relation $F(x, y, z) = 0$, the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ may be found using the implicit differentiation rule of Chapter 11. (See Problems 9 to 12.)

The partial derivatives defined above have simple geometric interpretations. Consider the surface $z = f(x, y)$ in Fig. 62-1. Let APB and CPD be sections of the surface cut by planes through P, parallel to xOz and yOz, respectively. As x varies while y is held fixed, P moves along the curve APB and the value of $\partial z/\partial x$ at P is the slope of the curve APB at P.

Similarly, as y varies while x is held fixed, P moves along the curve CPD and the value of $\partial z/\partial y$ at P is the slope of the curve CPD at P. (See Problem 13.)

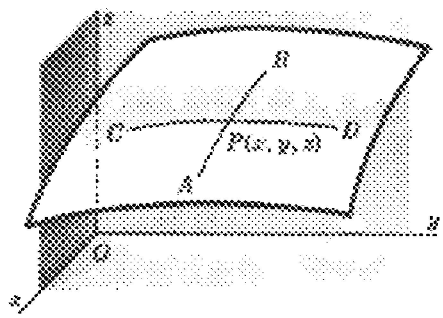

Fig. 62-3

PARTIAL DERIVATIVES OF HIGHER ORDERS. The partial derivative $\partial z/\partial x$ of $z = f(x, y)$ may in turn be differentiated partially with respect to x and y, yielding the second partial derivatives

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial^2 z}{\partial y\,\partial x} = f_{yx}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Similarly, from $\partial z/\partial y$ we may obtain

$$\frac{\partial^2 z}{\partial x\,\partial y} = f_{xy}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

If $z = f(x, y)$ and its partial derivatives are continuous, the order of differentiation turns out to be immaterial; that is, $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$. (See Problems 14 and 15.)

Solved Problems

1. Investigate $z = x^2 + y^2$ for continuity.

For any set of finite values $(x, y) = (a, b)$, we have $z = a^2 + b^2$. As $x \to a$ and $y \to b$, $x^2 + y^2 \to a^2 + b^2$. Hence, the function is continuous everywhere.

2. The following functions are continuous everywhere except at the origin $(0, 0)$, where they are not defined. Can they be made continuous there?

(a) $z = \dfrac{\sin(x + y)}{x + y}$

Let $(x, y) \to (0, 0)$ over the line $y = mx$; then $z = \dfrac{\sin(x + y)}{x + y} = \dfrac{\sin(1 + m)x}{(1 + m)x} \to 1$. The function may be made continuous everywhere by redefining it as $z = \dfrac{\sin(x + y)}{x + y}$ for $(x, y) \neq (0, 0)$; $z = 1$ for $(x, y) = (0, 0)$.

(b) $z = \dfrac{xy}{x^2 + y^2}$

Let $(x, y) \to (0, 0)$ over the line $y = mx$; the limiting value of $z = \dfrac{xy}{x^2 + y^2} = \dfrac{m}{1 + m^2}$ depends on the particular line chosen. Thus, the function cannot be made continuous at $(0, 0)$.

In Problems 3 to 7, find the first partial derivatives.

3. $z = 2x^2 - 3xy + 4y^2$

Treating y as a constant and differentiating with respect to x yield $\dfrac{\partial z}{\partial x} = 4x - 3y$.

Treating x as a constant and differentiating with respect to y yield $\dfrac{\partial z}{\partial y} = -3x + 8y$.

4. $z = \dfrac{x^2}{y} + \dfrac{y^2}{x}$

Treating y as a constant and differentiating with respect to x yield $\dfrac{\partial z}{\partial x} = \dfrac{2x}{y} - \dfrac{y^2}{x^2}$.

Treating x as a constant and differentiating with respect to y yield $\dfrac{\partial z}{\partial y} = -\dfrac{x^2}{y^2} + \dfrac{2y}{x}$.

5. $z = \sin(2x + 3y)$

$$\frac{\partial z}{\partial x} = 2\cos(2x + 3y) \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 3\cos(2x + 3y)$$

6. $z = \arctan x^2y + \arctan xy^2$

$$\frac{\partial z}{\partial x} = \frac{2xy}{1 + x^4y^2} + \frac{y^2}{1 + x^2y^4} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{x^2}{1 + x^4y^2} + \frac{2xy}{1 + x^2y^4}$$

7. $z = e^{x^2 + xy}$

$$\frac{\partial z}{\partial x} = e^{x^2 + xy}(2x + y) = z(2x + y) \qquad \text{and} \qquad \frac{\partial z}{\partial y} = e^{x^2 + xy}(x) = xz$$

8. The area of a triangle is given by $K = \frac{1}{2}ab \sin C$. If $a = 20$, $b = 30$, and $C = 30°$, find:
(a) The rate of change of K with respect to a, when b and C are constant.
(b) The rate of change of K with respect to C, when a and b are constant.
(c) The rate of change of b with respect to a, when K and C are constant.

(a) $\dfrac{\partial K}{\partial a} = \dfrac{1}{2}\, b \sin C = \dfrac{1}{2}(30)(\sin 30°) = \dfrac{15}{2}$

(b) $\dfrac{\partial K}{\partial C} = \dfrac{1}{2}\, ab \cos C = \dfrac{1}{2}(20)(30)(\cos 30°) = 150\sqrt{3}$

(c) $b = \dfrac{2K}{a \sin C}$ and $\dfrac{\partial b}{\partial a} = -\dfrac{2K}{a^2 \sin C} = -\dfrac{2(\frac{1}{2}ab \sin C)}{a^2 \sin C} = -\dfrac{b}{a} = -\dfrac{3}{2}$

In Problems 9 to 11, find the first partial derivatives of z with respect to the independent variables x and y.

9. $x^2 + y^2 + z^2 = 25$

Solution 1: Solve for z to obtain $z = \pm\sqrt{25 - x^2 - y^2}$. Then

$$\frac{\partial z}{\partial x} = \frac{-x}{\pm\sqrt{25 - x^2 - y^2}} = -\frac{x}{z} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{-y}{\pm\sqrt{25 - x^2 - y^2}} = -\frac{y}{z}$$

Solution 2: Differentiate implicitly with respect to x, treating y as a constant, to obtain

$$2x + 2z\frac{\partial z}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial z}{\partial x} = -\frac{x}{z}$$

Then differentiate implicitly with respect to y, treating x as a constant:

$$2y + 2z\frac{\partial z}{\partial y} = 0 \qquad \text{or} \qquad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

10. $x^2(2y+3z)+y^2(3x-4z)+z^2(x-2y)=xyz$

The procedure of Solution 1 of Problem 9 would be inconvenient here. Instead, differentiating implicitly with respect to x yields

$$2x(2y+3z)+3x^2\frac{\partial z}{\partial x}+3y^2-4y^2\frac{\partial z}{\partial x}+2z(x-2y)\frac{\partial z}{\partial x}+z^2=yz+xy\frac{\partial z}{\partial x}$$

so that
$$\frac{\partial z}{\partial x}=-\frac{4xy+6xz+3y^2+z^2-yz}{3x^2-4y^2+2xz-4yz-xy}$$

Differentiating implicitly with respect to y yields

$$2x^2+3x^2\frac{\partial z}{\partial y}+2y(3x-4z)-4y^2\frac{\partial z}{\partial y}+2z(x-2y)\frac{\partial z}{\partial y}-2z^2=xz+xy\frac{\partial z}{\partial y}$$

so that
$$\frac{\partial z}{\partial y}=-\frac{2x^2+6xy-8yz-2z^2-xz}{3x^2-4y^2+2xz-4yz-xy}$$

11. $xy+yz+zx=1$

Differentiating with respect to x yields $y+y\dfrac{\partial z}{\partial x}+x\dfrac{\partial z}{\partial x}+z=0$ and $\dfrac{\partial z}{\partial x}=-\dfrac{y+z}{x+y}$.

