Lecture

Systems of equations

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1. Systems of differential equations
2. First integral of a normal system
3. Integrable combinations
4. A symmetric form of a system of differential equations.

The system of differential equations

$$F_k(x, y_1, y_1', \ldots, y_1^{(k_1)}, y_2, y_2', \ldots, y_2^{(k_2)}, \ldots, y_n, y_n', \ldots, y_n^{(k_n)}) = 0, \tag{1}$$

$k = 1, 2, \ldots, n$ solved for the higher derivatives $y_1^{(k_1)}, y_2^{(k_2)}, \ldots, y_n^{(k_n)}$ is called a *canonical system.* It is of the form

$$\begin{cases} y_1^{(k_1)} = f_1(x, y_1, y_1', \ldots, y_1^{(k_1-1)}, y_2, y_2', \ldots, y_2^{(k_2-1)}, \ldots, y_n, y_n', \ldots, y_n^{(k_n-1)}), \\ y_2^{(k_2)} = f_2(x, y_1, y_1', \ldots, y_1^{(k_1-1)}, y_2, y_2', \ldots, y_2^{(k_2-1)}, \ldots, y_n, y_n', \ldots, y_n^{(k_n-1)}), \\ \vdots \\ y_n^{(k_n)} = f_n(x, y_1, y_1', \ldots, y_1^{(k_1-1)}, y_2, y_2', \ldots, y_2^{(k_2-1)}, \ldots, y_n, y_n', \ldots, y_n^{(k_n-1)}). \end{cases}$$

A number

$$p = k_1 + k_2 + \ldots + k_n$$

is called *the order of system (1)*

Example

Bring into canonical form the system of equations

$$\begin{cases} y_2 y_1' - \ln(y_1'' - y_1) = 0 \\ e^{y_2'} - y_1 - y_2 = 0. \end{cases}$$

Example

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Solution

$$\begin{cases} y_1'' = y_1 + e^{y_2 y_1'} \\ y_2'' = \ln(y_1 + y_2). \end{cases}$$

A system of first order differential equations of the form

$$\frac{dx_k}{dt} = f_k(t, x, x_1, x_2, \dots, x_n), \quad k = 1, 2, \dots, n \tag{2}$$

where t is an independent variable $x_1, x_2, \dots, x_n$ are unknown functions of t is called *a normal system*.

The number n is called *the order of the normal system*.

Two systems of differential equations are said to be *equivalent* if they have the same solutions.

Any canonical system can be reduced to the equivalent normal system, the order of the system remaining the same.

Example

Reduce to the normal system the following system of differential equations

$$\begin{cases} \frac{d^2x}{dt^2} - y = 0, \\ t^3\frac{dy}{dt} - 2x = 0. \end{cases}$$

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Solution

$$\begin{cases} \frac{dx_1}{dt} = x_2, \\ \frac{dx_2}{dt} = x_3, \\ \frac{dx_3}{dt} = \frac{2x_1}{t^3}. \end{cases}$$

The aggregate of any n functions

$$x_1 = \varphi_1(t),\ x_2 = \varphi_2(t), \ldots, x_n = \varphi_n(t),$$

defined and continuously differentiable in an interval (a, b) is said to be *a solution of system* (2) in the interval (a, b) if they transform the equations of system (2) into identities for all values of $t \in (a, b)$

The name of *the Cauchy problem* for system (2) is given to the problem of finding the solutions

$$x_1 = x_1(t),\ x_2 = x_2(t), \ldots, x_n = x_n(t),$$

of this system satisfying the initial conditions

$$x_1(t_0) = x_1^0,\ x_2(t_0) = x_2^0, \ldots, x_n(t_0) = x_n^0, \tag{3}$$

where $t_0, x_1^0, \ldots, x_n^0$ being the given numbers.

The existance and uniqueness theorem for the Cauchy problem.

Theorem

Consider a normal system of differential equations (2) and functions $f_i(t, x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$ *defined in some* $(n+1)$*-dimensional domain* D *of the variables* $t, x_1, x_2, \ldots, x_n$ *If there is a neighbourhood* Ω *of a point* $M_0(t_0, x_1^0, x_2^0, \ldots, x_n^0)$ *in which the functions* f_i

- *are continuous and*
- *have bounded partial derivatives with respect to the variables* $x_1, x_2, \ldots, x_n$,

then there is a range $t_0 - h < t < t_0 + h$ *of* t *in which there exists a unique solution of the normal system (2) satisfying initial conditions (3)*

A system of n independent functions

$$x_i = x_i(t, C_1, C_2, \ldots, C_n), \quad i = 1, 2, \ldots, n \tag{4}$$

of the independent variable t and n arbitrary constants $C_1, C_2, \ldots, C_n$ is said to be *the general solution* of the normal system (2) if

- for any allowed values of $C_1, C_2, \ldots, C_n$ the system of functions (4) transforms equations (2) into identities
- in the domain satisfying the conditions of the Cauchy theorem functions (4) give a solution to any Cauchy problem.

