Lecture

Integral Equations

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Definition

Relation

$$\Phi(x, f(x), \int_a^b F(x, y, f(y))dy) = 0 \qquad (1)$$

between an independent variable x, unknown function $f(x)$ and (Lebesgue's) integral, under which this unknown functions appears we call integral equation.

The classification of an integral equations centres on three basic characteristics which together describe their overall structure.

- The kind of an equation refers to the location of the unknown function
  1. First kind equations have unknown function present under the integral sign only
  2. Second and third kind equations also have unknown function outside the integral
- The historical descriptions Fredholm and Volterra are concerned with the integration interval
  1. In a Fredholm equation the integral is over a finite interval with fixed end-points.
  2. In a Volterra equation the integral is indefinite.

Classification of integral equations

1. Volterra equation of the first kind

$$\int_a^x N(x,y)f(y)dy = g(x) \tag{2}$$

2. Volterra equation of the second kind

$$f(x) - \int_a^x N(x,y)f(y)dy = g(x) \tag{3}$$

Functions $N(x,y)$, g are given, f - unknown function. We call the function $N(x,y)$ the Kernel of Volterra equation.

Classification of integral equations

3. Fredholm equation of the first kind

$$\int_a^b K(x,y)f(y)dy = g(x) \tag{4}$$

4. Fredholm equation of the second kind

$$f(x) - \int_a^b K(x,y)f(y)dy = g(x) \tag{5}$$

Functions $K(x,y)$, g are given, f - unknown function. We call the function $K(x,y)$ the Kernel of Fredholm equation.

Example

Example

Show that the function

$$\varphi(x) = (1 + x^2)^{-3/2}$$

is a solution of Volterra integral equations

$$\varphi(x) = \frac{1}{1 + x^2} - \int_0^x \frac{t}{1 + x^2} \varphi(t) dt$$

Example

Example

Show that the function

$$\varphi(x) = \frac{1}{\pi\sqrt{x}}$$

is a solution of Volterra integral equations

$$\int_0^x \frac{\varphi(t)}{\sqrt{x-t}} dt = 1$$

Example

Example

Show that the function

$$\varphi(x) = \cos(2x)$$

is a solution of Fredholm integral equations

$$\varphi(x) - 3\int_0^\pi K(x,t)\varphi(t)dt = \cos x$$

$$K(x,t) = \begin{cases} \sin x \cos t & \text{for} \quad 0 \le x \le t \\ \sin t \cos x & \text{for} \quad t < x \le \pi \end{cases}$$

Example

Example

Show that the function

$$\varphi(x) = xe^{-x}$$

is a solution of integral equations

$$\varphi(x) - 4\int_0^\infty e^{-(x+t)}\varphi(t)dt = (x-1)e^{-x}$$

There exists a fundamental relationship between Volterra integral equations and ordinary linear differential equations.

Let us consider second order linear differential equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x)$$

with initial condition

$$y(0) = C_0, \quad y'(0) = C_1$$

We set

$$\frac{d^2y}{dx^2} = \varphi(x).$$

Taking into account initial condition we observe that

$$\frac{dy}{dx} = \int_0^x \varphi(t)dt + C_1$$

$$y = \int_0^x (x-t)\varphi(t)dt + C_1 x + C_0$$

Thus

$$\varphi(x) = \int_0^x K(X,t)\varphi(t)dt + f(x),$$

where $-K(x,t) = a_1(x) + a_2(x)(x-t), \quad f(x) = F(x) - C_1 a_1(x) - C_0 a_2(x)$

Examples

Example

- $$y'' + xy' + y = 0, \quad y(0) = 1,\ y'(0) = 0$$
- $$y' + y = 0, \quad y(0) = 0$$
- $$y'' + y' = \cos x, \quad y(0) = y'(0) = 0$$

In applied mathematics equations can be often written as operator equations of the form

$$Tx = x \tag{6}$$

where T is an operator in a Hilbert space and x unknown.

Definition

A mapping f from a subset A of a normed space E into E is called a contraction mapping (or simply a contraction) if there exists a positive number $\alpha < 1$ such that

$$||f(x) - f(y)|| \leq \alpha ||x - y|| \tag{7}$$

for all $x, y \in A$

Theorem

Let S *be a closed subset of a Banach space, and let* $T: S \to S$ *be a contraction mapping. Then*

1. *the equation* $Tx = x$ *has one and only one solution in* S*, and*
2. *the unique solution* x *can be obtained as the limit of the sequence* (x_n) *of elements of* S *defined by*

$$x_n = Tx_{n-1}, \quad n = 1, 2, 3, \dots \tag{8}$$

where x_0 *is an arbitrary element of* S*:*

Then

$$x = \lim_{n \to \infty} T^n x_0 \tag{9}$$

Theorem

Let E be a Banach space, and let $T : E \to E$. If T^m is a contraction for some $m \in \mathbb{N}$, then T has a unique fixed point $x_0 \in E$ and $x_0 = \lim_{n\to\infty} T^n x$ for any $x \in E$.

