Lecture

Partial differential equations

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1. Partial differential equations
2. First order partial differential equations
3. Linear partial differential equations of second order.

Definition

The equation of the form

$$F\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots\right) = 0 \tag{1}$$

where u is unknown function of two (or more) variables and where at least one partial derivatives occurs we call a *partial differential equation.*

Definition

The order of higher derivative of the sought-for function occurring in partial differential equation we will call *the order of differential equation.*

Classification of equations

- according to the order
  - first order

$$F(x_1, x_2, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) = 0 \tag{2}$$

or

$$F(x_1, x_2, \dots, x_n, u, \frac{\partial u}{\partial x_j}) = 0, \;\; j = 1, 2, \dots, n \tag{3}$$

  - m-th order

$$F(x_1, x_2, \dots, x_n, u, \frac{\partial^k u}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}}) = 0, \tag{4}$$

$$k_1 + k_2 + \dots + k_n = k, \;\; k = 1, 2, \dots, m, \;\; (m \geq 1, n \geq 2)$$

- according to the number of variables

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = \cos x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

- according to the form of the function F
  - linear equations
  - quasilinear equations
  - almost linear equations
- homogeneous and nonhomogeneous

$$P(x,y,x)u_x + Q(x,y,x)u_y + R(x,y,x)u_z = f(x,y,z)$$

If $f(x,y,x) \equiv 0$ then we call the above equation homogeneous.

We call a partial differential equation *a* linear equation, if F is linear in u and its derivatives

Example

$$\sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + bu + c = 0$$

where a, b, c are functions of variables $(x_1, x_2, \ldots, x_n)$.

Example

$$\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{j=1}^{n} b_j \frac{\partial u}{\partial x_j} + cu + d = 0$$

We call a partial differential equation *a* quasilinear if F is linear in partial derivatives of m-th order of function u

Example

$$\sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n, u)\frac{\partial u}{\partial x_i} + b(x_1, x_2, \ldots .x_n, u) = 0$$

Example

$$x\frac{\partial u}{\partial y} + yu\frac{\partial u}{\partial x} = 0$$

We call a partial differential equation *an* almost linear equation, if it is quasilinear and its coefficients belonging to derivatives of m-th order are dependent only on $x_1, x_2, \ldots, x_n$

Example

$$\sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n) \frac{\partial u}{\partial x_i} + b(x_1, x_2, \ldots .x_n, u) = 0$$

Definition

The particular integral (particular solution) of a partial differential equation of n-th order in domain D, we will call a C^n function in domain D, satisfying the equation in every point of the domain D.

Definition

The general integral (general solution) of a partial differential equation, we will call the family of all particular integrals.

Remark

A general integral depends on some number of any regular enough functions, where each function depends on the same number of arguments which is one less then the number of arguments of a solution.

The Cauchy Problem (the Initial-value Problem)

Let us consider the equation of the form

$$\frac{\partial u}{\partial x} = f_1(x, y, z, u, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) \tag{5}$$

The Cauchy Problem we call a problem of finding a particular solution $u = u(x, y, z)$ of the above equation in the domain $D_1 \subset E_3$ satisfying initial-value condition of the form

$$u(x_0, y, z) = \varphi(y, z)$$

$f_1 = f_1(x, y, z, u, p, q)$ - some given function in the domain $\Omega_1 \subset E_6$
Both number x_0 and function $\varphi(y, z)$ defined in the domain D_1' are given. (D_1' is a projection of the domain D_1 onto Oyz)

Remark

The condition $u(x_0, y, x) = \varphi(y, z)$ means that the equation $u(x, y, z)$ takes given values on the plane $x = x_0$.

Remark

In case of an unknown function $u = u(x, y)$ the Cauchy Problem $u_x = f(x, y, u, u_y), u(a, y) = \varphi(y)$ means of finding an integral surface of the equation $u_x = f(x, y, u, u_y)$, which intersects the plane $x = a$ through given curve $u = \varphi(y)$.

