Line integrals

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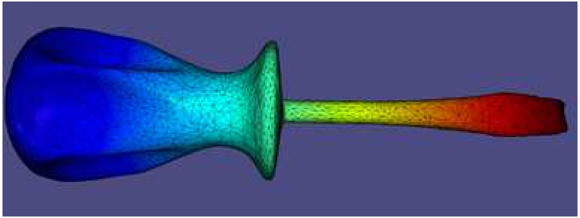
Scalar and vector fields

If a scalar function $u(r)$ is defined at each r in some region

- u is a scalar field in that region.

Examples: temperature, pressure, altitude, CO_2 concentration



Scalar and vector fields

Similarly, if a vector function v(r) is defined at each point, then

- v is a vector field in that region.

Examples: wind velocity, magnetic field, traffic flows, optical flow, electric fields

We introduce three field operators which reveal interesting collective field properties.

- the gradient of a scalar field,
- the divergence of a vector field, and
- the curl of a vector field.

The gradient of a scalar field

If $U(r)$ is a scalar field, its gradient is defined in Cartesians cords by

$$grad\ U = \frac{\partial U}{\partial x} i + \frac{\partial U}{\partial y} j + \frac{\partial U}{\partial z} k$$

It is usual to define the vector operator ∇

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]$$

which is called “del” or “nabla”. We can write $grad\ U = \nabla U$.

grad U or ∇U is a vector field!

The divergence of a vector field

Let a be a vector field:

$$a(x, y, z) = [a_1, a_2, a_3]$$

The divergence of a at any point is defined in Cartesian co-ordinates by

$$div\ \ a = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

The divergence of a vector field is a scalar field. We can write div as a scalar product with the ∇ vector differential operator:

$$div\ \ a = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \circ a = \nabla \circ a$$

The Laplacian: div (grad U) of a scalar field

grad U of any scalar field U is a vector field. We can take the *div* of any vector field. Thus we can certainly compute *div* (*grad U*)

$$\nabla \circ (\nabla U) = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \circ \left(\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] U\right) =$$

$$= \left(\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \circ \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\right) U =$$

$$= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

The operator ∇^2 (del-squared) is called the Laplacian

$$\nabla^2 U = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U$$

Appears in Engineering through, e.g., Laplace's equation

$$\nabla^2 U = 0$$

The curl of a vector field

So far we have applied the operator ∇ to a scalar field (∇U) and dotted with a vector field ($\nabla \circ a$).

- Now we cross it with a vector field

This gives the curl of a vector field

$$\nabla \times a = curl\ (a)$$

We can follow the pseudo-determinant recipe for vector products, so that

$$\nabla \times a = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \left[\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right), \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial y} \right), \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) \right]$$

Some definitions involving div, curl and grad

- A vector field with zero divergence is said to be solenoidal.
- A vector field with zero curl is said to be irrotational.
- A scalar field with zero gradient is said to be constant.

Line Integrals

By a line integral, we mean integrating a function along a curve C. We denote this by

$$\int_C f(x, y, z) ds$$

where s is the arc length parameter along the curve.

If C is smoothly parametrised by

$$r(t) = [x(t), y(t), z(t)],$$

then the line integral of $f(x, y, z)$ along C is given by

$$\int_C f(x, y, z)ds = \int_a^b f(x, y, z)||r'(t)||dt$$

Consider a vector field

$$F(x, y, z) = [f(x, y, z), g(x, y, z), h(x, y, z)]$$

for which we can find the line integral for each component in the same way as before. However, if we have a small length in each direction, this can be represented by

$$dr = [dx, dy, dz] .$$

Then, the line integral of F along C is denoted by

$$\int_C F \circ dr = \int_C (f(x, y, z)dx + g(x, y, z)dy + h(x, y, z)dz)$$

which we calculate by an analogous method to the simpler case as

$$\int_C F \circ dr = \int_a^b F(r(t)) \circ r'(t)dt$$

Line Integrals along smooth piecewise curves

If we have a smooth curves $C_1, C_2, \ldots, C_n$, then C represents the curve made by joining these together and the line integral is given by

$$\int_C F \circ dr = \int_{C_1} F \circ dr + \int_{C_2} F \circ dr + \ldots + \int_{C_n} F \circ dr$$

Path independence

If a vector field is conservative, line integral will be path independent. In other words, any path chosen will give the same result which is determined by the end points.
Conservative fields are those for which F is defined as the gradient of a function

$$F(x, y, z) = \nabla\varphi(x, y, z)$$

Then for the curve C from (x_0, y_0, z_0) to (x_1, y_1, z_1)

$$\int_C F \circ dr = \int_C \nabla\varphi(x, y, z) \circ dr = \varphi(x_0, y_0, z_0) - \varphi(x_1, y_1, z_1).$$

Conservative Field Test

If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first order partial derivatives on some open region D, and if the vector field $F(x, y) = [f(x, y), g(x, y)]$ is conservative on D, then

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

at each point in D. Conversely, if D is simply connected and the above condition holds at each point in D, then $F(x, y) = [f(x, y), g(x, y)]$ is conservative.

Theorem (Green's Theorem)

Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve C oriented anti-clockwise. If $f(x,y)$ and $g(x,y)$ are continuous and have continuous first partial derivatives on some open set containing R, then

$$\oint_C F(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

Finding area using Green's Theorem

We can use Green's theorem to find the area of a region via the line integral around its boundary. In the statement of Green's Theorem, if we choose $f(x, y)$ and $g(x, y)$ such that $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1$ we will get an area integral on the right hand side. Two useful choices are for $f(x, y) = 0$, $g(x, y) = x$ which gives

$$A = \iint_R dA = \oint xdy,$$

and for $f(x, y) = 0 - y$, $g(x, y) = 0$ which gives

$$A = -\iint_R dA = \oint ydx.$$

Often, combining these is also useful,

$$A = \frac{1}{2} \oint -ydx + xdy.$$