Solutions obtained from general solution for particular values of constants $C_1, C_2, \ldots C_n$ are called *particular solutions*.

Remark

Not every system of differential equations can be reduced to one equation.

Remark

If the number of equations in a system is equal to n and the number of desired functions is equal to N with $N > n$, then the system is *indeterminate*. In this case it is possible to choose arbitrarily $N - n$ desired functions and from them to find the remaining n functions

Remark

If a system consists of n equations and the number of desired functions is N, with $N < n$, then this system may be found to be *incompatible*, i.e. to have no solution.

The method of elimination

A particular case of a canonical system of differential equations is one equation of the n-th order solved for a higher derivatives

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)}).$$

Introducing new functions

$$x_1 = x'(t),\ x_2 = x''(t), \dots,\ x_{n-1} = x^{(n-1)}(t)$$

replaces this equation by a normal system of n equations

$$\begin{cases} \frac{dx}{dt} = x_1, \\ \frac{dx_1}{dt} = x_2, \\ \dots \\ \frac{dx_{n-2}}{dt} = x_{n-1}, \\ \frac{dx_{n-1}}{dt} = f(t, x, x_1, x_2, \dots, x_n). \end{cases}$$

The inverse statement is true , that in general a normal system of n first order equations

$$\begin{cases} \frac{dx_1}{dt} = f_1(t, x_1, x_2, \ldots, x_n), \\ \frac{dx_2}{dt} = f_2(t, x_1, x_2, \ldots, x_n), \\ \vdots \\ \frac{dx_n}{dt} = f_n(t, x_1, x_2, \ldots, x_n) \end{cases}$$

is equivalent to one equation of order n. This provides a basis for one of the methods of integrating systems of differential equations, for *the method of elimination*.

Example

Consider a system of two equations

$$\frac{dx}{dt} = ax + by + f(t), \quad \frac{dy}{dt} = cx + dy + g(t), \tag{5}$$

where a, b, c, d are constants, $f(t), g(t)$ are given functions $x(t), y(t)$ are desired functions. We find, that

$$y = \frac{1}{b}\left(\frac{dx}{dt} - ax - f(t)\right). \tag{6}$$

Substituting in the second equation we obtain an equation of the second order in $x(t)$

$$A\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Cx + P(t) = 0, \tag{7}$$

where A, B, C being constants. Hence

$$x = x(t, C_1, C_2)$$

Example

Solve the system of equations

$$\begin{cases} \frac{dx}{dt} = y + 1, \\ \frac{dy}{dt} = x + 1 \end{cases}$$

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$$\begin{cases} \frac{dx}{dt} = y + 1, \\ \frac{dy}{dt} = x + 1 \end{cases}$$

Solution

$$\begin{cases} x = C_1 e^t + C_2 e^{-t} - 1 \\ y = C_1 e^t - C_2 e^{-t} - 1. \end{cases}$$

Example

Example

Solve Cauchy problem for the system of equations

$$\begin{cases} \frac{dx}{dt} = 3x + 8y, \\ \frac{dy}{dt} = -x - 3y, \end{cases}$$

$$x(0) = 6, \quad y(0) = -2.$$

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Solution

$$\begin{cases} x = 4e^{t} + 2e^{-t} \\ y = -e^{t} - e^{-t}. \end{cases}$$

Exercise

1. $\begin{cases} \frac{dx}{dt} = 3 - 9y, \\ \frac{dy}{dt} = x, \end{cases}$
2. $\begin{cases} \frac{dx}{dt} = y + t, \\ \frac{dy}{dt} = x - t, \end{cases}$
3. $\begin{cases} \frac{dx}{dt} + 3x + 4y = 0, \\ \frac{dy}{dt} + 2x + 5y = 0, \end{cases}$
   $x(0) = 1, \quad y(0) = 4$
4. $\begin{cases} \frac{dx}{dt} = -y + z, \\ \frac{dy}{dt} = z, \\ \frac{dz}{dt} = -x + z, \end{cases}$

