1 Linear transformations

1. Is the following transformation linear?

   (a) $T\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad T([x_1, x_2, x_3]) = [4x_1 + 3x_2, x_1^2, x_2 - 4x_3]$.

   (b) $T\colon \mathbb{R}^6 \to \mathbb{R}^6, \quad T([x_1, x_2, x_3, x_4, x_5, x_6]) = [0, 2x_1 - x_2 + 3x_3, -x_2 - 4x_5, 0, x_1 + x_3, 0]$.

2. Which of the given transformations of the vector space of continuous real-valued functions is linear?

   (a) $T\colon C(\mathbb{R}) \to C(\mathbb{R}), \quad [T(g)](x) = g(\sin x)$,

   (b) $T\colon C(\mathbb{R}) \to C(\mathbb{R}), \quad [T(g)](x) = \sin(g(x))$,

   where $g \in C(\mathbb{R}),\ x \in \mathbb{R}$

3. Linear transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ is given by the following mapping. Find $\varphi([x_1, x_2])$.

   (a) $[3, 4] \to [3, 5, 7], \quad [4, 5] \to [4, 7, 9]$

   (b) $[3, -1] \to [2, 5, 5], \quad [5, -2] \to [3, 0, 9]$

4. Linear transformation $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by the following mapping. Find $\varphi([x_1, x_2, x_3])$.

   (a) $[1, 0, 0] \to [2, 1, 1], \quad [1, 1, 0] \to [7, 4, 2], \quad [1, 1, 1] \to [4, 0, 3]$

   (b) $[3, 1, 1] \to [4, 3, 3], \quad [4, 1, 4] \to [5, 9, 1], \quad [5, 1, 3] \to [6, 7, 3]$

5. Check if there exists linear transformation $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that:

   (a) $[2, 4, 3] \to [1, 3], \quad [1, 5, 4] \to [0, 3], \quad [9, 3, 1] \to [7, 6]$

   (b) $[3, 2, 1] \to [3, 1], \quad [4, 1, 5] \to [7, 2], \quad [7, 8, -5] \to [1, 1]$

6. Show that every linear transformation maps linearly dependent set of vectors onto linearly dependent set. Is the similarly formulated theorem for the linearly independent set true?

2 Kernel and image of a linear transformation

1. Find the kernel and the image and the rank of the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ given by the formula $T([x_1, x_2, x_3, x_4]) = [x_1 + 2x_3 + x_4, -2x_1 + x_2 - 3x_3 - 5x_4, x_1 - x_2 + x_3 + 4x_4]$

2. Find two different bases of the image of the linear transformation $T\colon \mathbb{R}^5 \to \mathbb{R}^4$ given by the formula wzorem $T([x_1, x_2, x_3, x_4, x_5]) = [x_1 + x_2 - x_3, -x_1 + 2x_2 + 3x_3 - x_4, 3x_2 + 2x_3 - x_4 - x_5, 2x_4]$

3. Write down the formula of the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ such that $T([-1, 1, -1, 1]) = [0, 2, 1]$, $T([1, 0, 1, 0]) = [1, 1, 2]$ and $\ker(T) = \{[x, 0, 0, t];\ x, t \in \mathbb{R}\}$.

3 Matrix representation of a linear transformation

1. Linear transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by the formula $\varphi[x_1, x_2]) = [4x_1 - x_2, 7x_1 - 3x_2]$ Find its matrix $M_{\mathcal{BC}}(\varphi)$, if the bases $\mathcal{B}$ i $\mathcal{C}$ are:

   (a) $([1, 0], [0, 1]), \quad ([1, 0], [0, 1])$

   (b) $([1, -1], [0, -1]), \quad ([1, 2], [1, 3])$

   (c) $([1, 1], [1, 2]), \quad ([2, 1], [3, 1])$

2. Linear transformation $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is given by the formula $\varphi[x_1, x_2, x_3]) = [x_1 + x_2, x_1 + 2x_2 - x_3]$ Find the matrix $M_{\mathcal{BC}}(\varphi)$, if bases $\mathcal{B}$ i $\mathcal{C}$ are equal to:

