# STRUCTURAL RELIABILITY ANALYSIS

*Nowak, A.S., Collins K.R. Reliability of structures.*
*McGraw-Hill Higher Education 2000*
*P. Thoft-Christensen, M. J. Baker*
*Structural reliability theory and its applications, 1982*

The term *structural reliability* should be considered as having two meanings - a general one and a mathematical one.

- In the most general sense, the *reliability* of a structure is its ability to fulfill its design purpose for some specified time.

- **In a narrow sense it is the *probability* that a structure will not attain each specified limit state (ultimate or serviceability) during a specified *reference period*.**

Here we shall be concerned with structural reliability in the narrow sense and shall generally be treating each limit state or failure mode separately and explicitly.

Most structures and structural elements have a **number of possible failure modes**, and in determining the overall reliability of a structural system this must be taken into account making due allowance for the correlations arising from common sources of loading and common material properties.

*Reference period* – in general, structural reliability is dependent on time of exposure to the loading environment.

It is also affected if material properties change with time.

## Example

Assume that an offshore structure is idealized as a uniform vertical cantilever rigidly connected to the sea bed.

The structure will fail when the moment $S$ induced at the root of the cantilever exceeds the flexural strength $R$.

Assume further that $R$ and $S$ are random variables whose statistical distributions are known very precisely as a result of a very long series of measurements.

$R$ is a variable representing the variations in strength between nominally identical structures, whereas $S$ represents the maximum load effects in successive $T$ year periods.

The distributions of $R$ and $S$ are both assumed to be stationary with time.

Under these assumptions, the probability that the structure will collapse during any reference period of duration *T* years is given by

$$P_f = P\left(M \le 0\right) = \int_{-\infty}^{+\infty} F_R\left(x\right) f_S\left(x\right) dx$$

where

$$M = R - S$$

and $F_R$ is the probability distribution function of *R* and $F_S$ the probability density function of *S*.

The reliability $\mathcal{R}$ is defined as

$$\mathcal{R} = 1 - P_f$$

may be interpreted as a long-run survival frequency or long-run reliability and is the percentage of a notionally infinite set of nominally identical structures which survive for the duration of the reference period *T*.

$\mathcal{R}$ may therefore be called a *frequencies reliability*.

$\mathcal{R}$ may also be interpreted as a measure of the reliability of that particular structure.

The associated reliability can be called a subjective or *Bayesian reliability*.

For a particular structure, the numerical value of this reliability changes as the state of knowledge about the structure changes, for example, if non-destructive tests were to be carried out on the structure to estimate the magnitude of $r$.

In the limit when $r$ becomes known exactly, the probability of failure given changes

$$P_f = P\left(r - S \leq 0\right) = 1 - F_S\left(r\right)$$

This special case may also be interpreted as a *conditional failure probability* with a relative frequency interpretation, i.e.

$$P_f = P\left(R - S \leq 0 \middle| R = r\right)$$

## Methods of Safety Checking

Methods of structural reliability analysis can be divided into two broad classes. These are:

***Level* 3:** Methods in which calculations are made to determine the “exact” probability of failure for a structure or structural component, making use of a full probabilistic description of the joint occurrence of the various quantities which affect the response of the structure and taking into account the true nature of the failure domain.

***Level* 2:** Methods involving certain approximate iterative calculation procedures to obtain an approximation to the failure probability of a structure or structural system, generally requiring an idealisation of failure domain and often associated with a simplified representation of the joint probability distribution of the variables.

For the sake of completeness, some mention should also be made of level 1 methods at this stage.

**These are not methods of reliability analysis, but are methods of design or safety checking.**

***Level 1:*** Design methods in which appropriate degrees of structural reliability are provided on a structural element basis (occasionally on a structural basis) by the use of a number of partial safety factors, or partial coefficients, related to pre-defined characteristic or nominal values of the major structural and loading variables.

## STRUCTURAL SAFETY ANALYSIS)

### Limit states - Definition of Failure

The term "failure" can have different - meanings. Before attempting a structural reliability analysis, failure must be clearly defined.

We could say that a structure fails if it cannot perform its intended function.

Consider a simply supported steel hot-rolled beam such as the one shown in the Figure.

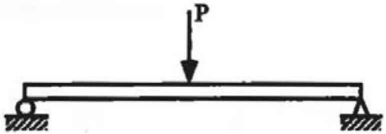
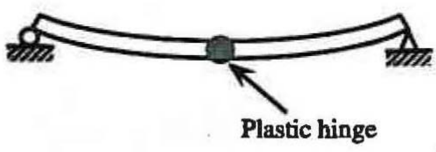


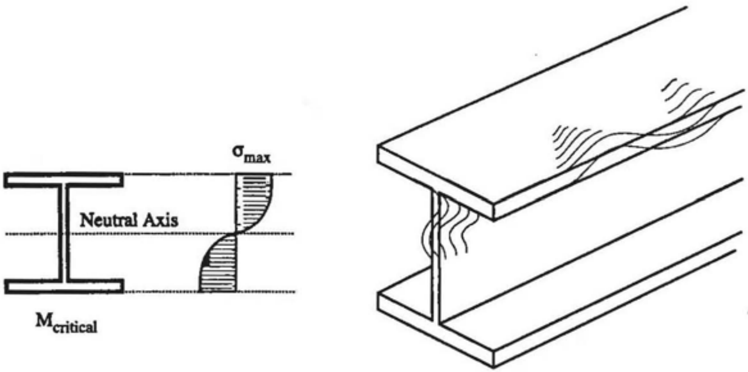

Figure. A simple supported beam.