Differentiating with respect to y yields $x+y\dfrac{\partial z}{\partial y}+z+x\dfrac{\partial z}{\partial y}=0$ and $\dfrac{\partial z}{\partial y}=-\dfrac{x+z}{x+y}$.

12. Considering x and y as independent variables, find $\dfrac{\partial r}{\partial x},\dfrac{\partial r}{\partial y},\dfrac{\partial \theta}{\partial x},\dfrac{\partial \theta}{\partial y}$ when $x=e^{2r}\cos\theta$, $y=e^{3r}\sin\theta$.

First differentiate the given relations with respect to x:

$$1=2e^{2r}\cos\theta\frac{\partial r}{\partial x}-e^{2r}\sin\theta\frac{\partial \theta}{\partial x}\qquad\text{and}\qquad 0=3e^{3r}\sin\theta\frac{\partial r}{\partial x}+e^{3r}\cos\theta\frac{\partial \theta}{\partial x}$$

Then solve simultaneously to obtain $\dfrac{\partial r}{\partial x}=\dfrac{\cos\theta}{e^{2r}(2+\sin^2\theta)}$ and $\dfrac{\partial \theta}{\partial x}=-\dfrac{3\sin\theta}{e^{2r}(2+\sin^2\theta)}$.

Now differentiate the given relations with respect to y:

$$0=2e^{2r}\cos\theta\frac{\partial r}{\partial y}-e^{2r}\sin\theta\frac{\partial \theta}{\partial y}\qquad\text{and}\qquad 1=3e^{3r}\sin\theta\frac{\partial r}{\partial y}+e^{3r}\cos\theta\frac{\partial \theta}{\partial y}$$

Then solve simultaneously to obtain $\dfrac{\partial r}{\partial y}=\dfrac{\sin\theta}{e^{3r}(2+\sin^2\theta)}$ and $\dfrac{\partial \theta}{\partial y}=\dfrac{2\cos\theta}{e^{3r}(2+\sin^2\theta)}$.

13. Find the slopes of the curves cut from the surface $z=3x^2+4y^2-6$ by planes through the point $(1,1,1)$ and parallel to the coordinate planes xOz and yOz.

The plane $x=1$, parallel to the plane yOz, intersects the surface in the curve $z=4y^2-3$, $x=1$. Then $\partial z/\partial y=8y=8\times 1=8$ is the required slope.

The plane $y=1$, parallel to the plane xOz, intersects the surface in the curve $z=3x^2-2$, $y=1$. Then $\partial z/\partial x=6x=6$ is the required slope.

In Problems 14 and 15, find all second partial derivatives of z.

14. $z=x^2+3xy+y^2$

$$\frac{\partial z}{\partial x}=2x+3y\qquad \frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=2\qquad \frac{\partial^2 z}{\partial y\,\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=3$$

$$\frac{\partial z}{\partial y}=3x+2y\qquad \frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=2\qquad \frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=3$$

15. $z = x\cos y - y\cos x$

$$\frac{\partial z}{\partial x} = \cos y + y\sin x \qquad \frac{\partial z}{\partial y} = -x\sin y - \cos x \qquad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = y\cos x$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = -\sin y + \sin x = \frac{\partial^2 z}{\partial x\,\partial y} \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = -x\cos y$$

Supplementary Problems

16. Investigate each of the following to determine whether or not it can be made continuous at $(0, 0)$: $(a)\ \frac{y^2}{x^2+y^2}$, $(b)\ \frac{x-y}{x+y}$, $(c)\ \frac{x^3+y^3}{x^2+y^2}$, $(d)\ \frac{x+y}{x^2+y^2}$. *Ans.* (*a*) no; (*b*) no; (*c*) yes; (*d*) no

17. For each of the following functions, find $\partial z/\partial x$ and $\partial z/\partial y$.

(*a*) $z = x^2 + 3xy + y^2$ *Ans.* $\frac{\partial z}{\partial x} = 2x + 3y;\ \frac{\partial z}{\partial y} = 3x + 2y$

(*b*) $z = \frac{x}{y^2} - \frac{y}{x^2}$ *Ans.* $\frac{\partial z}{\partial x} = \frac{1}{y^2} + \frac{2y}{x^3};\ \frac{\partial z}{\partial y} = -\frac{2x}{y^3} - \frac{1}{x^2}$

(*c*) $z = \sin 3x \cos 4y$ *Ans.* $\frac{\partial z}{\partial x} = 3\cos 3x\cos 4y;\ \frac{\partial z}{\partial y} = -4\sin 3x \sin 4y$

(*d*) $z = \arctan\frac{y}{x}$ *Ans.* $\frac{\partial z}{\partial x} = \frac{-y}{x^2+y^2};\ \frac{\partial z}{\partial y} = \frac{x}{x^2+y^2}$

(*e*) $x^2 - 4y^2 + 9z^2 = 36$ *Ans.* $\frac{\partial z}{\partial x} = -\frac{x}{9z};\ \frac{\partial z}{\partial y} = \frac{4y}{9z}$

(*f*) $z^3 - 3x^2y + 6xyz = 0$ *Ans.* $\frac{\partial z}{\partial x} = \frac{2y(x-z)}{z^2+2xy};\ \frac{\partial z}{\partial y} = \frac{x(x-2z)}{z^2+2xy}$

(*g*) $yz + xz + xy = 0$ *Ans.* $\frac{\partial z}{\partial x} = -\frac{y+z}{x+y};\ \frac{\partial z}{\partial y} = -\frac{x+z}{x+y}$

18. (*a*) If $z = \sqrt{x^2+y^2}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$.

(*b*) If $z = \ln\sqrt{x^2+y^2}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$.

(*c*) If $z = e^{x/y}\sin\frac{x}{y} + e^{y/x}\cos\frac{y}{x}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

(*d*) If $z = (ax+by)^2 + e^{ax+by} + \sin(ax+by)$, show that $b\frac{\partial z}{\partial x} = a\frac{\partial z}{\partial y}$.

