Lecture
Stability theory

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Lyapunov stability

Let the following system of differential equations be given

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n, t), \quad i = 1, 2, \ldots, n. \tag{1}$$

A solution $\varphi_i(t)$, $i = 1, 2, \ldots, n$ of system (1) satisfying the initial conditions

$$\varphi_i(t_0) = \varphi_{i0}, \quad i = 1, 2, \ldots, n$$

is said to be *a Lyapunov stable solution* as $t \to \infty$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each solution $x_i(t)$, $i = 1, 2, \ldots, n$ of system (1) whose initial values satisfy the conditions

$$|x_i(t_0) - \varphi_{i0}| < \delta, \quad i = 1, 2, \ldots, n \tag{2}$$

the inequalities

$$|x_i(t) - \varphi_i(t)| < \varepsilon, \quad i = 1, 2, \ldots, n \tag{3}$$

hold for all $t \geq t_0$.

If for an arbitrarily small $\delta > 0$ inequalities (3) fail to hold, then the solution $\varphi_i(t)$, $i = 1, 2, \ldots, n$ is said to be *unstable*.
If under condition (2) besides inequalities (3) the condition

$$\lim_{t \to \infty} |x_i(t) - \varphi_i(t)| = 0, \quad i = 1, 2, \ldots, n.$$

also holds, then the solution

$$\varphi_i(t), \quad i = 1, 2, \ldots, n$$

is said to be *asymptotically stable*.

Investigating a solution

$$\varphi_i(t), \quad i = 1, 2, \ldots, n$$

of system (1) for stability can be reduced to investigating for stability the zero solution $x_i \equiv 0, \quad i = 1, 2, \ldots, n$ of some system similar to system (1)

$$\frac{dx_i}{dt} = F_i(x_1, x_2, \ldots, x_n, t), \quad i = 1, 2, \ldots, n. \tag{4}$$

where

$$F_i(0, 0, \ldots, 0, t) \equiv 0, \quad i = 1, 2, \ldots, n.$$

A point $x_i = 0, \quad i = 1, 2, \ldots, n$ is said to be *a stationary point* of system (4)

The definitions of stability and instability can be reformulated as follows
A stationary point $x_i = 0, \quad i = 1, 2, \ldots, n$ is stable according to Lyapunov if whatever $\varepsilon > 0$ there exists $\delta > 0$ such that for any solution $x_i(t), \quad i = 1, 2, \ldots, n$ whose initial data $x_{i0} = x_i(t_0)$, $i = 1, 2, \ldots, n$ satisfy the condition

$$|x_{i0}| < \delta, \quad i = 1, 2, \ldots, n, \tag{5}$$

the inequalities

$$|x_i(t)| < \varepsilon, \quad i = 1, 2, \ldots, n, \tag{6}$$

hold for all $t \geq t_0$

If besides inequalities (3) the conditions

$$\lim_{t \to +\infty} |x_i(t)| = 0, \quad i = 1, 2, \dots, n \tag{7}$$

also holds, then the stability is asymptotic.
A stationary point is unstable if for an arbitrarily small $\delta > 0$ condition (6) does not hold for at least one solution $x_i(t), i = 1, 2, \dots, n$.

Examples

Example

Investigate for stability the solution of the equation

$$\frac{dx}{dt} = 1 + t - x$$

satisfying the initial conditions

$$x(0) = 0.$$

Example

$$\frac{dx}{dt} = \sin^2 x$$

Example

Show that the solution of the systems

$$\begin{cases} \frac{dx}{dt} = -y, \\ \frac{dy}{dt} = x \end{cases}$$

satisfying the initial conditions

$$x(0) = 0, \quad y(0) = 0$$

is stable.

Theorem

Solutions of a system of linear differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j + f_i(t), \quad i = 1, 2, \ldots, n \tag{8}$$

are all either simultaneously stable or unstable.

Exercises

1. $\frac{dx}{dt} = x + t, \quad x(0) = 1,$
2. $\frac{dx}{dt} = 2t(x + 1), \quad x(0) = 0,$
3. $\begin{cases} \frac{dx}{dt} = x - 13y, \\ \frac{dy}{dt} = \frac{1}{4}x - 2y \end{cases} \quad x(0) = y(0) = 0,$

Stationary points - the simplest case

Consider a system of two homogeneous linear differential equations with constant coefficients

$$\begin{cases} \frac{dx}{dt} = a_{11}x + a_{12}y, \\ \frac{dy}{dt} = a_{21}x + a_{22}y \end{cases} \tag{9}$$

with

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$$

A point $x = 0,\ y = 0$ in which the right-hand side of the equations of system (9) vanish is called a stationary point of system (9)

