Lecture
Probability

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Sample Space and Event

- In the study of probability, any process of observation is referred to as an experiment.
- The results of an observation are called the outcomes of the experiment. An experiment is called a random experiment if its outcome cannot be predicted.
- Typical examples of a random experiment are the roll of a die, the toss of a coin, drawing a card from a deck.

Sample Space

Definition

The set of all possible outcomes of a random experiment is called the sample space (or universal set), and it is denoted by S. An element in S is called a sample point. Each outcome of a random experiment corresponds to a sample point.

Example

Find the sample space for the experiment of tossing a coin

1. once,
2. twice.

Example

Find the sample space for the experiment of tossing a coin repeatedly and of counting the number of tosses required until the first head appears.

Example

Find the sample space for the experiment of measuring (in hours) the lifetime of a transistor.

Example

Consider a random experiment of tossing a coin three times.

1. Find the sample space S, if we wish to observe the exact sequences of heads and tails obtained.
2. Find the sample space S, if we wish to observe the number of heads in the three tosses.

Exercises

1. Consider an experiment of drawing two cards at random from a bag containing four cards marked with the integers 1 through 4.
    1. Find the sample space S_1, of the experiment if the first card is replaced before the second is drawn
    2. Find the sample space S_2, of the experiment if the first card is not replaced
2. An experiment consists of rolling a die until a 6 is obtained.
    1. Find the sample space S, if we are interested in all possibilities.
    2. Find the sample space S, if we are interested in the number of throws needed to get a 6.

Definition

Any subset of the sample space S is called an event. A sample point of S is often referred to as an elementary event.

Note that the sample space S is the subset of itself, that is, $S \subset S$. Since S is the set of all possible outcomes, it is often called the certain event.

Example

Consider the experiment of tossing a coin repeatedly and of counting the number of tosses required until the first head appears. Let A be the event that the number of tosses required until the first head appears is even. Let B be the event that the number of tosses required until the first head appears is odd. Let C be the event that the number of tosses required until the first head appears is less than 5. Express events A, B, and C.

Relative Frequency Definition

Suppose that the random experiment is repeated n times. If event A occurs $n(A)$ times, then the probability of event A, denoted $P(A)$, is defined as

$$P(a) = \lim_{n \to \infty} \frac{n(A)}{n}$$

where $n(A)/n$ is called the relative frequency of event A.

Axiomatic Definition

Definition

Let S be a finite sample space and A be an event in S. Then in the axiomatic definition, the, probability $P(A)$ of the event A is a real number assigned to A which satisfies the following three axioms :

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$

If the sample space S is not finite, then axiom 3 must be modified as follows:

3. If $A_1, A_2, \ldots$ is an infinite sequence of mutually exclusive events in S ($A_i \cap A_j = \emptyset$ for $i \neq j$), then
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

Elementary Properties of Probability

By using the above axioms, the following useful properties of probability can be obtained:

1. $P(A^C) = 1 - P(A)$
2. $P(\emptyset) = 0$
3. $P(A) \leq P(B)$ if $A \subset B$
4. $P(A) \leq 1$
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
6. If $A_1, A_2, \ldots, A_n$ are n arbitrary events in S, then
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i \neq j} P(A_i \cap A_j) + \sum_{i \neq j \neq k} P(A_i \cap A_j \cap A_k) + \\ + \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$
7. If $A_1, A_2, \ldots, A_n$, is a finite sequence of mutually exclusive events in S ($A_i \cap A_j = \emptyset$ for $i \neq j$), then
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$

Finite Sample Space

Consider a finite sample space S with n finite elements

$$S = \{\eta_1, \eta_2, \ldots, \eta_n\}$$

where η_i's are elementary events. Let $P(\eta_i) = p_i$. Then

1. $0 \leq p_i \leq 1, \; i = 1, 2, \ldots, n$
2. $\sum\limits_{i=1}^{n} p_i = p_1 + p_2 + \ldots + p_n = 1$
3. If $A = \bigcup\limits_{i \in I} \eta_i$, where I is a collection of subscripts, then

$$P(A) = \sum_{\eta_i \in A} P(\eta_i) = \sum_{i \in I} p_i$$

Equally Likely Events

When all elementary events η_i, $(i = 1, 2, \ldots, n)$ are equally likely, that is,

$$p_1 = p_2 = \ldots = p_n$$

then , we have

$$p_i = \frac{1}{n}, \quad i = 1, 2, \ldots, n$$

and

$$P(A) = \frac{n(A)}{n}$$

where $n(A)$ is the number of outcomes belonging to event A and n is the number of sample points in S.