19. Find the equation of the line tangent to
(*a*) The parabola $z = 2x^2 - 3y^2$, $y = 1$ at the point $(-2, 1, 5)$ *Ans.* $8x + z + 11 = 0$, $y = 1$
(*b*) The parabola $z = 2x^2 - 3y^2$, $x = -2$ at the point $(-2, 1, 5)$ *Ans.* $6y + z - 11 = 0$, $x = -2$
(*c*) The hyperbola $z = 2x^2 - 3y^2$, $z = 5$ at the point $(-2, 1, 5)$ *Ans.* $4x + 3y + 5 = 0$, $z = 5$
Show that these three lines lie in the plane $8x + 6y + z + 5 = 0$.

20. For each of the following functions, find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\,\partial y}$, $\frac{\partial^2 z}{\partial y\,\partial x}$, and $\frac{\partial^2 z}{\partial y^2}$.

(*a*) $z = 2x^2 - 5xy + y^2$ *Ans.* $\frac{\partial^2 z}{\partial x^2} = 4;\ \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = -5;\ \frac{\partial^2 z}{\partial y^2} = 2$

(*b*) $z = \frac{x}{y^2} - \frac{y}{x^2}$ *Ans.* $\frac{\partial^2 z}{\partial x^2} = -\frac{6y}{x^4};\ \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = 2\left(\frac{1}{x^3} - \frac{1}{y^3}\right);\ \frac{\partial^2 z}{\partial y^2} = \frac{6x}{y^4}$

(*c*) $z = \sin 3x \cos 4y$ *Ans.* $\frac{\partial^2 z}{\partial x^2} = -9z;\ \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = -12\cos 3x \sin 4y;\ \frac{\partial^2 z}{\partial y^2} = -16z$

(*d*) $z = \arctan\frac{y}{x}$ *Ans.* $\frac{\partial^2 z}{\partial x^2} = -\frac{\partial^2 z}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2};\ \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$

21. (*a*) If $z = \frac{xy}{x-y}$, show that $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$.

(*b*) If $z = e^{\alpha x}\cos \beta y$ and $\beta = \pm\alpha$, show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

(*c*) If $z = e^{-t}(\sin x + \cos y)$, show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial t}$.

(*d*) If $z = \sin ax \sin by \sin kt\sqrt{a^2+b^2}$, show that $\frac{\partial^2 z}{\partial t^2} = k^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$.

22. For the gas formula $\left(p + \frac{a}{v^2}\right)(v-b) = ct$, where a, b, and c are constants, show that

$$\frac{\partial p}{\partial v} = \frac{2a(v-b) - (p + a/v^2)v^3}{v^3(v-b)} \qquad \frac{\partial v}{\partial t} = \frac{cv^3}{(p + a/v^2)v^3 - 2a(v-b)}$$

$$\frac{\partial t}{\partial p} = \frac{v-b}{c} \qquad \frac{\partial p}{\partial v}\,\frac{\partial v}{\partial t}\,\frac{\partial t}{\partial p} = -1$$

[For the last result, see Problem 11 of Chapter 64.]

Chapter 63

Total Differentials and Total Derivatives

TOTAL DIFFERENTIALS. The differentials dx and dy for the function $y = f(x)$ of a single independent variable x were defined in Chapter 28 as

$$dx = \Delta x \qquad \text{and} \qquad dy = f'(x)\, dx = \frac{dy}{dx}\, dx$$

Consider the function $z = f(x, y)$ of the two independent variables x and y, and define $dx = \Delta x$ and $dy = \Delta y$. When x varies while y is held fixed, z is a function of x only and *the partial differential of z with respect to x* is defined as $d_x z = f_x(x, y)\, dx = \frac{\partial z}{\partial x}\, dx$. Similarly, *the partial differential of z with respect to y* is defined as $d_y z = f_y(x, y)\, dy = \frac{\partial z}{\partial y}\, dy$. The *total differential dz* is defined as the sum of the partial differentials,

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy \tag{63.1}$$

For a function $w = F(x, y, z, \ldots, t)$, the total differential dw is defined as

$$dw = \frac{\partial w}{\partial x}\, dx + \frac{\partial w}{\partial y}\, dy + \frac{\partial w}{\partial z}\, dz + \cdots + \frac{\partial w}{\partial t}\, dt \tag{63.2}$$

(See Problems 1 and 2.)

As in the case of a function of a single variable, the total differential of a function of several variables gives a good approximation of the total increment of the function when the increments of the several independent variables are small.

EXAMPLE 1: When $z = xy$, $dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy = y\, dx + x\, dy$; and when x and y are given increments $\Delta x = dx$ and $\Delta y = dy$, the increment Δz taken on by z is

$$\begin{aligned} \Delta z &= (x + \Delta x)(y + \Delta y) - xy = x\, \Delta y + y\, \Delta x + \Delta x\, \Delta y \\ &= x\, dy + y\, dx + dx\, dy \end{aligned}$$

A geometric interpretation is given in Fig. 63-1: dz and Δz differ by the rectangle of area $\Delta x\, \Delta y = dx\, dy$.

(See Problems 3 to 9.)

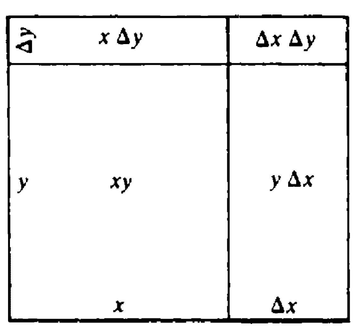

Fig. 63-1

THE CHAIN RULE FOR COMPOSITE FUNCTIONS. If $z = f(x, y)$ is a continuous function of the variables x and y with continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, and if x and y are differentiable functions $x = g(t)$ and $y = h(t)$ of a variable t, then z is a function of t and dz/dt, called the *total derivative* of z with respect to t, is given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{63.3}$$

Similarly, if $w = f(x, y, z, \ldots)$ is a continuous function of the variables $x, y, z, \ldots$ with continuous partial derivatives, and if $x, y, z, \ldots$ are differentiable functions of a variable t, the total derivative of w with respect to t is given by

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} + \cdots \tag{63.4}$$

(See Problems 10 to 16.)

If $z = f(x, y)$ is a continuous function of the variables x and y with continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, and if x and y are continuous functions $x = g(r, s)$ and $y = h(r, s)$ of the independent variables r and s, then z is a function of r and s with

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \tag{63.5}$$

Similarly, if $w = f(x, y, z, \ldots)$ is a continuous function of the variables $x, y, z, \ldots$ with continuous partial derivatives $\partial w/\partial x$, $\partial w/\partial y$, $\partial w/\partial z, \ldots$, and if $x, y, z, \ldots$ are continuous functions of the independent variables $r, s, t, \ldots$, then

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} + \cdots \\ \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} + \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{63.6}$$

(See Problems 17 to 19.)

Solved Problems

In Problems 1 and 2, find the total differential.