We could state that the beam fails when the maximum deflection exceeds $\delta_{critical}$

However, a steel beam may "fail" by developing a plastic hinge, losing overall stability, or by local buckling of the compression flange or web.




Development of a plastic hinge in a beam




Local buckling in a steel beam.

The concept of a **limit state** is used to help define failure in the context of structural reliability analyses.

A *limit state* is a boundary between desired and undesired performance of a structure.

This boundary is often represented mathematically by a *limit state function* or *performance function*.

This undesired performance can occur by many modes of failure: cracking, corrosion, excessive deformations, exceeding load-carrying capacity for shear or bending moment, or local or overall buckling.

Some members may fail in a brittle manner, whereas others may fail in a ductile fashion.

In the traditional approach, each mode of failure is considered separately, and each mode can be defined using the concept of a limit state.

## Three types of limit states are considered:

**1. Ultimate limit states** (ULSs) are mostly related to the loss of load-carrying capacity.

Examples of modes of failure in this category include:

Exceeding the moment carrying capacity

Formation of a plastic hinge.

Crushing of concrete in compression

Shear failure of the web in a steel beam

Loss of the overall stability

Buckling of Lange

Buckling of web

Weld rupture.

## 2. Serviceability limit states

(SLSs) are related to gradual deterioration, user's comfort, or maintenance costs.

Examples of modes of failure include:

***Excess deflection.***

Deflection is a rather controversial limit state.

The acceptable limits are subjective, and they may depend on human perception.

A building with visible deflections (horizontal or vertical) is not acceptable by the public, even though it may be structurally safe.

For example, for bridge girders, the current practice is to limit deflections to a fraction of the span length; for example, $L/800$, where L = span length.

The deflection limit often governs the design.

***Excess vibration.***

Vibration is another serviceability limit state that is difficult to quantify.

The acceptability criteria are also highly subjective and often depend on human perception.

In a building, the occupants may not tolerate excessive vibration; a vibrating bridge, however, may be acceptable if pedestrians are not involved.

The design for vibration may require a complicated dynamic analysis.

In many current design codes, vibration is not considered in a direct form.

Indirectly, the codes impose a limit on static deflection, and this is also intended to serve as a limit for vibration.

***Permanent deformations.***

Each time the load exceeds the elastic limit, a permanent deformation may result.

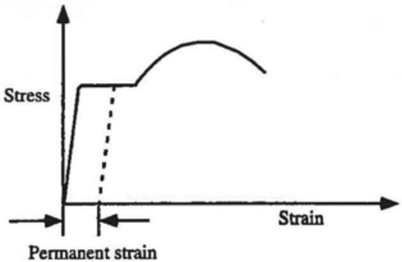
Accumulation of these permanent deformations can lead to serviceability problems.

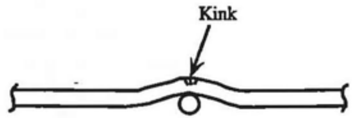
Therefore, in some design codes, a limit is imposed on permanent deformations.

For example, consider a multispan bridge with continuous girders as shown in Figure.

Each time the strain exceeds the yield strain, there is some permanent strain left in the section.

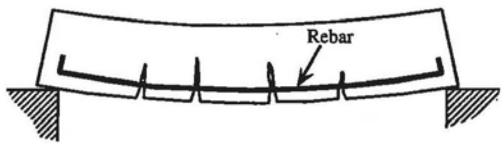
This strain accumulates and eventually causes the formation of a "kink," as shown in the next Figure.




***Cracking.***

Cracks, such as those shown in Figure, by themselves do not necessarily affect the structural performance of concrete structures.



However, they lead to steel corrosion, spalled concrete, salt (deicing agent) penetration, and irreversible loss of concrete tensile strength.

To define acceptable cracking standards, many questions must be answered.

What is acceptable with regard to cracking?

Are acceptable cracks limited by size?

Width?

Length?

How frequently can the cracks open?

## 3. Fatigue limit states

(FLSs) are related to loss of strength under repeated loads.

Fatigue limit states are related to the accumulation of damage and eventual failure under repeated loads.

It has been observed that a structural component can fail under repeated loads at a level lower than the ultimate load.

The failure mechanism involves the formation and propagation of cracks until their rupture.

Fatigue limit states occur in steel components.

Welding affects the fatigue resistance of components and connections.

Fatigue failures have also been reported in the prestressing strands. In any fatigue analysis, the critical factors are both the magnitude and frequency of load.

## LIMIT STATE FUNCTIONS (performance functions)

A traditional notion of the "safety margin" or "margin of safety" is associated with the ultimate limit states.

For example, a mode of beam failure could be when the moment due to loads exceeds the moment-carrying capacity.

Let $R$ represent the resistance (moment-carrying capacity) and Q represent the load effect (total moment applied to the considered beam).

It is sometimes helpful to think of $R$ as the "capacity" and $Q$ as the "demand."

A *performance function*, or *limit state function*, can be defined for this mode of failure as

$$g(R,Q) = R - Q \tag{0.1}$$

The *limit state*, corresponding to the boundary between desired and undesired performance, would be when $g = 0$.