Theorem

If A is a bounded linear operator on a Banach space E, and φ is an arbitrary element of E, then the operator defined by

$$Tf = \alpha Af + \varphi \tag{10}$$

has a unique fixed point for any sufficiently small $|\alpha|$. More precisely, if k is a positive constant such that

$$||A|| \leq k||f||$$

for all $f \in E$, then $Tf = f$ has a unique solution whenever $|\alpha|k < 1$.

Theorem

Let A be a bounded linear operator in a Banach space. Then the equation

$$x = x_0 + \alpha A x$$

has a unique solution given by

$$x = \sum_{n=0}^{\infty} \alpha^n A^n x_0$$

Theorem

Consider the initial value problem for the ordinary differential equation

$$\frac{dy}{dx} = f(x, y) \tag{11}$$

with the initial condition

$$y(x_0) = y_0 \tag{12}$$

where f is a continuous function in some closed domain

$$R = \{(x, y); \quad a \leq x \leq b, \; c \leq y \leq d\} \tag{13}$$

containing the point (x_0, y_0) in its interior. If f satisfies the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \tag{14}$$

for some $L \in \mathbb{R}$ and all $(x, y_1), (x, y_2) \in R$, then there exists a unique solution $y = \varphi(x)$of the problem (11)-(12) defined in some neighbourhood of x_0 .

We have already shown that (11)-(12) is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \tag{15}$$

Consider the operator T defined on $\mathcal{C}([a, b])$ by

$$(T\varphi)(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))dt \tag{16}$$

Let

$$M = \sup\{|f(x, y)|; \quad (x, y) \in R\},$$

and select $\varepsilon > 0$ such that $L\varepsilon < 1$ and $[x_0 - \varepsilon, x_0 + \varepsilon] \subset [a, b]$.

If

$$S = \{\varphi(x) \in \mathcal{C}([x_0 - \varepsilon, x_0 + \varepsilon]); \quad |\varphi(x) - y_0| \leq M\varepsilon$$
$$\text{for all } x \in [x_0 - \varepsilon, x_0 + \varepsilon]\}$$

then S is a closed subset of the Banach space $\mathcal{C}([x_0 - \varepsilon, x_0 + \varepsilon])$ with the norm

$$||\varphi|| = \sup_{[x_0 - \varepsilon, x_0 + \varepsilon]} |\varphi(x)|.$$

Furthermore, if $\varphi \in S$ and $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$, then

$$|(T\varphi)(x) - y_0| = \left| \int_{x_0}^{x} f(t, \varphi(t))dt \right| \leq M\varepsilon,$$

and thus T maps S onto itself.

Finally, for any $\varphi_1, \varphi_2 \in S$, we have

$$||T\varphi_1 - T\varphi_2|| = \sup_{[x_0-\varepsilon, x_0+\varepsilon]} \left| \int_{x_0}^{x} (f(t, \varphi_1(t)) - f(t, \varphi_2(t)))\, dt \right| \leq L\varepsilon ||\varphi_1 - \varphi_2||.$$

Thus, since $L\varepsilon < 1$, T is a contraction. Therefore, in view of previous Theorem there is a unique solution φ of the equation $T\varphi = \varphi$, that is, $y = \varphi$ is a unique solution of (15).

Volterra equations – examples

Example

- $\varphi(x) = 1 + \int\limits_0^x \varphi(t)dt, \quad \varphi_0(x) = 0$
- $\varphi(x) = x - \int\limits_0^x (x-t)\varphi(t)dt, \quad \varphi_0(x) = 0$
- $\varphi(x) = 1 + \int\limits_0^x (x-t)\varphi(t)dt, \quad \varphi_0(x) = 1$

Example

Consider the integral equation

$$f(x) = x + \frac{1}{2}\int_{-1}^{1}(t-x)f(t)dt$$

First we set $f_0(x) = x$. Then

$$f_1(x) = x + \frac{1}{2}\int_{-1}^{1}(t-x)tdt = x + \frac{1}{3}$$

Substituting f_1 back into the original equation, we find

$$f_2(x) = x + \frac{1}{2}\int_{-1}^{1}(t-x)(t+\frac{1}{3})dt = x + \frac{1}{3} - \frac{1}{3}x$$

Continuing this process, we obtain

$$f_3(x) = x + \frac{1}{3} - \frac{x}{3} - \frac{1}{3^2},$$

$$f_4(x) = x + \frac{1}{3} - \frac{x}{3} - \frac{1}{3^2} + \frac{x}{3^2},$$

$$\vdots$$

$$f_{2n}(x) = x + \sum_{m=1}^{n} (-1)^{m-1} 3^{-m} - x \sum_{m=1}^{n} (-1)^{m-1} 3^{-m}.$$

