1 Vector spaces (linear spaces)

1. Let us define in the set $\mathbb{R}^+$ addition $\oplus$ and scalar multiplication $\odot$ of elements of $\mathbb{R}^+$ by the real numbers by the following formulae
$$a \oplus b = ab, \quad r \odot a = a^r.$$
Verify that the set $(\mathbb{R}^+, \oplus, \odot)$ is a vector space over the field of real numbers.

2. Is the set $C^1[a, b]$ of real functions defined on the closed interval $[a, b]$, having a continuous derivative in (a, b), with addition and scalar multiplication, a vector space over the field of real numbers?

3. Let $V = \{(x_1, x_2, x_3) \in \mathbb{F}^3; \quad x_1 + 2x_2 + 2x_3 = 0\}$. Is V a vector space (over the field $\mathbb{F}$)?

4. Check, if $(\mathbb{R}^2, +, \odot)$, where multiplication $\odot$ of elements of the set $\mathbb{R}^2$ by real numbers is defined by the formula
$$a \odot (x_1, x_2) = (ax_1, x_2)$$
is a vector space over the field of real numbers.

2 Vector subspaces (linear subspaces)

1. Check, if the given subset of the vector space $\mathbb{R}^\infty$ is its subspace
   (a) the set of constant sequences
   (b) the set of convergent sequences
   (c) the set of non-decreasing sequences
   (d) the set of sequences with rational terms
   (e) the set of sequences convergent to 0

2. Let V be a vector space over a filed $\mathbb{F}$, and let W_1 and W_2 be two subspaces of V. Show, that the intersection $W_1 \cap W_2$ is also a subspace of V.

3. Let $\mathbb{F}[z]$ denotes the vector space of all polynomials with coefficients in a field $\mathbb{F}$. Let U be the subspace of $\mathbb{F}[z]$ defined as $U = \{az^6 + bz^3; \quad a, b \in \mathbb{F}\}$. Find the subspace W of the space $\mathbb{F}[z]$ such that $\mathbb{F}[z] = U \oplus W$.

3 Linear independence of vectors, spanning sets and bases

1. Check if in the vector space $\mathbb{R}^4$ the following statement are true
   (a) $[4, 6, 4, 5] \in \operatorname{span}([1, 4, 6, 5], [5, 6, 2, 4])$
   (b) $[4, 9, 9, 1] \in \operatorname{span}([1, 2, 3, 5], [3, 7, 9, 8], [1, 3, 4, 7])$

2. Check if the given vectors span the vector space $\mathbb{R}^3$
   (a) $[1, 3, 5], [1, 4, 7], [3, 8, 17]$
   (b) $[1, 2, 4], [7, 6, 4], [9, 7, 3]$

3. Check if the given vectors of the vector space $\mathbb{R}^3$ are linearly independent
   (a) $[3, -1, 2], [-9, 3, -6]$
   (b) $[1, 1, 1], [1, 0, 1], [1, 1, 0]$
   (c) $[1, 1 - 1], [3, 4, 1], [6, 7, 1], [2, 1, -5]$
   (d) $[3, 5, 1], [4, -1, 2], [8, 9, 4]$

4. Show that the vectors $1, \cos x, \cos^2 x, \cos^3 x, \cos^4 x, \ldots$ of the vector space $C(-\infty, \infty)$ are linearly independent.

5. Check if the given vectors form the basis of the vector space $\mathbb{R}^3$
   (a) $[1, 0, -1], [1, 1, 3], [4, 1, 1]$
   (b) $[1, 5, 0], [1, 2, 3], [1, 4, 1]$
   (c) $[4, 4, 5], [8, 7, 9], [5, 3, 4]$

6. Find the coordinates of the vector $[x, y] \in \mathbb{R}^2$ in the basis $([7, 5], [4, 3])$ of the vector space $\mathbb{R}^2$

7. Verify for which numbers $x \in \mathbb{R}$ the given triple forms the basis of the vector space $\mathbb{R}^3$
   (a) $[1, 3, 4], [2, 1, 5], [1, 8, x]$
   (b) $[1, 2, 3], [3, 4, 9], [1, x, 3]$