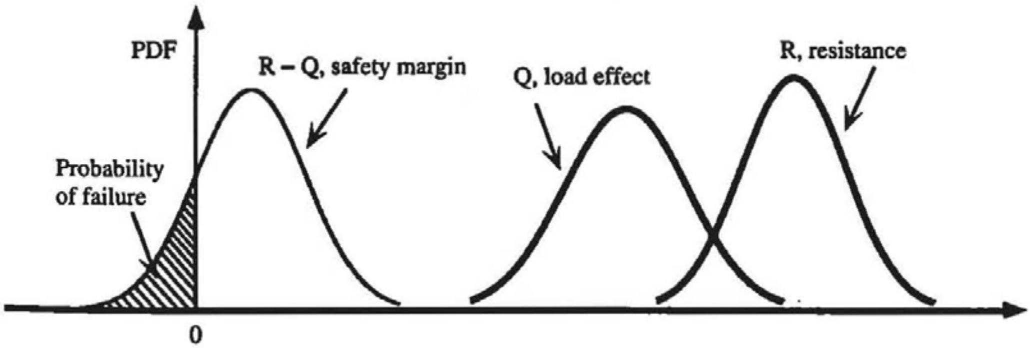
If $g \geq 0$, the structure is safe (desired performance); if g < 0, the structure is not safe (undesired performance).

The probability of failure, $P_f$, is equal to the probability that the undesired performance will occur.

Mathematically, this can be expressed in terms of the performance function as

$$P_f = P(R - Q < 0) = P(g < 0) \qquad (0.2)$$

If both *R* and *Q* are continuous random variables, then each has a probability density function (pDF) such as shown in Figure.




Furthermore, the quantity $R - Q$ is also a random variable with its own PDF.

This is also shown in Figure.

The probability of failure corresponds to the shaded area in Figure.

Now let's generalize the concepts just introduced.

All realizations of a structure can be put into one of two categories:
Safe (load effect ≤ resistance)
Failure (load effect > resistance)

The state of the structure can be described using various parameters $X_1, X_2, \ldots, X_n$, which are load and resistance parameters such as dead load, live load, length, depth, compressive strength, yield strength, and moment or inertia.

A limit state function, or performance function, is a function $g(X_1, X_2, \ldots, X_n)$ of these parameters such that

$g(X_1, X_2, \ldots, X_n) >$ for a safe structure

$g(X_1, X_2, \ldots, X_n) =$ border or boundary between safe and unsafe

$g(X_1, X_2, \ldots, X_n) <$ for failure

Each limit state function is associated with a particular limit state.

Different limit states may have different limit state functions.

Here are some examples of limit state functions:

l. Let $Q$ = total load effect (total demand) and $R$ = resistance (or capacity).
Then the limit state function can be defined as

$$g(R,Q) = R - Q \tag{0.3}$$

or

$$g(R,Q) = R / Q - 1 \tag{0.4}$$

2. Consider case 1 above for the moment capacity of a compact steel beam.
The moment capacity is $R = F_y Z$ where $F_y$ is the yield stress and $Z$ is the plastic section modulus.
Substituting into Eq. (0.3), we get

$$g(F_y, Z, Q) = F_y Z - Q \qquad (0.5)$$

3. Consider case 2 with a more definitive description of the demand.

Assume that the total demand or load effect on the beam is made up of contributions from dead load (*D*), live load (*L*), wind load (*W*), and earthquake load (*E*).
If $Q = D + L + W + E$,
then Eq. (0.5) is

$$g(F_y, Z, D, L, W, E) = F_y Z - D - L - W - E \qquad (0.6)$$

In general, the performance function (limit state function) can be a function of many variables: load components, influence factors, resistance parameters, material properties, dimensions, analysis factors, and so on.

A direct calculation of $P_f$ using Eq. (0.2) is often very difficult, if not impossible.

Therefore, it is convenient to measure structural safety in terms of a **reliability index**.

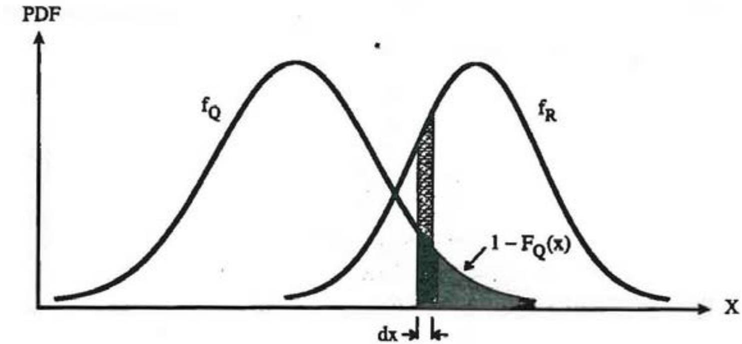
# FUNDAMENTAL CASE

## Probability or Failure

We now examine how to determine the probability of failure for the relatively simple performance function given earlier by

$$g(R,Q) = R - Q \qquad (0.7)$$

The probability of failure, $P_f$, can be derived by considering the PDFs of the random variables *R* and *Q* as shown in Figure.




PDFs of load (Q) and resistance (R).

The structure "fails" when the load exceeds the resistance.

If $R$ is equal to a specific value $r_i$, then the probability of failure is equal to the probability that the load is greater than the resistance, or $P(Q > r_i)$.

However, since $R$ is a random variable, there is a probability associated with each $r_i$ value.

Therefore, the probability of failure is composed of all possible combinations of $R = r_i$ and $Q > r_i$, which can be written as

$$P_f = \sum P(R = r_i \cap Q > r_i) = \sum P(Q > r_i \mid R = r_i) P(R = r_i) \qquad (0.8)$$

where the properties of conditional probability have been used in.