A function $\psi(t, x_1, x_2, \ldots, x_n)$ defined and continuous together with its first order partial derivatives $\frac{\partial \psi}{\partial t}$, $\frac{\partial \psi}{\partial x_k}$, $k = 1, 2, \ldots, n$ in domain D is said to be *the integral of a normal system* (2) if, when an arbitrary solution $x_1(t), x_2(t), \ldots, x_n(t)$ of system (2) is substituted in it the function takes a constant value, i.e. the function $\psi(t, x_1, x_2, \ldots, x_n)$ depends only on the choice of a solution $x_1(t), x_2(t), \ldots, x_n(t)$, not on the variable t.
A *first integral of a normal system* (2) is the equation

$$\psi(t, x_1, x_2, \ldots, x_n) = C, \tag{8}$$

where $\psi(t, x_1, x_2, \ldots, x_n)$ is an integral of system (2) and C is an arbitrary constant.

Example

Show that a function

$$\psi(t, x_1, x_2) = \frac{x_2}{t} - x_1,$$

defined in the domain $D: \ t \neq 0, \ -\infty < x_1, x_2 < +\infty$ is an integral of the system of equations

$$\begin{cases} \frac{dx_1}{dt} = \frac{x_1}{t} \\ \frac{dx_2}{dt} = x_1 + \frac{x_2}{t} \end{cases}$$

if the the general solution of the system is

$$x_1 = C_1 t, \quad x_2 = C_1 t^2 + C_2 t.$$

Theorem

For a function $\psi(t, x_1, x_2, \ldots, x_n)$ to be an integral of system (2) it is necessary and sufficient that the conditions

$$\frac{\partial \psi}{\partial t} + \sum_{k=1}^{n} f_k(t, x_1, x_2, \ldots, x_n) \frac{\partial \psi}{\partial x_k} = 0 \tag{9}$$

should hold in the domain D.

Example

Show that the function

$$\psi(t, x_1, x_2, \ldots, x_n) = \arctan\frac{x_1}{x_2} - t$$

is an integral of the system of equations

$$\frac{dx_1}{dt} = \frac{x_1^2}{x_2}, \quad \frac{dx_2}{dt} = -\frac{x_2^2}{x_1}.$$

Integrals $\psi_1, \psi_2, \ldots, \psi_n$ of system (2) are said to be *independent with respect to desired functions* $x_1, x_2, \ldots, x_n$ if there exists no relation of the form $F(\psi_1, \psi_2, \ldots, \psi_n) = 0$ between the functions $\psi_1, \psi_2, \ldots, \psi_n$ for any choice of the function F not depending explicitly on $x_1, x_2, \ldots, x_n$

Theorem

For function $\psi_1, \psi_2, \ldots, \psi_n$ *having partial derivatives* $\frac{\partial \psi_i}{\partial x_k}$, $i, k = 1, 2, \ldots, n$ *to be independent with respect to* $x_1, x_2, \ldots, x_n$ *in some domain* D *it is necessary and sufficient that Jacobian of these functions should be nonzero in the domain* D

$$\frac{D(\psi_1, \psi_2, \ldots, \psi_n)}{D(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial \psi_1}{\partial x_1} & \frac{\partial \psi_1}{\partial x_2} & \ldots & \frac{\partial \psi_1}{\partial x_n} \\ \frac{\partial \psi_2}{\partial x_1} & \frac{\partial \psi_2}{\partial x_2} & \ldots & \frac{\partial \psi_2}{\partial x_n} \\ \vdots & \ddots & \ldots & \vdots \\ \frac{\partial \psi_n}{\partial x_1} & \frac{\partial \psi_n}{\partial x_2} & \ldots & \frac{\partial \psi_n}{\partial x_n} \end{vmatrix} \neq 0.$$

The general integral of a normal system (2) is an aggregate of n independent first integrals of that system.

If k independent first integrals of system (2) are known, with $k < n$, then its order can be depressed by k.

