2.6 UNIVARIATE DISTRIBUTIONS

(*rozkłady jednej zmiennej losowej)*

Any random variable is defined by its cumulative distribution function (CDF), $F_X\left(x\right)$.

The probability density function $f_X\left(x\right)$,of a continuous random variable is the first derivative of $F_X\left(x\right)$.

The most important variables used in structural reliability analysis are as follows: uniform, normal, lognormal, Weibull, gamma, extreme type I, extreme type II, extreme type III, and Poisson.

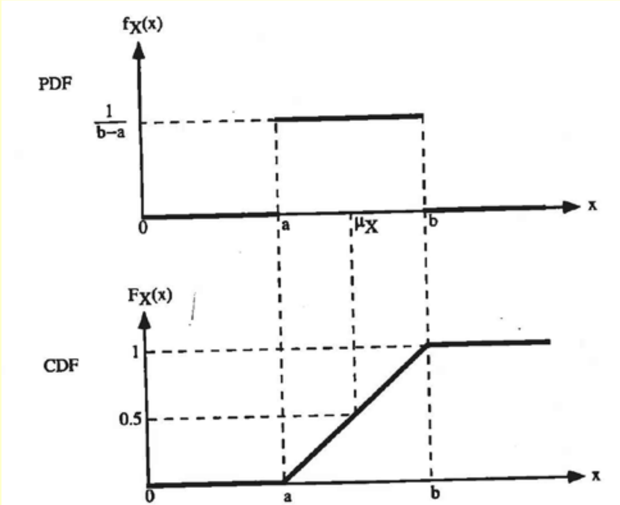
UNIFORM DISTRIBUTION

For a *uniform random var*iable or *uniform distribution* (rozkład równomiwerny), the PDF function has a constant value for all possible values of the random variable within a range [*a*, *b*]. This means that all numbers are equally likely to appear.

Mathematically, the PDF function is defined as follows:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{overwise} \end{cases}$$

where a and b define the lower and upper bounds of the random variable. The PDF and CDF for a uniform random variable are shown in Figure 2.9.



PDF and CDF of a uniform random variable

The mean and variance are as follows:

$$\mu_X = \frac{a+b}{2}$$

$$\sigma_X^2 = \frac{(b-a)^2}{12}$$

NORMAL DISTRIBUTION

The most important distribution is the *normal distribution* *(rozkład normalny)* also called the *Gaussian distribution* *(rozkład Gaussa)*. It is a two-parameter distribution defined by the density function *(funkcja gęstości prawdopodobieństwa)*

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right]$$

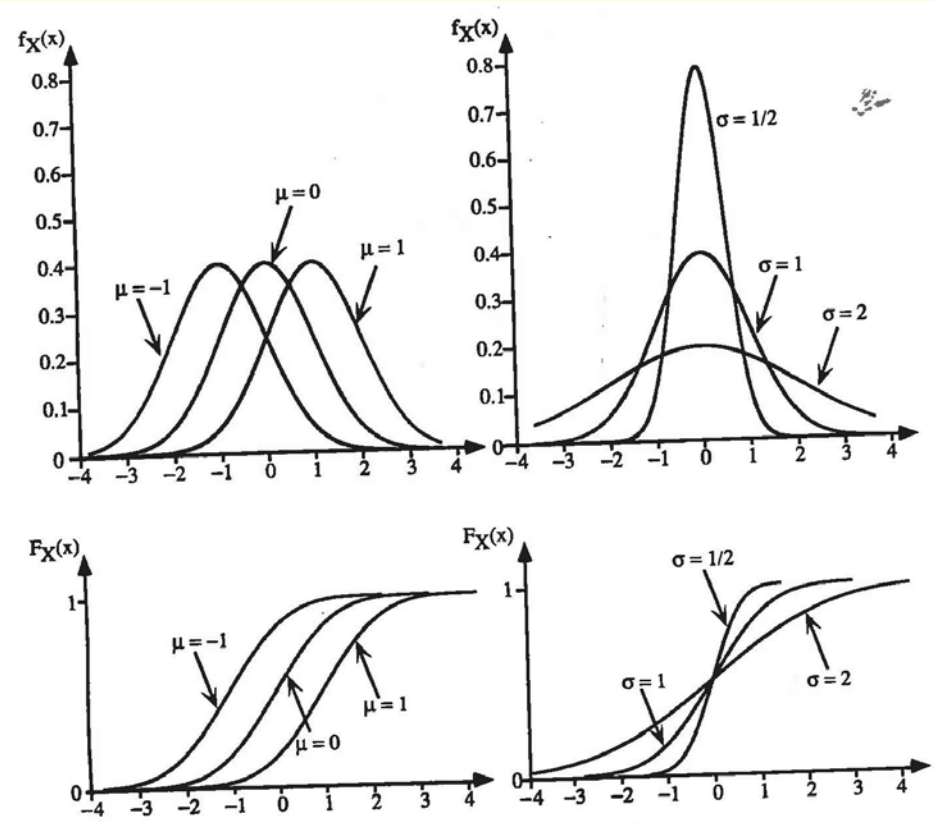
where μ and σ are parameters equal to μ_X *(expected value, wartość oczekiwana)* and σ_X (*standard deviation, odchylenie standardowe*).