For the continuous case, the summation becomes an integral.

The probability $P(Q > R \mid R = r_i)$ is simply

$1 - P(Q \le R \mid R = r_i) = 1 - F_Q(r_i)$

In the limit, the probability $P(R = r_i) \approx f_R(r_i)dr_i$ .

Combining all these modifications into Eq. (0.8) leads to

$$P_f = \int_{-\infty}^{\infty} [1 - F_Q(r_i)] f_R(r_i) dr_i = 1 - \int_{-\infty}^{\infty} F_Q(r_i)] f_R(r_i) dr \tag{0.9}$$

There is an alternative formulation that we can use.

If the load $Q$ is equal to a specific value $q_i$, then the probability of failure is equal to the probability that the resistance is less than the load, or $P(R < q_i)$.

However, since $Q$ is a random variable, there is a probability associated with each $q_i$ value.

Therefore, the probability of failure is composed of all possible combinations of $Q = q_i$ and $R < qi$, which can be written as

$$P_f = \sum P(Q = q_i \cap R < q_i) = \sum P(R < Q \middle| Q = q_i) P(Q = q_i) \qquad (0.10)$$

Following the same logic as earlier, this can be written in integral form as

$$P_f = \int_{-\infty}^{\infty} F_R(q_i) f_Q(q_i) dq_i \qquad (0.11)$$

Although Eqs. (0.9) and (0.11) appear rather straightforward, it is difficult to evaluate these integrals in general.

The integration requires special numerical techniques, and the accuracy of these techniques may not be adequate.

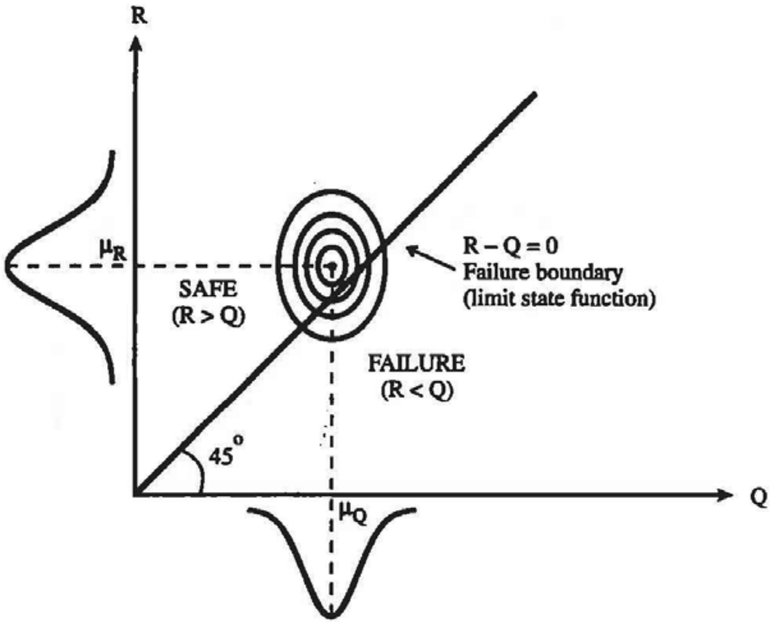
Therefore, in practice, the probability of failure is calculated indirectly using other procedures.

## Space of State Variables

The *state variables* are the basic load and resistance parameters used to formulate the performance function.

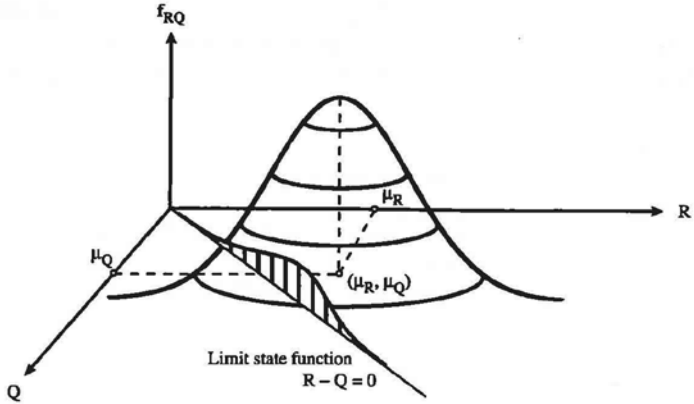
For *n* state variables, the limit stale function is a function of *n* parameters.

If all loads (or load effects) are represented by the variable *Q* and total resistance (or capacity) by *R* then the space of stale variables is a two-dimensional space as shown in Figure.




Within this space, we can separate the "safe domain" from the "failure domain"; the boundary between the two domains is described by the limit state function $g(R, Q) = 0$.

Since both *R* and *Q* are random variables, we can define a joint density function $f_{RQ}(r, q)$.
A general joint density function is plotted in Figure.




Three-dimensional representation of a possible joint density function $f_{RQ}$.
The probability of failure is calculated by integration of the joint density function over the failure domain i.e., the region in which $g(R, Q) < O$.

# RELIABILITY INDEX

## Reduced Variabies

Standard form, a nondimensional form of the random variables.

For the basic variables R and Q, the standard forms can be expressed as

$$Z_R = \frac{R - \mu_R}{\sigma_R}$$
$$Z_Q = \frac{Q - \mu_Q}{\sigma_Q} \qquad (0.12)$$

The variables $\underline{Z_R}$ and $Z_Q$ are sometimes called *reduced variables*.