By letting $n \to \infty$ we get

$$f(x) = \frac{3}{4}x + \frac{1}{4}$$

Theorem

The equation

$$f(x) = \alpha \int_a^b K(x,y) f(y) dy + \varphi(x) \tag{17}$$

has a unique solution $f \in L^2([a,b])$ *provided the kernel* K *is continuous in* $[a,b] \times [a,b]$, $\varphi \in L^2([a,b])$ *and* $|\alpha| k < 1$, *where*

$$k = \sqrt{\int_a^b \int_a^b |K(x,y)|^2 dxdy}.$$

Consider the operator

$$(Tf)(x) = \alpha \int_a^b K(x,y)f(y)dy + \varphi(x)$$

Since $\varphi \in L^2([a,b])$, $Tf \in L^2([a,b])$ if

$$\int_a^b K(x,y)f(y)dy \in L^2([a,b]) \tag{18}$$

By Schwarz's inequality, we find

$$\left|\int_a^b K(x,y)f(y)dy\right| \le \int_a^b |K(x,y)f(y)|dy \le$$

$$\le \left(\int_a^b |K(x,y)|^2 dy\right)^{1/2} \left(\int_a^b |f(y)|^2 dy\right)^{1/2}$$

Therefore,

$$\left|\int_a^b K(x,y)f(y)dy\right|^2 \leq \left(\int_a^b |K(x,y)|^2 dy\right)\left(\int_a^b |f(y)|^2 dy\right)$$

and

$$\int_a^b \left|\int_a^b K(x,y)f(y)dy\right|^2 dx \leq \int_a^b \left(\int_a^b |K(x,y)|^2 dy \int_a^b |f(y)|^2 dy\right) dx$$

$$\leq \int_a^b \int_a^b |K(x,y)|^2 dydx \int_a^b |f(y)|^2 dy$$

Since

$$\int_a^b \int_a^b |K(x,y)|^2 dydx < \infty \text{ and } \int_a^b |f(y)|^2 dy < \infty$$

(18) is satisfied and thus T maps $L^2([a,b])$ into itself. The above shows also that the operator defined by

$$(Af)(x) = \int_a^b K(x,y) f(y) dy$$

is bounded. Therefore, by appropriate Theorem, the equation $Tf = f$ has a unique solution whenever $|\alpha| k < 1$.

Example

Example

Consider the integral equation

$$f(x) = \alpha \int_a^b e^{(x-y)/2} f(y)dy + \varphi(x) \tag{19}$$

where φ is a given function. Since

$$\int_a^b \int_a^b (e^{(x-y)/2})^2 dxdy = \frac{(e^b - e^a)^2}{e^{a+b}}$$

Equation (19) has a unique solution whenever $|\alpha| < \frac{e^{(a+b)/2}}{e^b - e^a}$

Consider an operator equation

$$f = \varphi + \alpha Tf \tag{20}$$

If T is an integral operator with a kernel K, that is,

$$Tf(x) = \int_a^b K(x,t)f(t)dt \tag{21}$$

Then (20) leads to a Fredholm integral equation of the second kind

$$f(x) = \varphi(x) + \alpha \int_a^b K(x,t)f(t)dt \tag{22}$$

In such a case, we have

$$(T^2 f)(x) = T\left(\int_a^b K(x,t)f(t)dt\right) =$$

$$= \int_a^b K(x,z)\left(\int_a^b K(z,t)f(t)dt\right)dz =$$

$$= \int_a^b \left(\int_a^b K(x,z)K(z,t)dz\right)f(t)dt$$

Therefore T^2 is an integral operator whose kernel is

$$\int_a^b K(x,z)K(z,t)dz$$

Similarly

$$T^n f(x) = \int_a^b K_n(x,t) f(t) dt, \text{ dla } n \geq 2$$

where the kernel K_n of T^n is given by

$$K_n(x,t) = \int_a^b K(x,\xi) K_{n-1}(\xi,t) d\xi, \text{ dla } n \geq 2$$

The kernel can be also written as

$$K_n(x,t) = \int_a^b \dots \int_a^b K(x,\xi_{n-1}) K(\xi_{n-1},\xi_{n-2}) \dots K(\xi_1,t) d\xi_{n-1} d\xi_{n-2} \dots d\xi_1$$

Applying appropriate Theorem we conclude about solvability of (20) and hence also the integral equation (22)

If $|\alpha|||T|| < 1$, then Equation (20) has a unique solution given by the Neumann series

$$f = \varphi + \sum_{n=1}^{\infty} \alpha^n T^n \varphi \tag{23}$$

Hence, the integral equation (22) has a unique solution f given by

$$f(x) = \varphi(x) + \alpha \int_a^b \left[\sum_{n=1}^{\infty} \alpha^{n-1} K_n(x,t) \right] \varphi(t) dt. \tag{24}$$

If we adopt the notation

$$\Gamma(x, t; \alpha) = \sum_{n=1}^{\infty} \alpha^{n-1} K_n(x,t)$$

then the solution can be written in the form

$$f(x) = \varphi(x) + \alpha \int_a^b \Gamma(x, t, \alpha) \varphi(t) dt. \tag{25}$$

The function Γ is often called the resolvent kernel