Exercises

1. Examine if the given function ψ is the first integral of the given system of differential equation
$$\begin{cases} \frac{dx_1}{dt} = \frac{x_1^2}{x_2}, \\ \frac{dx_2}{dt} = x_2 - x_1; \end{cases}$$
$$\psi = x_1 x_2 e^{-t}.$$

2. Examine if the given pairs of functions form system of independent first integrals of the given system of differential equations
$$\begin{cases} \frac{dx}{dt} = \frac{t-y}{y-x}, \\ \frac{dy}{dt} = \frac{x-t}{y-x}; \end{cases}$$
$$x + y + t = C_1, \quad x^2 + y^2 + t^2 = C_2$$

Integrable combinations

This method of integrating a system of differential equations

$$\frac{dx_k}{dt} = f_k(t, x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n \tag{10}$$

is as follows: using suitable arithmetic operations (addition, subtraction, multiplication and division) one can form system (10) the so-called integrable combinations, i.e. equations of the form

$$F(t, u, \frac{du}{dt}) = 0 \tag{11}$$

that can be solved easily enough, u being some function of desired functions $x_1(t), x_2(t), \ldots, x_n(t)$. Each integrable combination gives one first integral. If n independent first integrals of system (10) are obtained, then its integration is complete. If m independent first integrals are obtained, where $m < n$ than system (10) is reduced to a system with smaller number of unknown functions.

Examples

Example

Solve the system

$$\begin{cases} \frac{dx_1}{dt} = 2(x_1^2 + x_2^2)t, \\ \frac{dx_2}{dt} = 4x_1 x_2 t; \end{cases}$$

Example

Solve the system

$$\begin{cases} \frac{dx_1}{dt} = \frac{x_1 - x_2}{x_3 - t}, \\ \frac{dx_2}{dt} = \frac{x_1 - x_2}{x_3 - t}, \\ \frac{dx_3}{dt} = x_1 - x_2 + 1. \end{cases}$$

Example

Find a particular solution of the system

$$\begin{cases} \frac{dx}{dt} = 1 - \frac{1}{y}, \\ \frac{dy}{dt} = \frac{1}{x-t} \end{cases}$$

satisfying the initial conditions $x(0) = 1, \quad y(0) = 1$

Exercises

Solve the following systems of differential equations

1. $$\begin{cases} \frac{dx}{dt} = x^2 + y^2, \\ \frac{dy}{dt} = 2xy; \end{cases}$$

2. $$\begin{cases} \frac{dx}{dt} = \frac{x}{y}, \\ \frac{dy}{dt} = \frac{y}{x}; \end{cases}$$

To find integrable combinations in solving the system of differential equations (10) it is sometimes convenient to write it in the symmetric form

$$\frac{dx_1}{f_1(t, x_1, x_2, \dots, x+n)} = \frac{dx_2}{f_2(t, x_1, x_2, \dots, x+n)} = \\ = \dots = \frac{dx_n}{f_n(t, x_1, x_2, \dots, x+n)} = \frac{dt}{1}. \tag{12}$$

In the system of differential equations written in the symmetric form, the variables $t, x_1, x_2, \dots, x_n$ are equivalent, which in some cases simplifies finding integrable combinations.

To solve system (12) one may either take pairs of relations allowing separation of the variables or use the derived proportions

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n} = \frac{\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n}{\lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_n b_n} \tag{13}$$

where coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ are arbitrary and chosen so that the numerator should be the differential of the denominator or so that numerator be the total differential and the denominator should be equal to zero.

Example

Find the general solution of the system of equations

$$\frac{dt}{2x} = \frac{dx}{-\ln t} = \frac{dy}{\ln t - 2x}.$$

Example

Solve the system of equations

$$\frac{dt}{4y - 5x} = \frac{dx}{5t - 3y} = \frac{dy}{3x - 4t}.$$

Examples

Example

Find the general solution of the system of equations

$$\frac{dt}{2x} = \frac{dx}{-\ln t} = \frac{dy}{\ln t - 2x}.$$

Solution: $x = \pm\sqrt{C_1 + t(\ln t - 1)}, \quad y = C_2 - t \pm \sqrt{C_1 + t(\ln t - 1)}.$

Example

Solve the system of equations

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Example

Solve the system of equations

$$\frac{dt}{4y - 5x} = \frac{dx}{5t - 3y} = \frac{dy}{3x - 4t}.$$

Solution: $3t + 4x + 5y = C_1, \quad t^2 + x^2 + y^2 = C_2$

Exercises

Solve the following systems of differential equations:

1. $$\frac{dt}{t}=\frac{dx}{x}=\frac{dy}{y},$$

2. $$\frac{dt}{xy}=\frac{dx}{yt}=\frac{dy}{xt},$$

3. $$\begin{cases}\frac{dx}{dt}=\frac{3t-4y}{2y-3x},\\ \frac{dy}{dt}=\frac{4x-2t}{2y-3x}.\end{cases}$$