By rearranging Eqs. (0.12), the resistance *R* and the load *Q* can be expressed in terms of the reduced variables as follows:

$$R = \mu_R + Z_R \sigma_R$$
$$Q = \mu_Q + Z_Q \sigma_Q \qquad (0.13)$$

The limit state function $g(R, Q) = R - Q$ can be expressed in terms of the reduced variables by using Eqs. 5.12.

The result is

$$g(Z_R, Z_Q) = \mu_R + Z_R\sigma_R - \mu_Q - Z_Q\sigma_Q = (\mu_R - \mu_Q) + Z_R\sigma_R - Z_Q\sigma_Q \tag{0.14}$$

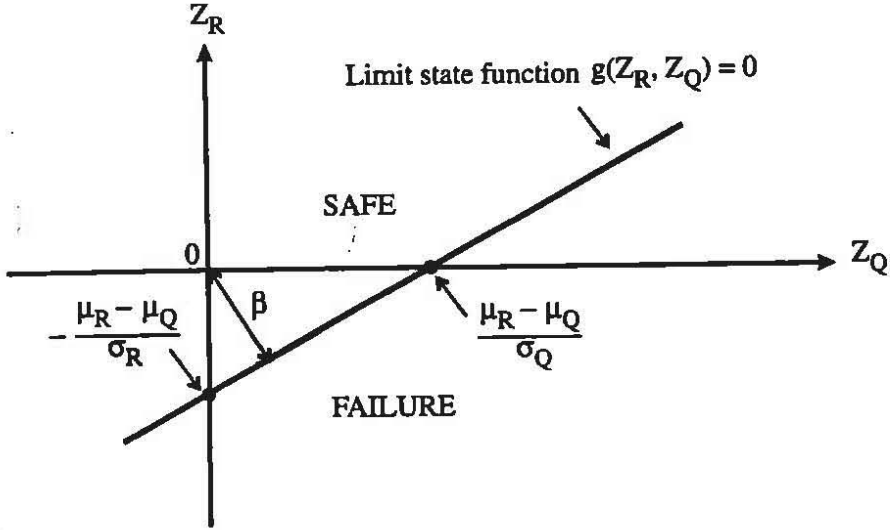
For any specific value of $g(Z_R, Z_Q)$, Eq. (0.14) represents a straight line in the space of reduced variables $Z_R$ and $Z_Q$.

The line of interest to us in reliability analysis is the line corresponding to $g(Z_R, Z_Q) = 0$ because this line separates the safe and failure domains in the space of reduced variables.

## General Definition of the Reliability Index

The reliability index is defined as the *shortest* distance from the origin of reduced variables to the line $g(Z_R, Z_Q) = 0$.

This definition, which was introduced by Hasofer and Lind (1974), is illustrated in Figure.



Using geometry, we can calculate the reliability index (shortest distance) from the following formula:

$$\beta = \frac{\mu_R - \mu_Q}{\sqrt{\sigma_R^2 - \sigma_Q^2}} \tag{0.15}$$

where $\beta$ is the inverse of the coefficient of variation of the function $g(R, Q) = R - Q$ when $R$ and $Q$ are uncorrelated.

For normally distributed random variables $R$ and $Q$, it can be shown that the reliability index is related to the probability of failure by

$$\beta = -\Phi^{-1}(P_f) \qquad or \qquad P_f = \Phi(-\beta) \tag{0.16}$$

Table provides an indication of how $\beta$ varies with $P_f$ and vice versa based on Eq. 5.15.

| $P_f$ | $\beta$ |
|---|---|
| $10^{-1}$ | 1.28 |
| $10^{-2}$ | 2.33 |
| $10^{-3}$ | 3.09 |
| $10^{-4}$ | 3.71 |
| $10^{-5}$ | 4.26 |
| $10^{-6}$ | 4.75 |
| $10^{-7}$ | 5.19 |
| $10^{-8}$ | 5.62 |
| $10^{-9}$ | 5.99 |

The definition for a two-variable case can be generalized for n variables as follows.

Consider a limit state function $g(X_1, X_2 \ldots X_n)$ where the $X_i$ variables are all uncorrelated. The Hasofer-Lind reliability index is defined as follows:

1. Define the set of reduced variables $\{Z_1, Z_2 \ldots Z_n\}$ using

$$Z_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \qquad (0.17)$$

2. Redefine the limit state function by expressing it in terms of the reduced variables $\{Z_1, Z_2 \ldots Z_n\}$

3. The reliability index is the shortest distance from the origin in the n-dimensional space of reduced variables to the curve described by $g(Z_1, Z_2 \ldots Z_n) = 0$.

## First-Order Second-Moment Reliability Index

## Linear limit state functions

Consider a *linear* limit state function of the form

$$g(X_1, X_2, \ldots, X_n) = a_0 + a_1 X_1 + a_2 X_2 + \ldots + a_n X_n = a_0 + \sum_{i=1}^{n} a_i X_i \quad (0.18)$$

where the $a_i$ terms ($i = 0, 1, 2 \ldots n$) are constants and the $X_i$ terms are *uncorrelated* random variables.

If we apply the three-step procedure outlined above for determining the Hasofer-Lind reliability index. we would obtain the following expression for $\beta$:

$$\beta = \frac{a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}}{\sqrt{\sum_{i=1}^{n} (a_i \sigma_{X_i})^2}} \quad (0.19)$$

Observe that the reliability index, $\beta$, in Eq. (0.19) depends only on the means and standard deviations of the random variables.