Definition

We call a function f analytical in the neighbourhood of some point x_0, if

$$\underset{x \in U}{\forall} f(x) = \sum_{n=0}^{\infty} a_0 (x - x_0)^n \tag{6}$$

Remark

An analytical function in the neighbourhood of the point x_0 has in that neighbourhood derivatives of all orders, moreover for every natural number

$$a_n = \frac{f^{(n)}}{n!}$$

Then the series (6) we call the Taylor series.

Theorem

If

1. *the function* $f_1(v_1, \ldots, v_6)$ *is analytical in some neighbourhood of the point* $(x_0, y_0, z_0, x_4^0, x_5^0, x_6^0) \in \Omega_1$
2. *the function* $\varphi(y, z)$ *is analytical in some neighbourhood of the point* $(y_0, t_0) \in D_1$

then there exists a neighbourhood of the point $(x_0, y_0, z_0) \in D_1$ *in which the Cauchy Problem of the equation (5) has a unique analytical solution.*

Theorem

If

1. *the function* $f_1(v_1, \ldots, v_{18})$ *is analytical in some neighbourhood of the point* $(x_0, y_0, z_0, x_4^0, x_5^0, \ldots, x_{18}^0) \in \Omega_2$
2. *the functions* $\varphi(y, z, t)$, $\psi(y, z, t)$ *are analytical in some neighbourhood of the point* $(x_0, y_0, x_0, t_0) \in D_2$

then there exists a neighbourhood of the point $(x_0, y_0, z_0, t_0) \in D_2$ *where the Cauchy Problem of equation (5) has a unique analytical solution.*

(Homogeneous) Linear Equation

Let us consider the equation

$$P(x,y,z)\frac{\partial u}{\partial x} + Q(x,y,z)\frac{\partial u}{\partial y} + R(x,y,x)\frac{\partial u}{\partial z} = 0 \tag{7}$$

where at least one of the functions P, Q, R is not identically equal zero in the domain D. Let us consider the following system of ordinary differential equations, which is closely related with the system of equations (7)

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \tag{8}$$

Theorem

If the function $u = \varphi(x,y,z)$ is a solution of partial differential equation (7) in the domain D, then the function $\varphi(x,y,z)$ is a first integral of the system (8)

Theorem

If the C^1 function $\varphi(x, y, z)$ is a first integral of the system of ordinary differential equations (8), then the function $u = \varphi(x, y, z)$ is a solution of equation (7).

Theorem

If the C^1 functions $\varphi_1(x, y, z)$, $\varphi_2(x, y, z)$ are independent first integrals of the system (8), then $u = F\left(\varphi_1(x, y, z), \varphi_2(x, y, z)\right)$, where $F(\xi_1, \xi_2)$ is any C^1 function in some plane domain, is an integral (a solution) of differential equation (7).

Remark

We call the system of ordinary differential equations (8) *the system of characteristics of the equation* (7) and the solutions of the system (first integrals) *characteristics of that equation.*

Example

Find the general solution of the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z^2 y\frac{\partial u}{\partial z} = 0$$

with unknown function u, where $u = u(x, y, z)$.

Remark

In case of a homogeneous linear equation of the form

$$P(x,y)\frac{\partial u}{\partial x} + Q(x,y)\frac{\partial u}{\partial y} = 0$$

with an unknown function $u = u(x,y)$ the system of characteristics reduces to one differential equation

$$\frac{dx}{P} = \frac{dy}{Q}$$

Example

Find the integral surface of the equation

$$y\frac{\partial u}{\partial x} + 2x\frac{\partial u}{\partial y} = 0$$

passing through the curve

$$u = \frac{1}{2}y^2, \text{ for } x = 0$$

Let us consider the equation of the form

$$P(x, y, u)\frac{\partial u}{\partial x} + Q(x, y, u)\frac{\partial u}{\partial y} - R(x, y, u) = 0 \tag{9}$$