1. $z = x^3y + x^2y^2 + xy^3$

We have $\quad \dfrac{\partial z}{\partial x} = 3x^2y + 2xy^2 + y^3 \quad$ and $\quad \dfrac{\partial z}{\partial y} = x^3 + 2x^2y + 3xy^2$

Then $\quad dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = (3x^2y + 2xy^2 + y^3)\,dx + (x^3 + 2x^2y + 3xy^2)\,dy$

2. $z = x\sin y - y\sin x$

We have $\quad \dfrac{\partial z}{\partial x} = \sin y - y\cos x \quad$ and $\quad \dfrac{\partial z}{\partial y} = x\cos y - \sin x$

Then $\quad dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = (\sin y - y\cos x)\,dx + (x\cos y - \sin x)\,dy$

3. Compare dz and Δz, given $z = x^2 + 2xy - 3y^2$.

$$\frac{\partial z}{\partial x} = 2x + 2y \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 2x - 6y\,. \qquad \text{So} \qquad dz = 2(x + y)\,dx + 2(x - 3y)\,dy$$

Also,
$$\begin{aligned}\Delta z &= [(x + dx)^2 + 2(x + dx)(y + dy) - 3(y + dy)^2] - (x^2 + 2xy - 3y^2)\\ &= 2(x + y)\,dx + 2(x - 3y)\,dy + (dx)^2 + 2\,dx\,dy - 3(dy)^2\end{aligned}$$

Thus dz and Δz differ by $(dx)^2 + 2\,dx\,dy - 3(dy)^2$.

4. Approximate the area of a rectangle of dimensions 35.02 by 24.97 units.

For dimensions x by y, the area is $A = xy$ so that $dA = \frac{\partial A}{\partial x}\,dx + \frac{\partial A}{\partial y}\,dy = y\,dx + x\,dy$. With $x = 35$, $dx = 0.02$, $y = 25$, and $dy = -0.03$, we have $A = 35(25) = 875$ and $dA = 25(0.02) + 35(-0.03) = -0.55$. The area is approximately $A + dA = 874.45$ square units.

5. Approximate the change in the hypotenuse of a right triangle of legs 6 and 8 inches when the shorter leg is lengthened by $\frac{1}{4}$ inch and the longer leg is shortened by $\frac{1}{8}$ inch.

Let x, y, and z be the shorter leg, the longer leg, and the hypotenuse of the triangle. Then

$$z = \sqrt{x^2 + y^2} \qquad \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} \qquad dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$$

When $x = 6$, $y = 8$, $dx = \frac{1}{4}$, and $dy = -\frac{1}{8}$, then $dz = \frac{6(\frac{1}{4}) + 8(-\frac{1}{8})}{\sqrt{6^2 + 8^2}} = \frac{1}{20}$ inch. Thus the hypotenuse is lengthened by approximately $\frac{1}{20}$ inch.

6. The power consumed in an electrical resistor is given by $P = E^2/R$ (in watts). If $E = 200$ volts and $R = 8$ ohms, by how much does the power change if E is decreased by 5 volts and R is decreased by 0.2 ohm?

We have
$$\frac{\partial P}{\partial E} = \frac{2E}{R} \qquad \frac{\partial P}{\partial R} = -\frac{E^2}{R^2} \qquad dP = \frac{2E}{R}\,dE - \frac{E^2}{R^2}\,dR$$

When $E = 200$, $R = 8$, $dE = -5$, and $dR = -0.2$, then

$$dP = \frac{2(200)}{8}(-5) - \left(\frac{200}{8}\right)^2(-0.2) = -250 + 125 = -125$$

The power is reduced by approximately 125 watts.

7. The dimensions of a rectangular block of wood were found to be 10, 12, and 20 inches, with a possible error of 0.05 in in each of the measurements. Find (approximately) the greatest error in the surface area of the block and the percentage error in the area caused by the errors in the individual measurements.

The surface area is $S = 2(xy + yz + zx)$; then

$$dS = \frac{\partial S}{\partial x}\,dx + \frac{\partial S}{\partial y}\,dy + \frac{\partial S}{dz}\,dz = 2(y + z)\,dx + 2(x + z)\,dy + 2(y + x)\,dz$$

The greatest error in S occurs when the errors in the lengths are of the same sign, say positive. Then

$$dS = 2(12 + 20)(0.05) + 2(10 + 20)(0.05) + 2(12 + 10)(0.05) = 8.4 \text{ in}^2$$

The percentage error is (error/area)(100) = (8.4/1120)(100) = 0.75%.

8. For the formula $R = E/C$, find the maximum error and the percentage error if $C = 20$ with a possible error of 0.1 and $E = 120$ with a possible error of 0.05.

Here $$dR = \frac{\partial R}{\partial E}\,dE + \frac{\partial R}{\partial C}\,dC = \frac{1}{C}\,dE - \frac{E}{C^2}\,dC$$

The maximum error will occur when $dE = 0.05$ and $dC = -0.1$; then $dR = \frac{0.05}{20} - \frac{120}{400}(-0.1) = 0.0325$ is the approximate maximum error. The percentage error is $\frac{dR}{R}(100) = \frac{0.0325}{8}(100) = 0.40625 = 0.41\%$.

9. Two sides of a triangle were measured as 150 and 200 ft, and the included angle as 60°. If the possible errors are 0.2 ft in measuring the sides and 1° in the angle, what is the greatest possible error in the computed area?

Here $$A = \frac{1}{2}xy\sin\theta \qquad \frac{\partial A}{\partial x} = \frac{1}{2}y\sin\theta \qquad \frac{\partial A}{\partial y} = \frac{1}{2}x\sin\theta \qquad \frac{\partial A}{\partial \theta} = \frac{1}{2}xy\cos\theta$$

and $$dA = \frac{1}{2}y\sin\theta\,dx + \frac{1}{2}x\sin\theta\,dy + \frac{1}{2}xy\cos\theta\,d\theta$$

When $x = 150$, $y = 200$, $\theta = 60°$, $dx = 0.2$, $dy = 0.2$, and $d\theta = 1° = \pi/180$, then

$$dA = \tfrac{1}{2}(200)(\sin 60°)(0.2) + \tfrac{1}{2}(150)(\sin 60°)(0.2) + \tfrac{1}{2}(150)(200)(\cos 60°)(\pi/180) = 161.21\ \text{ft}^2$$

10. Find dz/dt, given $z = x^2 + 3xy + 5y^2$; $x = \sin t$, $y = \cos t$.

Since $$\frac{\partial z}{\partial x} = 2x + 3y \qquad \frac{\partial z}{\partial y} = 3x + 10y \qquad \frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = -\sin t$$

we have $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x + 3y)\cos t - (3x + 10y)\sin t$$

11. Find dz/dt, given $z = \ln(x^2 + y^2)$; $x = e^{-t}$, $y = e^t$.

Since $$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2} \qquad \frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2} \qquad \frac{dx}{dt} = -e^{-t} \qquad \frac{dy}{dt} = e^t$$

we have $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{2x}{x^2 + y^2}(-e^{-t}) + \frac{2y}{x^2 + y^2}e^t = 2\,\frac{ye^t - xe^{-t}}{x^2 + y^2}$$

12. Let $z = f(x, y)$ be a continuous function of x and y with continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, and let y be a differentiable function of x. Then z is a differentiable function of x. Find a formula for dz/dx.

By (63.3), $$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

The shift in notation from z to f is made here to avoid possible confusion arising from the use of dz/dx and $\partial z/\partial x$ in the same expression.