Therefore, this p is called a *second-moment* measure of structural safety because only the first two moments (mean and variance) are required to calculate $\beta$.

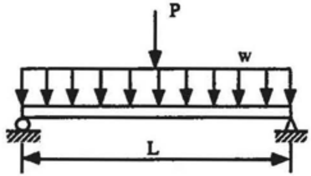
There is no explicit relationship between $\beta$ and the type of probability distributions of the random variables.

If the random variables are all normally distributed and uncorrelated, then this formula is *exact* in the sense that $p$ and $P_f$ are related by Eq. (0.16).

Otherwise, Eq. (0.16) provides only an approximate means of relating $\beta$ to a probability of failure.

## EXAMPLE

Consider the simply supported beam shown in Figure.




The beam is subjected to a concentrated live load *P* and a uniformly distributed dead load *w*.

The loads are random variables.

Assume that *P*, *w*, and the yield stress, $F_y$, are random quantities; the length *L* and the plastic section modulus *Z* are assumed to be precisely known (deterministic).

The distribution parameters for *P*, *w*, and $F_y$ are given below.

A quantity known as the "bias factor" (denoted by $\lambda$) is specified for each of the random variables.

It is defined as the ratio of the mean value of a variable to its nominal value (i.e., the value specified in a standard or code).

The length $L$ is 18 ft, and the plastic section modulus is 80 in$^3$.

Nominal (design) value of $w = w_n$ = 3.0 k/ft = 0.25 k/in
Bias factor for $w = \lambda_W$ = 1.0
$\mu_W = \lambda_W w_n$ = 3.0 k/ft = 0.25 k/in
$Vw$ = 10% => $\sigma_W = V_w \mu_W$ = 0.3 k/ft = 0.025 k/in

Nominal (design) value of $P = p_n$ = 12.0 k
Bias factor for $P = \lambda_P$ = 0.85
$\mu_P = \lambda_P P_n$ = 10.2 k
$V_P$ = 11% => $\sigma_P = V_P \mu_P$ = 1.12k

Nominal (design) value of $F_y = f_y = 36$ ksi
Bias factor for $F_y = \lambda_F = 1.12$
$\mu_F = \lambda_F f_y = 40.3$ ksi
$V_F = 11.5\%$ => $\sigma_F = V_F \mu_F = 4.64$ ksi
Calculate the reliability index.

**Solution**.
The limit state function for beam bending can be expressed as

$$g(P, w, F_y) = F_y Z - \frac{PL}{4} - \frac{wL^2}{8}$$

Substituting for $L$ and $Z$ (and converting all units to inches), the limit state function can be rewritten as

$$g(P, w, F_y) = 80F_y - 54P - 5832w \quad [k, in]$$

Since the limit state function is linear, Eq. (0.19) can be used to determine the reliability index $\beta$:

$$\beta = \frac{a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}}{\sqrt{\sum_{i=1}^{n} (a_i \sigma_{X_i})^2}} =$$

$$= \frac{80(40.3) - 54(10.2) - 5832(0.25)}{\sqrt{[(80)(4.640]^2 + [(-54)(1.12))]^2 + [(-5832)(0.025)]^2}} =$$

$$= \frac{1215.2}{403.37} = 3.01$$

## Nonlinear limit state functions

When the function is nonlinear, we can obtain an approximate answer by linearizing the nonlinear function using a Taylor series expansion.

The result is

$$g(X_1, X_2, \ldots, X_n) = \approx g(x_1^*, x_2^*, \ldots, x_n^*) + \sum_{i=1}^{n} (X_i - x_i^*) \frac{\partial g}{\partial X_i}\bigg|_{evaluated\ at\ (x_1^*, x_2^*, \ldots, x_n^*)} \quad (0.20)$$

where $(x_1^*, x_2^*, \ldots, x_n^*)$ is the point about which the expansion is performed.

One choice for this linearization point is the point corresponding to the mean values of the random variables.

Thus Eq. (0.20) becomes

$$g(X_1, X_2, \dots, X_n) = \approx g(\mu_{X_1}, \mu_{X_2}, \dots, \mu_{X_n}) + \sum_{i=1}^{n} (X_i - \mu_{X_i}) \frac{\partial g}{\partial X_i}\bigg|_{\text{evaluated at mean values}} \quad (0.21)$$

Since Eq. (0.21) is a linear function of the $X_i$ variables, it can be rewritten to look exactly like Eq. (0.18).

Thus Eq. (0.19) can be used as an approximate solution for the reliability index $\beta$.

After some algebraic manipulations, the following expression for $\beta$ results:

$$\beta = \frac{g(\mu_{X_1}, \mu_{X_2}, \dots, \mu_{X_n})}{\sqrt{\sum_{i=1}^{n} (a_i \sigma_{X_i})^2}} \quad \text{where} \quad a_i = \frac{\partial g}{\partial X_i}\bigg|_{\text{evaluated at mean values}} \quad (0.22)$$

The reliability index defined in Eq. (0.22) is called *a first-order second-moment mean value reliability index.*

It is a long name, but the underlying meaning of each part of the name is very important:

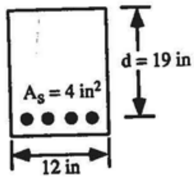
*First order* because we use first-order terms in the Taylor series expansion.

*Second moment* because only means and variances are needed.

*Mean value* because the Taylor series expansion is about the mean values.