This normal distribution will be denoted $N(\mu, \sigma)$.

The distribution function *(dystrybuanta)* corresponding to (2.45) is given by

$$F_X(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2 \right] dt$$

This integral cannot be evaluated on a closed form.



STANDARD NORMAL DISTRIBUTION FUNCTION

Let X be a random variable.

The standard form of X, denoted by Z, is defined as

$$Z = \frac{X - \mu_X}{\sigma_X}$$

The mean of Z is calculated as follows.

We note that the mathematical expectation (mean value) of an arbitrary function, $g(X)$, of the random variable X is defined as

$$\mu_{g(X)} = E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Using this definition with Z = g(X), we can show that

$$\mu_Z = E\left[\frac{X - \mu_X}{\sigma_X}\right] = \frac{1}{\sigma_X}\left[E(X) - E(\mu_X)\right] = \frac{1}{\sigma_X}(\mu_X - \mu_X) = 0$$

and

$$\sigma_Z^2 = E(Z^2) - \mu_Z^2 = E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)^2\right] - 0 = \frac{1}{\sigma_X^2}\left[E(X - \mu_X)^2\right] = \frac{\sigma_X^2}{\sigma_X^2} = 1$$

Thus the mean of the standard form of a random variable is 0 and its variance is 1.

The distribution function *(dystrybuanta)*

$$F_X(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right] dt$$

By the substitution

$$s = \frac{t-\mu}{\sigma}, \quad dt = \sigma ds$$

the equation (2.46) becomes

$$F_X(x) = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}s^2\right] ds = \Phi_X\left(\frac{x-\mu}{\sigma}\right)$$

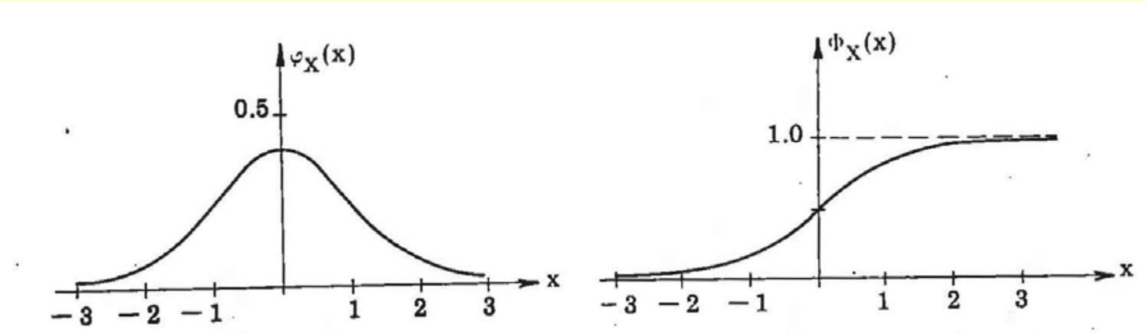
where Φ_X is the *standard normal distribution function* defined by

$$\Phi_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right] dt$$

The corresponding *standard normal density function* *(standardowa funkcja gęstości prawdopodobieństwa)* is

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right]$$

The functions φ_X and Φ_X are shown in figure 2.8.