13. Find dz/dx, given $z = f(x, y) = x^2 + 2xy + 4y^2$, $y = e^{ax}$.

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = (2x + 2y) + (2x + 8y)ae^{ax} = 2(x + y) + 2a(x + 4y)e^{ax}$$

14. Find (*a*) dz/dx and (*b*) dz/dy, given $z = f(x, y) = xy^2 + x^2y$, $y = \ln x$.

(*a*) Here x is the independent variable:

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = (y^2 + 2xy) + (2xy + x^2)\frac{1}{x} = y^2 + 2xy + 2y + x$$

(*b*) Here y is the independent variable:

$$\frac{dz}{dy} = \frac{\partial f}{\partial x}\frac{dx}{dy} + \frac{\partial f}{\partial y} = (y^2 + 2xy)x + (2xy + x^2) = xy^2 + 2x^2y + 2xy + x^2$$

15. The altitude of a right circular cone is 15 inches and is increasing at 0.2 in/min. The radius of the base is 10 inches and is decreasing at 0.3 in/min. How fast is the volume changing?

Let x be the radius, and y the altitude of the cone (Fig. 63-2). From $V = \frac{1}{3}\pi x^2 y$, considering x and y as functions of time t, we have

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = \frac{\pi}{3}\left(2xy\frac{dx}{dt} + x^2\frac{dy}{dt}\right) = \frac{\pi}{3}[2(10)(15)(-0.3) + 10^2(0.2)] = -70\pi/3 \text{ in}^3/\text{min}$$

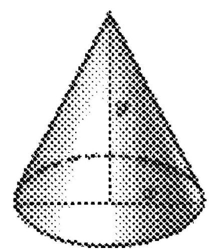

Fig. 63-2

16. A point P is moving along the curve of intersection of the paraboloid $\frac{x^2}{16} - \frac{y^2}{9} = z$ and the cylinder $x^2 + y^2 = 5$, with x, y, and z expressed in inches. If x is increasing at 0.2 in/min, how fast is z changing when $x = 2$?

From $z = \frac{x^2}{16} - \frac{y^2}{9}$, we obtain $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{x}{8}\frac{dx}{dt} - \frac{2y}{9}\frac{dy}{dt}$. Since $x^2 + y^2 = 5$, $y = \pm 1$ when $x = 2$; also, differentiation yields $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$.

When $y = 1$, $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{2}{1}(0.2) = -0.4$ and $\frac{dz}{dt} = \frac{2}{8}(0.2) - \frac{2}{9}(-0.4) = \frac{5}{36}$ in/min.

When $y = -1$, $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = 0.4$ and $\frac{dz}{dt} = \frac{2}{8}(0.2) - \frac{2}{9}(-1)(0.4) = \frac{5}{36}$ in/min.

17. Find $\partial z/\partial r$ and $\partial z/\partial s$, given $z = x^2 + xy + y^2$; $x = 2r + s$, $y = r - 2s$.

Here $\frac{\partial z}{\partial x} = 2x + y \qquad \frac{\partial z}{\partial y} = x + 2y \qquad \frac{\partial x}{\partial r} = 2 \qquad \frac{\partial x}{\partial s} = 1 \qquad \frac{\partial y}{\partial r} = 1 \qquad \frac{\partial y}{\partial s} = -2$

Then
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (2x + y)(2) + (x + 2y)(1) = 5x + 4y$$

and
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2x + y)(1) + (x + 2y)(-2) = -3y$$

18. Find $\frac{\partial u}{\partial \rho}$, $\frac{\partial u}{\partial \beta}$, and $\frac{\partial u}{\partial \theta}$, given $u = x^2 + 2y^2 + 2z^2$; $x = \rho \sin\beta\cos\theta$, $y = \rho\sin\beta\sin\theta$, $z = \rho\cos\beta$.

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \rho} = 2x\sin\beta\cos\theta + 4y\sin\beta\sin\theta + 4z\cos\beta$$

$$\frac{\partial u}{\partial \beta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \beta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \beta}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial \beta}=2x\,\rho\cos\beta\cos\theta+4y\,\rho\cos\beta\sin\theta-4z\,\rho\sin\beta$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}=-2x\,\rho\sin\beta\sin\theta+4y\,\rho\sin\beta\cos\theta$$

19. Find du/dx, given $u=f(x,y,z)=xy+yz+zx$; $y=1/x$, $z=x^2$.

From (*63.6*),

$$\frac{du}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}+\frac{\partial f}{\partial z}\frac{dz}{dx}=(y+z)+(x+z)\left(-\frac{1}{x^2}\right)+(y+x)2x=y+z+2x(x+y)-\frac{x+z}{x^2}$$

20. If $z=f(x,y)$ is a continuous function of x and y possessing continuous first partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, derive the basic formula

$$\Delta z=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y+\epsilon_1\Delta x+\epsilon_2\Delta y \tag{1}$$

where ϵ_1 and $\epsilon_2\to 0$ as Δx and $\Delta y\to 0$.

When x and y are given increments Δx and Δy respectively, the increment given to z is

$$\begin{aligned}\Delta z&=f(x+\Delta x,y+\Delta y)-f(x,y)\\&=[f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)]+[f(x,y+\Delta y)-f(x,y)]\end{aligned} \tag{2}$$

In the first bracketed expression, only x changes; in the second, only y changes. Thus, the law of the mean (*26.5*) may be applied to each:

$$f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)=\Delta x\,f_x(x+\theta_1\Delta x,y+\Delta y) \tag{3}$$

$$f(x,y+\Delta y)-f(x,y)=\Delta y\,f_y(x,y+\theta_2\Delta y) \tag{4}$$

where $0<\theta_1<1$ and $0<\theta_2<1$. Note that here the derivatives involved are partial derivatives.

Since $\partial z/\partial x=f_x(x,y)$ and $\partial z/\partial y=f_y(x,y)$ are, by hypothesis, continuous functions of x and y,

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f_x(x+\theta_1\Delta x,y+\Delta y)=f_x(x,y) \quad\text{and}\quad \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f_y(x,y+\theta_2\Delta y)=f_y(x,y)$$

Then $\quad f_x(x+\theta_1\Delta x,y+\Delta y)=f_x(x,y)+\epsilon_1 \quad$ and $\quad f_y(x,y+\theta_2\Delta y)=f_y(x,y)+\epsilon_2$

where $\epsilon_1\to 0$ and $\epsilon_2\to 0$ as Δx and $\Delta y\to 0$.

After making these replacements in (*3*) and (*4*) and then substituting in (*1*), we have, as required,

$$\Delta z=[f_x(x,y)+\epsilon_1]\Delta x+[f_y(x,y)+\epsilon_2]\Delta y=f_x(x,y)\Delta x+f_y(x,y)\Delta y+\epsilon_1\Delta x+\epsilon_2\Delta y$$

Note that the total derivative dz is a fairly good approximation of the total increment Δz when $|\Delta x|$ and $|\Delta y|$ are small.