# EXAMPLE

Consider the reinforced concrete beam shown in Figure.




The moment-carrying capacity of the section is calculated using

$$M = A_s f_y \left( d - 0.59 \frac{A_s f_y}{f_c' b} \right) = A_s f_y d - 0.59 \frac{(A_s f_y)^2}{f_c' b}$$

where $A_s$ is the area of steel, $f_y$ is the yield strength of the steel, $f_c'$ is the compressive strength of the concrete, *b* is the width of the section, and *d* is the depth of the section.

We want to examine the limit state of exceeding the beam capacity in bending.

The limit state function would be

$$g\left(A_s, f_y, f_c', Q\right) = A_s f_y d - 0.59 \frac{A_s f_y}{f_c' b} - Q$$

where $Q$ is the moment (load effect) due to the applied load. The random variables in the problem are Q, $f_y$, $f_c'$, and $A_s$.

The distribution parameters and design parameters are given in Table, where $\lambda$ is the bias factor (ratio of mean value to nominal value).

| | Mean | Nominal | λ | σ | V |
|---|---|---|---|---|---|
| $f_y$ | 44 ksi | 40 ksi | 1.10 | 4.62 ksi | 0.105 |
| $A_s$ | 4.08 in$^2$ | 4 in$^2$ | 1.02 | 0.08 in$^2$ | 0.02 |
| $f_c'$ | 3.12 ksi | 3 ksi | 1.04 | 0.44 ksi | 0.14 |
| Q | 2052 k-in | 2160 k-in | 0.95 | 246 k-in | 0.12 |

The values of $d$ and $b$ are assumed to be deterministic constants. Calculate the reliability index, $\beta$.

***Solution.***

For this problem, the limit state function is nonlinear, so we need to apply Eq.(0.20) or (0.21). The Taylor expansion about the mean values yields the following linear function:

$$g(A_s,\ f_y, f_c', Q) \approx \left[ \mu_{A_s}\mu_{f_y} d - 0.59\frac{(\mu_{A_s}\mu_{f_y})^2}{\mu_{f_c'} b} - \mu_Q \right] +$$

$$+(A_s - \mu_{A_s})\frac{\partial g}{\partial A_s}\bigg|_{evaluated\ at\ mean\ values} +$$

$$+(f_y - \mu_{f_y})\frac{\partial g}{\partial f_y}\bigg|_{evaluated\ at\ mean\ values} + (f_c' - \mu_{f_c'})\frac{\partial g}{\partial f_c'}\bigg|_{evaluated\ at\ mean\ values} +$$

$$(Q - \mu_Q)\frac{\partial g}{\partial Q}\bigg|_{evaluated\ at\ mean\ values}$$

To calculate $\beta$, the partial derivatives must be determined and the limit state function must be evaluated at the mean values of the random variables:

$$g\left(\mu_{A_s},\mu_{f_y},\mu_{f_c'},\mu_Q\right)=\mu_{A_s}\mu_{f_y}\mu_d-0.59\frac{(\mu_{A_s}\mu_{f_y})^2}{\mu_{f_c'}b}-\mu_Q=851.0\ kin$$

$$a_1=\left.\frac{\partial g}{\partial A_s}\right|_{mean\ values}=\left.\left[f_y d-0.59\frac{\left(2A_s f_y^2\right)}{f_c' b}\right]\right|_{mean\ values}=587.1\quad k/in$$

$$a_2=\left.\frac{\partial g}{\partial f_y}\right|_{mean\ values}=\left.\left[A_s d-0.59\frac{\left(2f_y A_s^2\right)}{f_c' b}\right]\right|_{mean\ values}=54.44\quad in^3$$

$$a_3=\left.\frac{\partial g}{\partial f_c'}\right|_{mean\ values}=\left.\left[0.59\frac{\left(A_s f_y\right)^2}{(f_c')^2 b}\right]\right|_{mean\ values}=162.8\quad in^3$$

$$a_4 = \frac{\partial g}{\partial Q}\bigg|_{mean\ values} = -1\bigg|_{mean\ values} = -1$$

Substituting these resu1ts into Eq. (0.22), we get

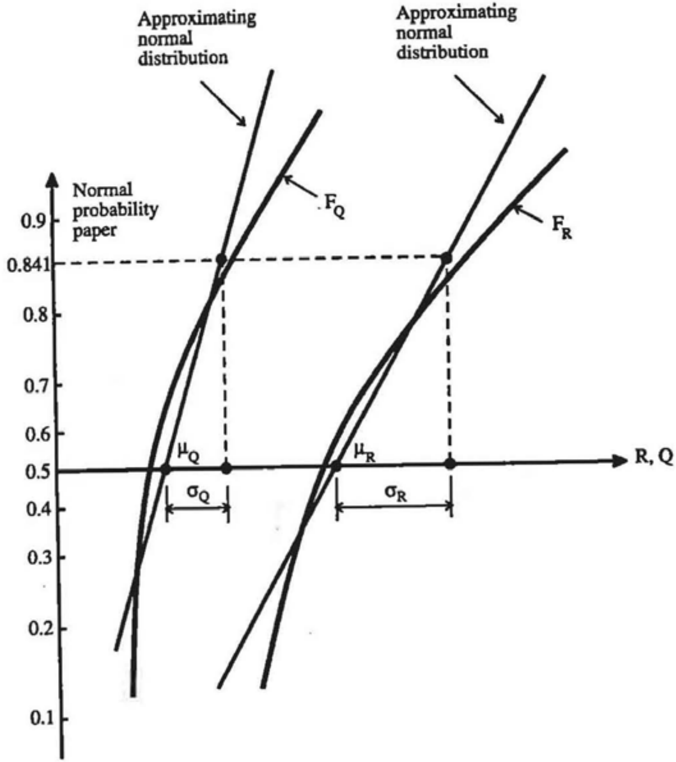
$$\beta = \frac{g\left(\mu_{A_s}, \mu_{f_y}, \mu_{f_c'}, \mu_Q\right)}{\sqrt{[(587.1)(\sigma_{A_s})]^2 + [(54.44)(\sigma_{f_y})]^2 + [(162.8)(\sigma_{f_c'})]^2 + [(-1)(\sigma_Q)]^2}}$$