Due to the important relation (2.48) only a standard normal table is necessary.

Many popular mathematics and spreadsheet programs have a standard normal CDF function built in.

LOGARITHMIC NORMAL DISTRIBUTION

Let the random variable $Y = \ln X$ be normally distributed $N(\mu_Y, \sigma_Y)$. Then the random variable X is said to follow a *logarithmic normal distribution* *(rozkład lognormalny)* with the parameters $\mu_Y \in R$ and $\sigma_Y > 0$.

The log-normal density function is

$$f_X(x) = \frac{1}{\sigma_Y \sqrt{2\pi}} \frac{1}{x} \exp\left[-\frac{1}{2} \left(\frac{\ln x - \mu_Y}{\sigma_Y} \right)^2 \right]$$

where $x > 0$.

Let X be log-normally distributed with the parameters μ_Y and σ_Y.

Note that μ_Y and σ_Y are not equal to μ_X and σ_X.

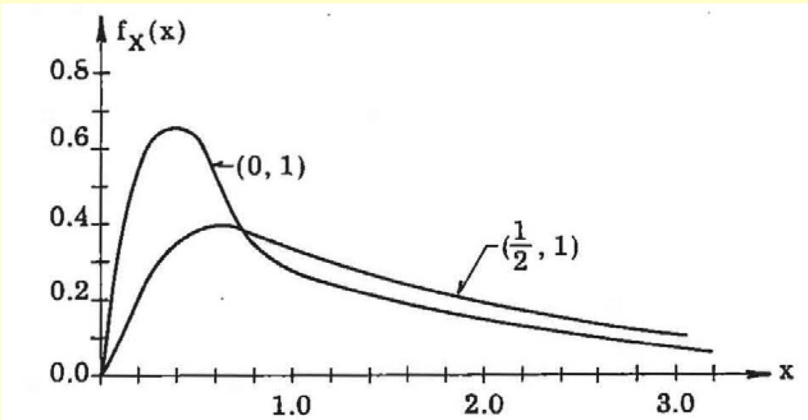
It can be shown that

$$\mu_X(x) = \exp\left(\mu_Y + \frac{1}{2}\sigma_Y^2 \right)$$

$$\sigma_X = \sqrt{\mu_X^2 \left(e^{\sigma_Y^2} - 1\right)}$$

$$F_X(x) = P(X \le x) = \Phi\left(\frac{\ln x - \mu_Y}{\sigma_Y}\right)$$

The log-normal density functions with the parameters $(\mu_Y, \sigma_Y) = (0,1)$ and (1/2, 1) are illustrated in figure 2.9.



Example 2.12.

Let the compressive strength X for concrete be log-normally distributed with the parameters $(\mu_Y, \sigma_Y) = (3\ \text{MPa},\ 0.2\,\text{MPa})$.

Then

$$\mu_X = \exp\left(3 + \frac{1}{2} \cdot 0.04\right) = 20.49\ \text{MPa}$$

$$\sigma_X^2 = 20.49^2 (1.0408 - 1) = 17.14\ (\text{MPa})^2$$

$$\sigma_X = 4.14\ \text{MPa}$$

and

$$P(X \le 10\ \text{MPa}) = \Phi((\ln 10 - 3)/2) = \Phi(-3.487) = 2.4 \cdot 10^{-4}$$

WEIBULL DISTRIBUTION

An important distribution is the so-called *Weibull distribution* *(rozkład Weibulla)* with 3 parameters β, ε and k.