Example

Consider a telegraph source generating two symbols, dots and dashes. We observed that the dots were twice as likely to occur as the dashes. Find the probabilities of the dot's occurring and the dash's occurring.

Example

The sample space S of a random experiment is given by

$$S = \{a, b, c, d\}$$

with probabilities $P(a) = 0.2$, $P(b) = 0.3$, $P(c) = 0.4$, and $P(d) = 0.1$. Let A denote the event $\{a, b\}$, and B the event $\{b, c, d\}$. Determine the following probabilities: (a) $P(A)$; (b) $P(B)$; (c) $P(A^C)$; (d) $P(A \cup B)$; and (e) $P(A \cap B)$.

Exercises

1. An experiment consists of observing the sum of the dice when two fair dice are thrown. Find (a) the probability that the sum is 7 and (b) the probability that the sum is greater than 10.
2. There are n persons in a room
    1. What is the probability that at least two persons have the same birthday?
    2. Calculate this probability for $n = 50$.
    3. How large need n be for this probability to be greater than 0.5?

1. A committee of 5 persons is to be selected randomly from a group of 5 men and 10 women.
   1. Find the probability that the committee consists of 2 men and 3 women.
   2. Find the probability that the committee consists of all women.

The conditional probability of an event A given event B, denoted by $P(A|B)$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

where $P(A \cap B)$ is the joint probability of A and B. Similarly,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

is the conditional probability of an event B given event A.
From above equations, we have

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

The equation is often quite useful in computing the joint probability of events.

Bayes' Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

The events $A_1, A_2, \ldots, A_n$, are called mutually exclusive and exhaustive if

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = S \text{ and } A_i \cap A_j = \emptyset, \;\; i \neq j$$

Let B be any event in S. Then

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

which is known as the total probability of event B.

Theorem (Bayes' theorem)

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

Example

Consider the experiment of throwing the two fair dice behind you; you are then informed that the sum is not greater than 3.

1. Find the probability of the event that two faces are the same without the information given.
2. Find the probability of the same event with the information given.

Example

Two cards are drawn at random from a deck. Find the probability that both are aces.

Example

Suppose that a laboratory test to detect a certain disease has the following statistics. Let

$$A = \text{ event that the tested person has the disease}$$

$$B = \text{ event that the test result is positive}$$

It is known that

$$P(B|A) = 0.99 \quad P(B|A^C) = 0.005$$

and 0.1 percent of the population actually has the disease. What is the probability that a person has the disease given that the test result is positive?

Exercises

1. Two manufacturing plants produce similar parts. Plant 1 produces $1,000$ parts, 100 of which are defective. Plant 2 produces $2,000$ parts, 150 of which are defective. A part is selected at random and found to be defective. What is the probability that it came from plant 1?
2. Two numbers are chosen at random from among the numbers 1 to 10 without replacement. Find the probability that the second number chosen is 5.

Two events A and B are said to be (statistically) independent if and only if

$$P(A \cap B) = P(A)P(B)$$

It follows immediately that if A and B are independent, then

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

If two events A and B are independent, then it can be shown that A and B^C are also independent; that is

$$P(A \cap B^C) = P(A)P(B^C)$$

Then

$$P(A|B^C) = \frac{P(A \cap B^C)}{P(B^C)} = P(A)$$

Thus, if A is independent of B, then the probability of A's occurrence is unchanged by information as to whether or not B has occurred.

Three events A, B, C are said to be independent if and only if

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cap C) = P(A)P(C)$$

$$P(B \cap C) = P(B)P(C)$$

We may also extend the definition of independence to more than three events. The events $A_1, A_2, \ldots, A_n$ are independent if and only if for every subset $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$ $(2 \leq k \leq n)$ of these events,

$$P(A_{i_!} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \ldots P(A_{i_k})$$

Finally, we define an infinite set of events to be independent if and only if every finite subset of these events is independent.

Example

Consider the experiment of throwing two fair dice. Let A be the event that the sum of the dice is 7, B be the event that the sum of the dice is 6, and C be the event that the first die is 4. Show that events A and C are independent, but events B and C are not independent.

Exercises

1. In the experiment of throwing two fair dice, let A be the event that the first die is odd, B be the event that the second die is odd, and C be the event that the sum is odd. Show that events A, B, and C are pairwise independent, but A, B, and C are not independent.