In order for a stationary point of system (9) to be investigated it is necessary to set up the characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 \tag{10}$$

and find its roots λ_1 and λ_2

The roots λ_1, λ_2 of the characteristic equation (10) are real and distinct

a. $\lambda_1 < 0, \lambda_2 < 0$. The stationary point is asymptotically stable (a stable node)

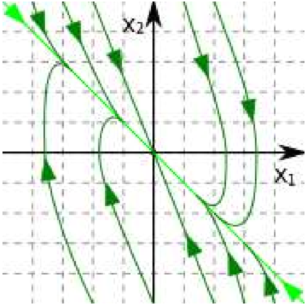

Figure: Stable node

b. $\lambda_1 > 0, \lambda_2 > 0$. The stationary point is unstable (an unstable node)

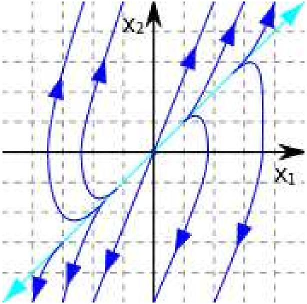

Figure: Unstable node

c. $\lambda_1 > 0, \lambda_2 < 0$. The stationary point is unstable (a saddle point)

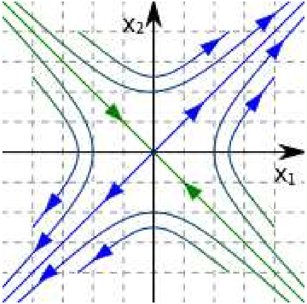

Figure: Saddle point

Case II

The roots of the characteristic equation (10) are complex

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta$$

a. $\alpha < 0, \beta \neq 0$. The stationary point is asymptotically stable (a stable focus)

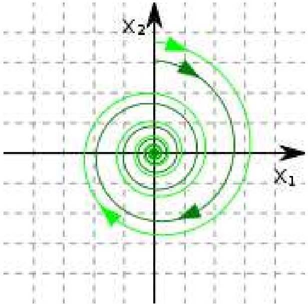

Figure: Stable focus

b. $\alpha > 0, \beta \neq 0$. The stationary point is unstable stable (an unstable focus)

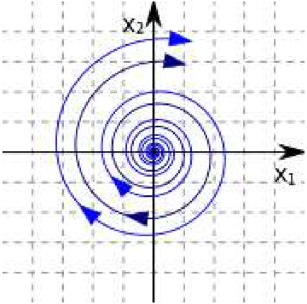

Figure: Unstable focus

c. $\alpha = 0, \beta \neq 0$. The stationary point is stable stable (a midpoint)

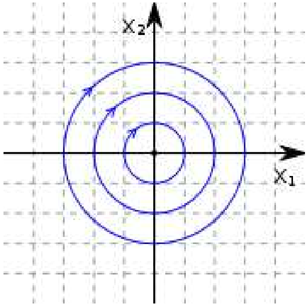

Figure: Midpoint

Case III

The roots $\lambda_1 = \lambda_2$ are multiple

a. $\lambda_1 = \lambda_2 < 0$. The stationary point is asymptotically stable (a stable node)

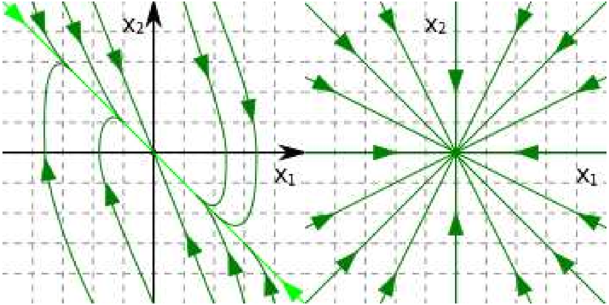

Figure: Stable node

b. $\lambda_1 = \lambda_2 > 0$. The stationary point is unstable (an unstable node)

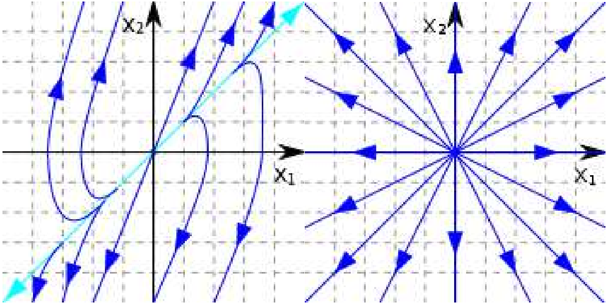

Figure: Unstable node

Example

Example

Determine the character of stationary point $(0, 0)$ of the system

$$\begin{cases} \frac{dx}{dt} = 5x - y, \\ \frac{dy}{dt} = 2x + y \end{cases}$$

Exercises

1. $\begin{cases} \frac{dx}{dt} = 3x + y, \\ \frac{dy}{dt} = -2x + y \end{cases}$
2. $\begin{cases} \frac{dx}{dt} = -x + 2y, \\ \frac{dy}{dt} = x + 2y \end{cases}$
3. $\begin{cases} \frac{dx}{dt} = -x + 3y, \\ \frac{dy}{dt} = -x + y \end{cases}$

Consider the system of homogeneous linear differential equations with constant coefficients

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, n \ (n \geq 2) \tag{11}$$

Theorem

If all roots of the characteristic equation for system (11) have a negative real part, then the stationary point of system (11) $x_i = 0, \ i = 1, 2, \ldots, n$ is asymptotically stable. If at least one root of the characteristic equation has a positive real part, then the stationary point is unstable.