Supplementary Problems

21. Find the total differential, given:

(*a*) $z=x^3y+2xy^3$ *Ans.* $dz=(3x^2+2y^2)y\,dx+(x^2+6y^2)x\,dy$

(*b*) $\theta=\arctan(y/x)$ *Ans.* $d\theta=\dfrac{x\,dy-y\,dx}{x^2+y^2}$

(*c*) $z=e^{x^2-y^2}$ *Ans.* $dz=2z(x\,dx-y\,dy)$

(*d*) $z=x(x^2+y^2)^{-1/2}$ *Ans.* $dz=\dfrac{y(y\,dx-x\,dy)}{(x^2+y^2)^{3/2}}$

22. The fundamental frequency of vibration of a string or wire of circular section under tension T is $n = \frac{1}{2rl}\sqrt{\frac{T}{\pi d}}$, where l is the length, r the radius, and d the density of the string. Find (a) the approximate effect of changing l by a small amount dl, (b) the effect of changing T by a small amount dT, and (c) the effect of changing l and T simultaneously.

Ans. (a) $-(n/l)\,dl$; (b) $(n/2T)\,dT$; (c) $n(-dl/l + dT/2T)$

23. Use differentials to compute (a) the volume of a box with square base of side 8.005 and height 9.996 ft; (b) the diagonal of a rectangular box of dimensions 3.03 by 5.98 by 6.01 ft.

Ans. (a) 640.544 ft^3; (b) 9.003 ft

24. Approximate the maximum possible error and the percentage of error when z is computed by the given formula:

(a) $z = \pi r^2 h$; $r = 5 \pm 0.05$, $h = 12 \pm 0.1$ *Ans.* 8.5π; 2.8%

(b) $1/z = 1/f + 1/g$; $f = 4 \pm 0.01$, $g = 8 \pm 0.02$ *Ans.* 0.0067; 0.25%

(c) $z = y/x$; $x = 1.8 \pm 0.1$, $y = 2.4 \pm 0.1$ *Ans.* 0.13; 10%

25. Find the approximate maximum percentage of error in:

(a) $\omega = \sqrt[3]{g/b}$ if there is a possible 1% error in measuring g and a possible $\frac{1}{2}$% error in measuring b. $\left(\textit{Hint}: \ln \omega = \frac{1}{3}(\ln g - \ln b);\ \frac{d\omega}{\omega} = \frac{1}{3}\left(\frac{dg}{g} - \frac{db}{b}\right);\ \left|\frac{dg}{g}\right| = 0.01;\ \left|\frac{db}{b}\right| = 0.005\right)$ *Ans.* 0.005

(b) $g = 2s/t^2$ if there is a possible 1% error in measuring s and $\frac{1}{4}$% error in measuring t.
Ans. 0.015

26. Find du/dt, given:

(a) $u = x^2y^3$; $x = 2t^3$, $y = 3t^2$ *Ans.* $6xy^2t(2yt + 3x)$

(b) $u = x\cos y + y\sin x$; $x = \sin 2t$, $y = \cos 2t$
Ans. $2(\cos y + y\cos x)\cos 2t - 2(-x\sin y + \sin x)\sin 2t$

(c) $u = xy + yz + zx$; $x = e^t$, $y = e^{-t}$, $z = e^t + e^{-t}$ *Ans.* $(x + 2y + z)e^t - (2x + y + z)e^{-t}$

27. At a certain instant the radius of a right circular cylinder is 6 inches and is increasing at the rate 0.2 in/sec, while the altitude is 8 inches and is decreasing at the rate 0.4 in/s. Find the time rate of change (a) of the volume and (b) of the surface at that instant.

Ans. (a) 4.8π in^3/sec; (b) 3.2π in^2/sec

28. A particle moves in a plane so that at any time t its abscissa and ordinate are given by $x = 2 + 3t$, $y = t^2 + 4$ with x and y in feet and t in minutes. How is the distance of the particle from the origin changing when $t = 1$? *Ans.* $5/\sqrt{2}$ ft/min

29. A point is moving along the curve of intersection of $x^2 + 3xy + 3y^2 = z^2$ and the plane $x - 2y + 4 = 0$. When $x = 2$ and is increasing at 3 units/sec, find (a) how y is changing, (b) how z is changing, and (c) the speed of the point.

Ans. (a) increasing 3/2 units/sec; (b) increasing 75/14 units/sec at $(2, 3, 7)$ and decreasing 75/14 units/sec at $(2, 3, -7)$; (c) 6.3 units/sec

30. Find $\partial z/\partial s$ and $\partial z/\partial t$, given:

(a) $z = x^2 - 2y^2$; $x = 3s + 2t$, $y = 3s - 2t$ *Ans.* $6(x - 2y)$; $4(x + 2y)$

(b) $z = x^2 + 3xy + y^2$; $x = \sin s + \cos t$, $y = \sin s - \cos t$ *Ans.* $5(x + y)\cos s$; $(x - y)\sin t$

(c) $z = x^2 + 2y^2$; $x = e^s - e^t$, $y = e^s + e^t$ *Ans.* $2(x + 2y)e^s$; $2(2y - x)e^t$

(d) $z = \sin(4x + 5y)$; $x = s + t$, $y = s - t$ *Ans.* $9\cos(4x + 5y)$; $-\cos(4x + 5y)$

(e) $z = e^{xy}$; $x = s^2 + 2st$, $y = 2st + t^2$ *Ans.* $2e^{xy}[tx + (s + t)y]$; $2e^{xy}[(s + t)x + sy]$

31. (*a*) If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

(*b*) If $u = f(x, y)$ and $x = r\cosh s$, $y = r\sinh s$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 - \frac{1}{s^2}\left(\frac{\partial u}{\partial s}\right)^2$$

32. (*a*) If $z = f(x + \alpha y) + g(x - \alpha y)$, show that $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial^2 z}{\partial y^2}$. (*Hint*: Write $z = f(u) + g(v)$, $u = x + \alpha y$, $v = x - \alpha y$.)

(*b*) If $z = x^n f(y/x)$, show that $x\,\partial z/\partial x + y\,\partial z/\partial y = nz$.

(*c*) If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$, show that, subject to continuity conditions

$$\frac{d^2 z}{dt^2} = f_{xx}(g')^2 + 2f_{xy}g'h' + f_{yy}(h')^2 + f_x g'' + f_y h''$$

(*d*) If $z = f(x, y)$; $x = g(r, s)$, $y = h(r, s)$, show that, subject to continuity conditions

$$\frac{\partial^2 z}{\partial r^2} = f_{xx}(g_r)^2 + 2f_{xy}g_r h_r + f_{yy}(h_r)^2 + f_x g_{rr} + f_y h_{rr}$$

$$\frac{\partial^2 z}{\partial r\,\partial s} = f_{xx}g_r g_s + f_{xy}(g_r h_s + g_s h_r) + f_{yy}h_r h_s + f_x g_{rs} + f_y h_{rs}$$

$$\frac{\partial^2 z}{\partial s^2} = f_{xx}(g_s)^2 + 2f_{xy}g_s h_s + f_{yy}(h_s)^2 + f_x g_{ss} + f_y h_{ss}$$

33. A function $f(x, y)$ is called *homogeneous of order n* if $f(tx, ty) = t^n f(x, y)$. (For example, $f(x, y) = x^2 + 2xy + 3y^2$ is homogeneous of order 2; $f(x, y) = x\sin(y/x) + y\cos(y/x)$ is homogeneous of order 1.) Differentiate $f(tx, ty) = t^n f(x, y)$ with respect to t and replace t by 1 to show that $xf_x + yf_y = nf$. Verify this formula using the two given examples. See also Problem 32(*b*).