The density function f_X is defined by

$$f_X(x) = \frac{\beta}{k-\varepsilon}\left(\frac{x-\varepsilon}{k-\varepsilon}\right)^{\beta-1} \exp\left[-\left(\frac{x-\varepsilon}{k-\varepsilon}\right)^{\beta}\right]$$

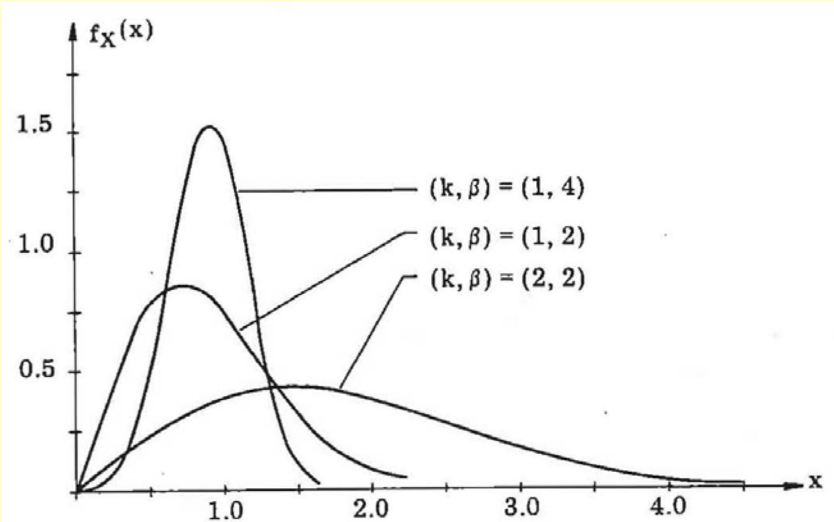
where $x \geq \varepsilon$ and $\beta > 1$, $k > \varepsilon$.

If $\varepsilon = 0$ equation (2.55) is

$$f_X(x) = \frac{\beta}{k}\left(\frac{x}{k}\right)^{\beta-1} \exp\left[-\left(\frac{x}{k}\right)^{\beta}\right], \quad x \geq 0$$

The density function (2.56) is called a two-parameter Weibull density function and is shown in Figure 2.10. If $\varepsilon = 0$ and $\beta = 2$ in (2.55) the density function is identical with the so-called *Rayleigh* density function.

$$f_X(x) = \frac{2x}{k^2} \exp\left[-\left(\frac{x}{k}\right)^2\right]$$



Gamma Distribution

The PDF of a gamma random variable is useful for modeling sustained live load, such as in buildings. It is defined by

$$f_X(x) = \frac{\lambda(\lambda x)^{k-1} e^{-\lambda x}}{\Gamma(k)} \qquad \text{for } k \geq 0$$

where λ and k are distribution parameters.

The function $\Gamma(k)$ is the gamma function, which is defined as

$$\Gamma(k) = \int_0^\infty e^{-u} u^{k-1} du$$

and

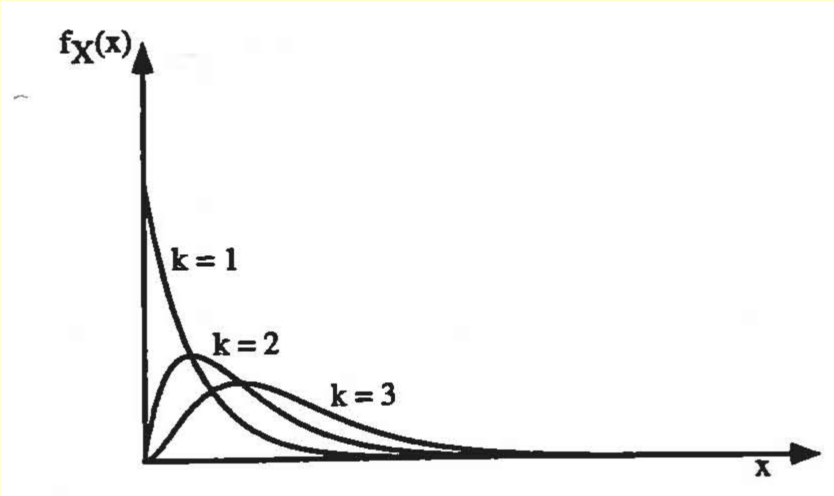
$$\Gamma(k) = (k-1)(k-2)\ldots(2)(1) = (k-1)!$$

$$\Gamma(k+1) = \Gamma(k)k$$

Values of $\Gamma(k)$ for $1 \leq k \leq 2$ are tabulated.

The mean and variance can be calculated as follows:

$$\mu_X = \frac{k}{\lambda}, \qquad \sigma_X^2 = \frac{k}{\lambda^2}$$



PDFs of gamma random variables