   (a) $([2, -1, 0], [1, 3, 2], [0, 4, 1]), \quad ([1, 0], [0, 1])$

   (b) $([3, 1, 1], [5, 1, 6], [4, -1, 2]), \quad ([-1, 1], [1, 0])$

3. Linear transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by the matrix $M_{\mathcal{BC}}(\varphi)$. Calculate $\varphi([x_1, x_2])$ if

   (a) $\mathcal{B} = \mathcal{C} = ([1, 0], [0, 1])$, $M_{\mathcal{BC}}(\varphi) = \begin{bmatrix} 1 & -3 \\ 8 & -5 \end{bmatrix}$

   (b) $\mathcal{B} = ([8, 2], [7, 1])$, $\mathcal{C} = ([6, 7], [4, 5])$, $M_{\mathcal{BC}}(\varphi) = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

4 Change of basis matrix

1. Using the definition find the change of basis matrix from the basis $\mathcal{B}$ to the basis $\mathcal{B}'$ of the vector space $\mathbb{R}^2$, if

   (a) $\mathcal{B} = ([1,1],\ [3,2]),\quad \mathcal{B}' = ([3,4],\ [9,8])$

   (b) $\mathcal{B} = ([7,3],\ [9,4]),\quad \mathcal{B}' = ([1,0],\ [6,7])$

2. Find the change of basis matrix from the basis $\mathcal{B}$ to the basis $\mathcal{B}'$ of the vector space $\mathbb{R}^2$, if

   (a) $\mathcal{B} = ([4,5],\ [7,8]),\quad \mathcal{B}' = ([7,11],\ [1,5])$

   (b) $\mathcal{B} = ([8,3],\ [3,5]),\quad \mathcal{B}' = ([1,0],\ [4,1])$

3. Change of basis matrices $\mathcal{B}$ from the basis $\mathcal{B}'$ and from the basis $\mathcal{B}$ to the basis $\mathcal{B}''$ are equal to $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ i $\begin{bmatrix} 8 & 5 \\ 1 & 7 \end{bmatrix}$. Find the change of basis matrix from the basis $\mathcal{B}'$ to the basis $\mathcal{B}''$.

4. Change of basis matrices from $\mathcal{B}$ to $\mathcal{B}''$ and from $\mathcal{B}'$ to $\mathcal{B}''$ are equal to $\begin{bmatrix} 3 & 9 \\ 7 & 11 \end{bmatrix}$ i $\begin{bmatrix} 5 & 1 \\ -1 & 1 \end{bmatrix}$. Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}''$.

5 Matrix of linear transformation in different bases

1. Linear transformation $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is given by the matrix $M_{\mathcal{B}\mathcal{C}}(\varphi) = \begin{bmatrix} 4 & 3 & 5 \\ 1 & 1 & -1 \end{bmatrix}$ Find the matrix $M_{\mathcal{B}'\mathcal{C}'}(\varphi)$, if

   (a) $\mathcal{B} = ([4,5,1],[1,1,1],[3,0,5]),\quad \mathcal{C} = ([4,4],[-4,-5])\ \mathcal{B}' = ([6,4,5],[4,1,6],[5,2,7]),\quad \mathcal{C}' = ([0,1],[4,3])$

   (b) $\mathcal{B} = ([4,4,1],[3,0,1],[4,3,1]),\quad \mathcal{C} = ([1,2],[2,-1])\ \mathcal{B}' = ([4,4,1],[5,8,1],[3,1,1]),\quad \mathcal{C}' = ([3,1],[-5,0])$

2. Linear transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ is given by the following mapping. Find $\varphi([x_1,x_2])$

   (a) $[4,5] \to [-1,2,5],\quad [5,7] \to [-2,1,4]$

   (b) $[1,-2] \to [1,3,1],\quad [3,-5] \to [6,10,4]$