34. If $z = \phi(u, v)$, where $u = f(x, y)$ and $v = g(x, y)$, and if $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, show that

(*a*) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ (*b*) $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = \left\{\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2\right\}\left(\dfrac{\partial^2 \phi}{\partial u^2} + \dfrac{\partial^2 \phi}{\partial v^2}\right)$

35. Use (*1*) of Problem 20 to derive the chain rules (*63.3*) and (*63.5*). (*Hint*: For (*63.3*), divide by Δt.)

Chapter 64

Implicit Functions

THE DIFFERENTIATION of a function of one variable, defined implicitly by a relation $f(x, y) = 0$, was treated intuitively in Chapter 11. For this case, we state without proof:

Theorem 64.1: If $f(x, y)$ is continuous in a region including a point (x_0, y_0) for which $f(x_0, y_0) = 0$, if $\partial f/\partial x$ and $\partial f/\partial y$ are continuous throughout the region, and if $\partial f/\partial y \neq 0$ at (x_0, y_0), then there is a neighborhood of (x_0, y_0) in which $f(x, y) = 0$ can be solved for y as a continuous differentiable function of x, $y = \phi(x)$, with $y_0 = \phi(x_0)$ and $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$.

(See Problems 1 to 3.)

Extending this theorem, we have the following:

Theorem 64.2: If $F(x, y, z)$ is continuous in a region including a point (x_0, y_0, z_0) for which $F(x_0, y_0, z_0) = 0$, if $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, and $\frac{\partial F}{\partial z}$ are continuous throughout the region, and if $\partial F/\partial z \neq 0$ at (x_0, y_0, z_0), then there is a neighborhood of (x_0, y_0, z_0) in which $F(x, y, z) = 0$ can be solved for z as a continuous differentiable function of x and y, $z = \phi(x, y)$, with $z_0 = \phi(x_0, y_0)$ and $\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$, $\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$.

(See Problems 4 and 5.)

Theorem 64.3: If $f(x, y, u, v)$ and $g(x, y, u, v)$ are continuous in a region including the point (x_0, y_0, u_0, v_0) for which $f(x_0, y_0, u_0, v_0) = 0$ and $g(x_0, y_0, u_0, v_0) = 0$, if the first partial derivatives of f and of g are continuous throughout the region, and if at (x_0, y_0, u_0, v_0) the determinant $J\left(\frac{f, g}{u, v}\right) \equiv \begin{vmatrix} \partial f/\partial u & \partial f/\partial v \\ \partial g/\partial u & \partial g/\partial v \end{vmatrix} \neq 0$, then there is a neighborhood of (x_0, y_0, u_0, v_0) in which $f(x, y, u, v) = 0$ and $g(x, y, u, v) = 0$ can be solved simultaneously for u and v as continuous differentiable functions of x and y, $u = \phi(x, y)$ and $v = \psi(x, y)$. If at (x_0, y_0, u_0, v_0) the determinant $J\left(\frac{f, g}{x, y}\right) \neq 0$, then there is a neighborhood of (x_0, y_0, u_0, v_0) in which $f(x, y, u, v) = 0$ and $g(x, y, u, v) = 0$ can be solved for x and y as continuous differentiable functions of u and v, $x = h(u, v)$ and $y = k(u, v)$.

(See Problems 6 and 7.)

Solved Problems

1. Use Theorem 64.1 to show that $x^2 + y^2 - 13 = 0$ defines y as a continuous differentiable function of x in any neighborhood of the point $(2, 3)$ that does not include a point of the x axis. Find the derivative at the point.

Set $f(x, y) = x^2 + y^2 - 13$. Then $f(2, 3) = 0$, while in any neighborhood of $(2, 3)$ in which the function is defined, its partial derivatives $\partial f/\partial x = 2x$ and $\partial f/\partial y = 2y$ are continuous, and $\partial f/\partial y \neq 0$. Then

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{x}{y} = -\frac{2}{3} \text{ at } (2, 3)$$

2. Find dy/dx, given $f(x, y) = y^3 + xy - 12 = 0$.

We have $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = 3y^2 + x$. So $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{y}{3y^2 + x}$

3. Find dy/dx, given $e^x \sin y + e^y \sin x = 1$.

Put $f(x, y) = e^x \sin y + e^y \sin x - 1$. Then $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{e^x \sin y + e^y \cos x}{e^x \cos y + e^y \sin x}$.

4. Find $\partial z/\partial x$ and $\partial z/\partial y$, given $F(x, y, z) = x^2 + 3xy - 2y^2 + 3xz + z^2 = 0$.

Treating z as a function of x and y defined by the relation and differentiating partially with respect to x and again with respect to y, we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = (2x + 3y + 3z) + (3x + 2z)\frac{\partial z}{\partial x} = 0 \qquad (1)$$

and

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = (3x - 4y) + (3x + 2z)\frac{\partial z}{\partial y} = 0 \qquad (2)$$

From (1), $\dfrac{\partial z}{\partial x} = -\dfrac{\partial F/\partial x}{\partial F/\partial z} = -\dfrac{2x + 3y + 3z}{3x + 2z}$. From (2), $\dfrac{\partial z}{\partial y} = -\dfrac{\partial F/\partial y}{\partial F/\partial z} = -\dfrac{3x - 4y}{3x + 2z}$.

5. Find $\partial z/\partial x$ and $\partial z/\partial y$, given $\sin xy + \sin yz + \sin zx = 1$.

Set $F(x, y, z) = \sin xy + \sin yz + \sin zx - 1$; then

$$\frac{\partial F}{\partial x} = y \cos xy + z \cos zx \qquad \frac{\partial F}{\partial y} = x \cos xy + z \cos yz \qquad \frac{\partial F}{\partial z} = y \cos yz + x \cos zx$$

and

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{y \cos xy + z \cos zx}{y \cos yz + x \cos zx} \qquad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{x \cos xy + z \cos yz}{y \cos yz + x \cos zx}$$

6. If u and v are defined as functions of x and y by the equations

$$f(x, y, u, v) = x + y^2 + 2uv = 0 \qquad g(x, y, u, v) = x^2 - xy + y^2 + u^2 + v^2 = 0$$

find (a) $\partial u/\partial x$, $\partial v/\partial x$ and (b) $\partial u/\partial y$, $\partial v/\partial y$.