where P, Q, R are some C^1 functions in the domain $D \subset E_3$, and $u = u(x, y)$ is an unknown function. Let us assume that one of the functions P or Q is not equal to zero in the whole domain D. Let the function $v(x, y, u)$ be the $C^1(D)$ function, and $\frac{\partial v}{\partial u} \neq 0$ in some neighbourhood of the point $p_0 \in D$. Let the function $u = u(x, y)$ be implicitly defined by the formula $v(x, y, u) = 0$ satisfying the equation

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} = R \tag{10}$$

Then the function $v(x, y, u)$ is a solution (in the neighbourhood of $p_0 \in D$) of a homogeneous equation

Remark

If the C^1 functions $\varphi_1(x, y, u)$ and $\varphi_2(x, y, u)$ are in the neighbourhood of the point p_0 independent first integrals of the systems of characteristics of the equation $P(x, y, u)\frac{\partial v}{\partial x} + Q(x, y, z)\frac{\partial v}{\partial y} + R(x, y, u)\frac{\partial v}{\partial u} = 0$ then the relation $F(\varphi_1(x, y, u), \varphi_2(x, y, u)) = 0$, where $F(\xi_1, \xi_2)$ is some C^1 function in some plane domain, consisting of all solutions of the equation $P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} = R$ in sufficiently small neighbourhood of the point p_0.

Example

Solve the equation

$$(1 + \sqrt{u - x - y})\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2$$

Remark

The nonhomogeneous linear equation $P(x, y)\frac{\partial u}{\partial x} + Q(x, y)\frac{\partial u}{\partial y} = R(x, y)$ with an unknown function $u(x, y)$, we can solve similarly to the equation of the form $P(x, y, u)\frac{\partial u}{\partial x} + Q(x, y, u)\frac{\partial u}{\partial y} = R(x, y, u)$

Example

Solve the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1$$

Example

Find the solution passing through the curve

$$2yu\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} + y = 0$$

$$y^2 - 4u^2 = 36, \quad x + u = 5$$

Definition

The equation of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + \\ D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u + G(x,y) = 0 \tag{11}$$

we call a *l*inear partial differential equations of second order with an unknown function $u = u(x,y)$, where A, B, C, D, E, F, G are $C^1(\Omega, \mathbb{R})$ functions, and $A^2 + B^2 + C^2 > 0$ (i.e. functions A, B, C do not vanish simultaneously at any point of the domain Ω

Classification of equations of the above form according to coefficients belonging to derivatives of the second order (A, B, C)
Let us introduce the following notation

$$\delta(x,y) = A(x,y)C(x,y) - B^2(x,y) \quad \text{equation discriminant} \tag{12}$$

Definition

1. If $\forall_{(x,y)\in\Omega}\ \delta(x,y) < 0$ then we call the equation (11) *a* hyperbolic equation.
2. If $\forall_{(x,y)\in\Omega}\ \delta(x,y) = 0$ then we call the equation (11) *a* parabolic equation.
3. If $\forall_{(x,y)\in\Omega}\ \delta(x,y) > 0$ then we call the equation (11) *an* elliptic equation.

Examples

Example

The wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \quad u = u(x, y)$$

is an example of a hyperbolic equation.

$$(A = 1, B = 0, C = -1) \quad \delta = -1 - 0^2 = -1 < 0$$

Example

The heat equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \quad u = u(x, t)$$

is an example of a parabolic equation.

$$(A = 1, B = 0, C = 0) \quad \delta = 0 - 0^2 = 0$$

Example

The Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u = u(x, y)$$

is an example of an elliptic equation.

$$(A = C = 1, B = 0) \quad \delta = 1$$

Remark

If discriminant of equation has no fixed sign in the domain Ω than we say that equation (11) is of mixed type.