Example

Is the stationary point $(0,0,0)$ of the system

$$\begin{cases} \frac{dx}{dt} = -x + z, \\ \frac{dy}{dt} = -2y - z, \\ \frac{dz}{dt} = y - z \end{cases}$$

stable?

The method of Lyapunov functions is to investigate directly the stability of the equilibrium position of the system

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \dots, x_n, t), \quad i = 1, 2, \dots, n. \tag{12}$$

with the help of a suitably selected function $V(t, x_1, \dots, x_n)$, *the Lyapunov function* , this being done without finding beforehand any solutions of this system.

Let us consider an autonomous system

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n. \tag{13}$$

for which $x_i \equiv 0$, $i = 1, 2, \ldots, n$ is a stationary point. The function $V(x_1, x_2, \ldots, x_n)$ defined in

some neighbourhood of the origin of coordinates is said to be of fixed sign if in the domain

$$|x_i| \leq h, \quad i = 1, 2, \ldots, n$$

h being a sufficiently small positive number, it can take values of only one definite sign and vanish only when $x_1 = x_2 = \ldots = x_n = 0$

Example

$n = 3$

$$V = x_1^2 + x_2^2 + x_3^2$$

$$V = x_1^2 + 2x_1x_2 + 2x_2^2 + x_3^2$$

are positive definite.

The function $V(x_1, x_2, \ldots, x_n)$ is said to be of constant signs if in domain (10) it can take values of only one definite sign but can also vanish when $x_1^2 + x_2^2 + \ldots + x_n^2 \neq 0$

Example

$$V(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_1x_2 + x_3^2$$

Let $V(x_1, x_2, \ldots, x_n)$ be a differentiable function of its variables and let $x_1, x_2, \ldots, x_n$ be some functions of time satisfying the system of differential equations (9). Then the total derivative of V with respect to time t is of the form

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \cdot \frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial V}{x_i} f_i(x_1, x_2, \ldots, x_n). \tag{14}$$

The quantity $\frac{dV}{dt}$ defined by formula (14) is called the total derivative of the function V with respect to time composed by virtue of the system of equations (9).

Theorem (Lyapunov's stability theorem)

If for a system of differential equations (9) there exists a function of fixed sign $V(x_1, x_2, \ldots, x_n)$ *(a Lyapunov function) whose total derivative* $\frac{dV}{dt}$ *with respect to time composed by virtue of system (9) is a function of constant sign, of sign opposite to that of* V *or identically equal to zero, then the stationary point* $x_i = 0$, $i = 1, 2, \ldots .n$ *of system (9) is stable.*

Theorem (Lyapunov's asymptotic-stability theorem)

If for a system of differential equations (9) there exists a function of fixed sign $V(x_1, x_2, \ldots, x_n)$ *whose total derivative* $\frac{dV}{dt}$ *with respect to time composed by virtue of system (9) is also a function of fixed sign, of sign opposite to that of* V*, then the stationary point* $x_i = 0$, $i = 1, 2, \ldots .n$ *of system (9) is asymptotically stable.*

Example

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -x \end{cases}$$

$$V = x^2 + y^2$$

Example

$$\begin{cases} \frac{dx}{dt} = y - x^3, \\ \frac{dy}{dt} = -x - 3y^3 \end{cases}$$

$$V = x^2 + y^2$$

Example

Investigate for stability the trivial solution $x \equiv 0,\ \ y \equiv 0$

$$\begin{cases} \frac{dx}{dt} = -x - 2y + x^2y^2, \\ \frac{dy}{dt} = x - \frac{y}{2} - \frac{x^3y}{2} \end{cases}$$

Theorem (Lyapunov's instability theorem)

Let there exist for the system of differential equations (9) a function differentiable in the neighbourhood of the origin of coordinates, $V(x_1, x_2, \ldots, x_n)$*, such that* $V(0, 0, \ldots, 0) = 0$*. If its total derivative* $\frac{dV}{dt}$ *composed by virtue of system (9) is a positive definite function and arbitrarily close to the origin of coordinates there are points in which the function* $V(x_1, x_2, \ldots, x_n)$ *takes positive values, then the stationary point* $x_i = 0$*,* $i = 1, 2, \ldots, n$ *is unstable.*