(a) Differentiating f and g partially with respect to x, we obtain

$$1 + 2v\frac{\partial u}{\partial x} + 2u\frac{\partial v}{\partial x} = 0 \qquad \text{and} \qquad 2x - y + 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

Solving these relations simultaneously for $\partial u/\partial x$ and $\partial v/\partial x$, we find

$$\frac{\partial u}{\partial x} = \frac{v + u(y - 2x)}{2(u^2 - v^2)} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = \frac{v(2x - y) - u}{2(u^2 - v^2)}$$

(b) Differentiating f and g partially with respect to y, we obtain

$$2y + 2v\frac{\partial u}{\partial y} + 2u\frac{\partial v}{\partial y} = 0 \qquad \text{and} \qquad -x + 2y + 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

Then

$$\frac{\partial u}{\partial y} = \frac{u(x - 2y) + 2vy}{2(u^2 - v^2)} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{v(2y - x) - 2uy}{2(u^2 - v^2)}$$

7. Given $u^2 - v^2 + 2x + 3y = 0$ and $uv + x - y = 0$, find (a) $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$ and (b) $\dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}$.

(a) Here x and y are to be considered as independent variables. Differentiate the given equations partially with respect to x, obtaining

$$2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} + 2 = 0 \qquad \text{and} \qquad v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} + 1 = 0$$

Solve these relations simultaneously to obtain $\frac{\partial u}{\partial x} = -\frac{u+v}{u^2+v^2}$ and $\frac{\partial v}{\partial x} = \frac{v-u}{u^2+v^2}$.

Differentiate the given equations partially with respect to y, obtaining

$$2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} + 3 = 0 \qquad \text{and} \qquad v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} - 1 = 0$$

Solve simultaneously to obtain $\frac{\partial u}{\partial y} = \frac{2v-3u}{2(u^2+v^2)}$ and $\frac{\partial v}{\partial y} = \frac{2u+3v}{2(u^2+v^2)}$.

(*b*) Here u and v are to be considered as independent variables. Differentiate the given equations partially with respect to u, obtaining $2u + 2\frac{\partial x}{\partial u} + 3\frac{\partial y}{\partial u} = 0$ and $v + \frac{\partial x}{\partial u} - \frac{\partial y}{\partial u} = 0$. Then $\frac{\partial x}{\partial u} = -\frac{2u+3v}{5}$ and $\frac{\partial y}{\partial u} = \frac{2(v-u)}{5}$.

Differentiate the given equations partially with respect to v, obtaining $-2v + 2\frac{\partial x}{\partial v} + 3\frac{\partial y}{\partial v} = 0$ and $u + \frac{\partial x}{\partial v} - \frac{\partial y}{\partial v} = 0$. Then $\frac{\partial x}{\partial v} = \frac{2v-3u}{5}$ and $\frac{\partial y}{\partial v} = \frac{2u(u+v)}{5}$.

Supplementary Problems

8. Find dy/dx, given

(*a*) $x^3 - x^2y + xy^2 - y^3 = 1$ (*b*) $xy - e^x \sin y = 0$ (*c*) $\ln(x^2+y^2) - \arctan y/x = 0$

Ans. (*a*) $\frac{3x^2-2xy+y^2}{x^2-2xy+3y^2}$; (*b*) $\frac{e^x \sin y - y}{x - e^x \cos y}$; (*c*) $\frac{2x+y}{x-2y}$

9. Find $\partial z/\partial x$ and $\partial z/\partial y$, given

(*a*) $3x^2 + 4y^2 - 5z^2 = 60$ *Ans.* $\partial z/\partial x = 3x/5z$; $\partial z/\partial y = 4y/5z$

(*b*) $x^2 + y^2 + z^2 + 2xy + 4yz + 8zx = 20$ *Ans.* $\frac{\partial z}{\partial x} = -\frac{x+y+4z}{4x+2y+z}$; $\frac{\partial z}{\partial y} = -\frac{x+y+2z}{4x+2y+z}$

(*c*) $x + 3y + 2z = \ln z$ *Ans.* $\frac{\partial z}{\partial x} = \frac{z}{1-2z}$; $\frac{\partial z}{\partial y} = \frac{3z}{1-2z}$

(*d*) $z = e^x \cos(y+z)$ *Ans.* $\frac{\partial z}{\partial x} = \frac{z}{1+e^x \sin(y+z)}$; $\frac{\partial z}{\partial y} = \frac{-e^x \sin(y+z)}{1+e^x \sin(y+z)}$

(*e*) $\sin(x+y) + \sin(y+z) + \sin(z+x) = 1$

Ans. $\frac{\partial z}{\partial x} = -\frac{\cos(x+y)+\cos(z+x)}{\cos(y+z)+\cos(z+x)}$; $\frac{\partial z}{\partial y} = -\frac{\cos(x+y)+\cos(y+z)}{\cos(y+z)+\cos(z+x)}$

10. Find all the first and second partial derivatives of z, given $x^2 + 2yz + 2zx = 1$.

Ans. $\frac{\partial z}{\partial x} = -\frac{x+z}{x+y}$; $\frac{\partial z}{\partial y} = -\frac{z}{x+y}$; $\frac{\partial^2 z}{\partial x^2} = \frac{x-y+2z}{(x+y)^2}$; $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{x+2z}{(x+y)^2}$; $\frac{\partial^2 z}{\partial y^2} = \frac{2z}{(x+y)^2}$

11. If $F(x, y, z) = 0$ show that $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$.

12. If $z = f(x, y)$ and $g(x, y) = 0$, show that $\frac{dz}{dx} = \frac{\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} = \frac{1}{\frac{\partial g}{\partial y}} J\left(\frac{f, g}{x, y}\right)$.

13. If $f(x, y) = 0$ and $g(z, x) = 0$, show that $\dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x}\dfrac{\partial y}{\partial z} = \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial z}$.

14. Find the first partial derivatives of u and v with respect to x and y and the first partial derivatives of x and y with respect to u and v, given $2u - v + x^2 + xy = 0$, $u + 2v + xy - y^2 = 0$.

Ans. $\dfrac{\partial u}{\partial x} = -\dfrac{1}{5}(4x + 3y)$; $\dfrac{\partial v}{\partial x} = \dfrac{1}{5}(2x - y)$; $\dfrac{\partial u}{\partial y} = \dfrac{1}{5}(2y - 3x)$; $\dfrac{\partial v}{\partial y} = \dfrac{4y - x}{5}$; $\dfrac{\partial x}{\partial u} = \dfrac{4y - x}{2(x^2 - 2xy - y^2)}$;
$\dfrac{\partial y}{\partial u} = \dfrac{y - 2x}{2(x^2 - 2xy - y^2)}$; $\dfrac{\partial x}{\partial v} = \dfrac{3x - 2y}{2(x^2 - 2xy - y^2)}$; $\dfrac{\partial y}{\partial v} = \dfrac{-4x - 3y}{2(x^2 - 2xy - y^2)}$

15. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, and $w = x^3 + y^3 + z^3$, show that

$$\frac{\partial x}{\partial u} = \frac{yz}{(x - y)(x - z)} \qquad \frac{\partial y}{\partial v} = \frac{x + z}{2(x - y)(y - z)} \qquad \frac{\partial z}{\partial w} = \frac{1}{3(x - z)(y - z)}$$