1 Fredholm Equations of the Second Kind. Iterated Kernels. Constructing the Resolvent Kernel with the Aid of Iterated Kernels.

Suppose we have a Fredholm integral equation

$$\varphi(x) - \lambda \int_a^b K(x,t)\varphi(t)dt = f(x) \tag{1}$$

The integral equation (1) may be solved by the method of successive approximations To do this, put

$$\varphi(x) = f(x) + \sum_{n=1}^{\infty} \psi_n(x)\lambda^n \tag{2}$$

where the $\psi_n(x)$ are determined from the formulas

$$\psi_1(x) = \int_a^b K(x,t)f(t)dt$$

$$\psi_{n+1}(x) = \int_a^b K_n(x,t)f(t)dt$$

$$K_1(x,t) = K(x,t)$$

$$K_n(x,t) = \int_a^b K(x,z)K_{n-1}(z,t)dz \tag{3}$$

The functions $K_n(x,t)$ determined from formulas (3) are called iterated kernels.

The resolvent kernel of the integral equation (1) is determined in terms of iterated kernels by the formula

$$R(x,t;\lambda) = \sum_{n=1}^{\infty} K_n(x,t)\lambda^{n-1} \tag{4}$$

where the series on the right is called the Neumann series of the kernel $K(x,t)$.

The solution of the Fredholm equation of the second kind (1) is expressed by the formula

$$\varphi(x) = f(x) + \lambda \int_a^b R(x,t;\lambda)f(t)dt \tag{5}$$

Ex. Find the iterated kernels for the kernel

- $K(x,t) = x - t$ if $a = -1$, $b = 1$.
- $K(x,t) = x + \sin t$; $a = -\pi$, $b = \pi$.
- $K(x,t) = xe^t$; $a = 0$, $b = 1$.
- $K(x,t) = x - t$ if $a = 0$, $b = 1$.

Ex. Construct resolvent kernels for the following kernels:

- $K(x,t) = e^{x+t}$; $a = 0$, $b = 1$
- $K(x,t) = xe^t$; $a = -1$, $b = 1$.

2 Resolvent Kernel of Volterra Integral Equation. Solution of Integral Equation by Resolvent Kernel

Suppose we have a Volterra integral equation of the second kind:

$$\varphi(x) = f(x) + \lambda \int_0^x K(x,t)\varphi(t)dt \tag{6}$$

where $K(x,t)$ is a continuous function for $0 \le x \le a$, $0 \le t \le x$, and $f(x)$ is continuous for $0 \le x \le a$. We shall seek the solution of integral equation (6) in the form of an infinite power in series λ:

$$\varphi(x) = \varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \ldots + \lambda^n\varphi_n(x) + \ldots \tag{7}$$

where

$$\varphi_n(x) = \int_0^x K_n(x,t)f(t)dt, \quad (n = 1, 2, \ldots) \tag{8}$$

The functions $K_n(x,t)$ are called iterated kernels. It can readily be shown that they are determined with the aid of the recursion formulas

$$K_1(x,t) = K(x,t)$$

$$K_{n+1}(x,t) = \int_t^x K(x,z)K_n(z,t)dz, \quad (n = 1, 2, \ldots) \tag{9}$$

The function $R(x,t;\lambda)$ defined by means of the series

$$R(x,t,\lambda) = \sum_{l=0}^{\infty} \lambda^l K_{l+1}(x,t) \tag{10}$$

is called the resolqent kernel (or reciprocal kernel) for the integral equation (6). Series converges absolutely and uniformly in the case of a continuous kernel $K(x,t)$. Iterated kernels and also the resolvent kernel do not depend on the lower limit in an integral equation.

With the aid of the resolvent kernel, the solution of integral equation (6) may be written in the form

$$\varphi(x) = f(x) + \lambda \int_0^x R(x,t;\lambda)f(t)dt \tag{11}$$

Ex. Find the resolvent kernels for Volterra-type integral equations with the following kernels:

- $K(x,t) = e^{x-t}$
- $K(x,t) = e^{x^2-t^2}$
- $K(x,t) = x - t$

Ex. Using the results of the preceding examples, find (by means of resolvent kernels) solutions of the following integral equations:

- $\varphi(x) = e^x + \int_0^x e^{x-t}\varphi(t)dt$
- $\varphi(x) = e^{x^2} + \int_0^x e^{x^2-t^2}\varphi(t)dt$