1 Linear independence of vectors, spanning sets and bases

1. Which of the following sets of vectors in $\mathbb{R}^4$ are linearly independent, (a generating set, a basis)?
   (a) $(1, 1, 1, 1), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 1)$
   (b) $(1, 0, 0, 0), (2, 0, 0, 0)$
   (c) $(17, 39, 25, 10), (13, 12, 99, 4), (16, 1, 0, 0)$
   (d) $(1, \frac{1}{2}, 0, 0), (0, 0, 1, 1), (0, \frac{1}{2}, \frac{1}{2}, 1), (\frac{1}{4}, 0, 0, \frac{1}{4})$

   Extend the linearly independent sets to bases.

2. Are the vectors $x_1 = (1, 0, 1)$; $x_2 = (i, 1, 0)$; $x_3 = (i, 2, 1 + i)$ linearly independent in $\mathbb{C}^3$? Express $x = (1, 2, 3)$ and $y = (i, i, i)$ as linear combinations of x_1, x_2, x_3.

3. Let S be any set and consider the set of maps
   $$f\colon S \to \mathbb{F}^n$$
   such that $f(x) = 0$ for all but finitely many $x \in S$. Make this set into vector space (denoted by $C(S, \mathbb{F}^n)$). Construct a basis for this vector space.

4. Consider the set of polynomial functions $f\colon \mathbb{R} \to \mathbb{R}$,
   $$f(x) = \sum_{i=0}^{n} \alpha_i x^i.$$
   Make this set into a vector space, and construct a natural basis.

5. Let (ξ^1, ξ^2, ξ^3) be an arbitrary vector in $\mathbb{F}^3$. Which of the following subsets are subspaces?
   (a) all vectors with $\xi^1 = \xi^2 = \xi^3$
   (b) all vectors with $\xi^3 = 0$
   (c) all vectors with $\xi^1 = \xi^2 - \xi^3$
   (d) all vectors with $\xi^1 = 1$

6. Find subspaces F_a, F_b, F_c, F_d generated by the sets of previous exercise, and construct bases for these subspaces.

7. Find complementary spaces for subspaces of previous problem and construct bases for these complementary spaces. Show that there exists more then one complementary space for each given subspace.

8. Show that
   (a) $\mathbb{F}^3 = F_a + F_b$
   (b) $\mathbb{F}^3 = F_b + F_c$
   (c) $\mathbb{F}^3 = F_a + F_c$

   Find the intersections $F_a \cap F_b$, $F_b \cap F_c$, $F_a \cap F_c$ and decide in which cases the sums above are direct.

9. Let (x_1, x_2) be a basis of a 2-dimensional vector space. Show that the vectors
   $$\widetilde{x_1} = x_1 + x_2, \quad \widetilde{x_2} = x_1 - x_2$$
   again form a basis. Let (ξ^1, ξ^2) and $(\widetilde{\xi}^1, \widetilde{\xi}^2)$ be the components of a vector x relative to the bases (x_1, x_2) and $(\widetilde{x}_1, \widetilde{x}_2)$ respectively. Express the components $(\widetilde{\xi}^1, \widetilde{\xi}^2)$ in terms of the components (ξ^1, ξ^2).

10. Consider an n-dimensional complex vector space E. Since the multiplication with real coefficients in particular is defined in E, this space may also be considered as a real vector space. Let $(z_1, \ldots, z_n)$ be a basis of E. Show that the vectors $z_1, \ldots, z_n, iz_1, \ldots, iz_n$ form a basis of E if E considered as a real vector space.