Theorem (Chetayev's instability theorem)

Let for the system of differential equations (9) there exists a function $v(x_1, x_2, \ldots)$ *continuously differentiable in some neighbourhood of a stationary point* $x_i = 0$, $i = 1, 2, \ldots, n$ *satisfying the following conditions in some closed neighbourhood of the stationary point*

1. *in an arbitrarily small neighbourhood* Ω *of the stationary point* $x_i = 0$, $i = 1, 2, \ldots, n$ *there exists a domain* Ω_1 *in which* $v(x_1, x_2, \ldots, x_n > 0$, *with* $v = 0$ *in the boundary points of* Ω_1 *that are interior for* Ω
2. *the stationary point* $O(0, 0, \ldots, 0)$ *is a boundary point of the domain* Ω_1
3. *the derivative* $\frac{dv}{dt}$ *composed by virtue of system (9) is positive definite in the domain* Ω_1

Then the stationary point $x_i = 0$, $i = 1, 2, \ldots, n$ *of system (9) is unstable.*

Example

Investigate the stationary point $x = 0,\ y = 0$ of the system

$$\begin{cases} \frac{dx}{dt} = x, \\ \frac{dy}{dt} = -y \end{cases}$$

for stability

Example

Investigate the stationary point $x = 0,\ y = 0$ of the system

$$\begin{cases} \frac{dx}{dt} = y^3 + x^5, \\ \frac{dy}{dt} = x^3 + y^5 \end{cases}$$

for stability

Exercises

1. Investigate the stationary point $x = 0,\ y = 0$ of the system

$$\begin{cases} \frac{dx}{dt} = -3y - 2x^3, \\ \frac{dy}{dt} = 2x - 3y^3 \end{cases}$$

   for stability.

2. Investigate the stationary point $x = 0,\ y = 0$ of the system

$$\begin{cases} \frac{dx}{dt} = -xy^4, \\ \frac{dy}{dt} = x^4 y \end{cases}$$

   for stability.

Let the following system of differential equations be given

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \dots, x_n), \ \ i = 1, 2, \dots, n \tag{15}$$

and let $x_i \equiv 0, ß = 1, 2, \dots, n$ be a stationary point of the system (15 i.e. $f_i(0, 0, \dots, 0) = 0, \ \ i = 1, 2, \dots, n$. We shall assume that functions $f_i(x_1, x_2, \dots, x_n)$ can be differentiated a sufficiently large number of times at the origin of coordinates.

We expand the functions f_i in the Taylor series of x in the neighbourhood of the origin of coordinates

$$f_i(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} a_{ij} x_j + R_i(x_1, x_2, \ldots, x_n), \tag{16}$$

where $a_{ij} = \frac{\partial f_i(0,0,\ldots,0)}{\partial x_j}$ and R_i are terms of second order smallness with respect to $x_1, x_2, \ldots, x_n$. The original system (15) will be written as

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j + R_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n \tag{17}$$

Instead of system (17) we shall consider the system

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, n, \ a_{ij} = const \tag{18}$$

called the system of equations of the first approximation for system (15).

The following proposition hold

1. If all roots of the characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} \tag{19}$$

have negative real parts, then zero solution $x_i \equiv 0$, $i = 1, 2, \dots, n$ of system (18) and system (17) are asymptotically stable.

2. If at least one root of the characteristic equation (19) has a positive real part, then zero solution of system (18) and system (17) are unstable. It is said that the investigation for stability in the first approximation is possible on cases 1 and 2.

In critical cases when real parts of all roots of the characteristic equation (19) are nonpositive, with the real part of at least one root being zero, investigating for stability in the first approximation is in general impossible (nonlinear terms R_i starting to exert influence).

Example

Investigate the stationary point $x = 0, \; y = 0$ of the system

$$\begin{cases} \frac{dx}{dt} = 2x + y - 5y^2, \\ \frac{dy}{dt} = 3x + y + \frac{x^3}{2} \end{cases}$$

for stability in the first approximation.

Exercises

Investigate the zero solution $x = 0, \; y = 0$ of the following system for stability in the first approximation

1. $$\begin{cases} \frac{dx}{dt} = x + 2y - \sin y^2 \\ \frac{dy}{dt} = -x - 3y + x(e^{\frac{x^2}{2}} - 1) \end{cases}$$

2. $$\begin{cases} \frac{dx}{dt} = -x + 3y + x^2 \sin y \\ \frac{dy}{dt} = -x - 4y + 1 - \cos y^2 \end{cases}$$

3. $$\begin{cases} \frac{dx}{dt} = 7x + 2 \sin y - y^4 \\ \frac{dy}{dt} = e^x - 3y - 1 + \frac{5}{2}x^2 \end{cases}$$