Example

The Triconny equation

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is a mixed type equation

$$A(x, y) = y, B = 0, C = 1$$

1. $y < 0 \Rightarrow \delta(x, y) < 0$ hyperbolic equation,
2. $y = 0 \Rightarrow \delta(x, y) = 0$ parabolic equation,
3. $y > 0 \Rightarrow \delta(x, y) > 0$ elliptic equation.

Definition

We call a real map such that $\xi = \varphi_1(x, y), \eta = \varphi_2(x, y)$ *an* affine map in the domain Ω if and only if the following conditions in the domain are fulfilled

1. φ_1, φ_2 are C^2 functions
2. Jacobian of the map $\xi = \varphi_1(x, y), \eta = \varphi_2(x, y)$ $\frac{D(\xi,\eta)}{D(x,y)} \neq 0$ at every point of Ω

Theorem

The type of the equation (11) is invariant of an affine transformation.

Definition

As characteristics of the equation (11) we call integral curves of the ordinary differential equations

$$Ady^2 - 2Bdxdy + Cdx^2 = 0 \tag{13}$$

so the equations

1. $\delta < 0$

$$\frac{dy}{dx} = \frac{-B - \sqrt{-\delta}}{A}, \quad \frac{dy}{dx} = \frac{-B + \sqrt{-\delta}}{A}, \quad A \neq 0$$

$$\frac{dx}{dy} = \frac{-B - \sqrt{-\delta}}{C}, \quad \frac{dx}{dy} = \frac{-B + \sqrt{-\delta}}{C}, \quad C \neq 0$$

$$y = C_1, \quad x = C_2, \quad A = C = 0$$

Definition

2. $\delta = 0$

$$\frac{dy}{dx} = \frac{B}{A}$$

3. $\delta > 0$

$$\frac{dy}{dx} = \frac{-B - i\sqrt{\delta}}{A}, \quad \frac{dy}{dx} = \frac{-B + i\sqrt{\delta}}{A}, \quad A \neq 0$$

Example

Find characteristics of the equation

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial u}{\partial y^2} = 0$$

Example

Find characteristics of the equation

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial u}{\partial y^2} = 0$$

Example

Find characteristics of the equation

$$y^2 \frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial u}{\partial y^2} = 0$$

Canonical form of the equation with two variables

Theorem

There exists affine mapping $\xi = \varphi_1(x, y)$, $\eta = \varphi_2(x, y)$, *which transforms equation (11) to one of the following canonical forms*

1. $\delta < 0$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \cdots = 0, \quad \frac{\partial^2 u}{\partial \xi \partial \eta} + \cdots = 0$$

2. $\delta = 0$

$$\frac{\partial^2 u}{\partial \xi^2} + \cdots = 0, \quad \frac{\partial^2 u}{\partial \eta^2} + \cdots = 0$$

3. $\delta > 0$

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \cdots = 0$$

$\cdots$ – *denotes components not containing partial derivatives of second order of the unknown function*

Remark

Introducing new variables

$$\mu = \frac{1}{2}(\xi + \eta), \quad \nu = \frac{1}{2}(\xi - \eta)$$

and using previous canonical form we can transform the equation (11) to the second canonical form

$$\frac{\partial^2 u}{\partial \mu^2} - \frac{\partial^2 u}{\partial \nu^2} + \cdots = 0 \tag{14}$$

Example

Transform the equation to the canonical form

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 y}{\partial y^2} = 0$$

Example

Transform the equation to the canonical form

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 y}{\partial y^2} = 0$$

Example

Transform the equation to the canonical form

$$\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 y}{\partial y^2} = 0$$

Example

Example

Transform the equation to the canonical form

$$\frac{\partial^2 u}{\partial x^2} - 2\sin x \frac{\partial^2 u}{\partial x \partial y} - \cos^2 x \frac{\partial^2 y}{\partial y^2} - \cos x \frac{\partial u}{\partial y} = 0$$