$$= \frac{851.0}{\sqrt{[(587.1)(0.08)]^2 + [(54.44)(4.62)]^2 + [(162.8)(0.44)]^2 + [(-1)(246)]^2}}$$

$$= \frac{851.0}{362.1} = 2.35$$

## Comments
## on the First-Order Second-Moment Mean Value Index

The first-order second-moment mean value method is based on approximating nonnormal CDFs of the state variables by normal variables, as shown in Figure for the simple case in which $g(R, Q) = R - Q$.




# Mean value second-moment formu1ation

The method has both advantages and disadvantages in structural reliability analysis.

Among its advantages,

l. It is easy to use.

2. It does not require knowledge of the distributions of the random variables

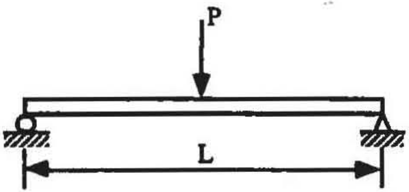
Among its disadvantages

l. Results are inaccurate if the tails of the distribution functions cannot be approximated by a normal distribution.

2. There is an invariance problem: the value of the reliability index depends on the specific form of the limit state function.

**EXAMPLE.**

Consider the steel beam shown in Figure.




The steel beam is assumed to be compact with parameters $Z$ (plastic modulus) and yield stress $f_y$.

There are four random variables to consider: $P, L, Z, F_y$.

It is assumed that the four variables are uncorrelated.

The means and covariance matrix are given as

$$\{\mu_X\} = \begin{Bmatrix} \mu_P \\ \mu_L \\ \mu_Z \\ \mu_{F_y} \end{Bmatrix} = \begin{Bmatrix} 100\ kN \\ 8\ m \\ 100 \times 10^{-6}\ m^3 \\ 600 \times 10^3\ kN/m^2 \end{Bmatrix}$$

$$\{C_X\} == \begin{Bmatrix} 4\ kN & 0 & 0 & 0 \\ 0 & 100\times10^{-6}\ m^2 & 0 & 0 \\ 0 & 0 & 400\times10^{-12}\ m^6 & 0 \\ 0 & 0 & 0 & 10\times10^{9}(kN/m^2)^2 \end{Bmatrix}$$

To begin, consider a limit state function in terms of moments. We can write

$$g_1\left(Z, F_y, P, L\right) = ZF_y - \frac{PL}{4}$$

Now recall that the purpose of the limit state function is to define the boundary between the safe and unsafe domains, and the boundary corresponds to $g = 0$.

So if we divide $g_1$ by a positive quantity (e.g., $Z$), then we are not changing the boundary or the regions in which the limit state function is positive or negative.

Thus an alternative limit state function (with units of stress) would be

$$g_2\left(Z,F_y,P,L\right)=F_y-\frac{PL}{4Z}=\frac{g_1\left(Z,F_y,P,L\right)}{Z}$$

Since both functions satisfy the requirements for a limit state function, both are valid, and we want to calculate the reliability index for both functions.

***Solution.***

For the function $g_1$, which is nonlinear, the calculation of the reliability index is given by Eq. (0.22).

The limit state function is linearized about the means.

The results are

$$g_1 \approx \left[ \mu_Z \mu_{F_y} - \frac{\mu_P \mu_L}{4} \right] + \mu_{F_y} \left( Z - \mu_Z \right) + \mu_Z \left( F_y - \mu_{F_y} \right) - \frac{\mu_L}{4} \left( P - \mu_P \right)$$

$$- \frac{\mu_P}{4} \left( L - \mu_L \right)$$

$$\beta = 2.48$$

For $g_2$, which is also nonlinear, we use Eq. (0.22)and again linearize about the mean values.

The results are

$$g_1 \approx \left[ \mu_{F_y} - \frac{\mu_P \mu_L}{4\mu_Z} \right] + \frac{\mu_P \mu_L}{4(\mu_Z)^2}(Z - \mu_Z) + (1)\left(F_y - \mu_{F_y}\right) - \frac{\mu_L}{4\mu_Z}(P - \mu_P)$$

$$- \frac{\mu_P}{4\mu_Z}(L - \mu_L)$$

$$\beta = 3.48$$

This example clearly demonstrates the "invariance" in the mean value second-moment reliability index.

In this example, the same fundamental limit state forms the basis for both limit state functions.

Therefore, the probability of failure (as reflected by the reliability index) should be the same.

# RELIABILITY INDEXES - SUMMARY

## Cornell reliability index

$$g(r,s) = r - s$$

$g(r,s)$ – limit state function

$r$ – total uncertainty contribution of material and structural strength (resistance)

$s$ – total uncertainty contribution of structural actions (including loads)

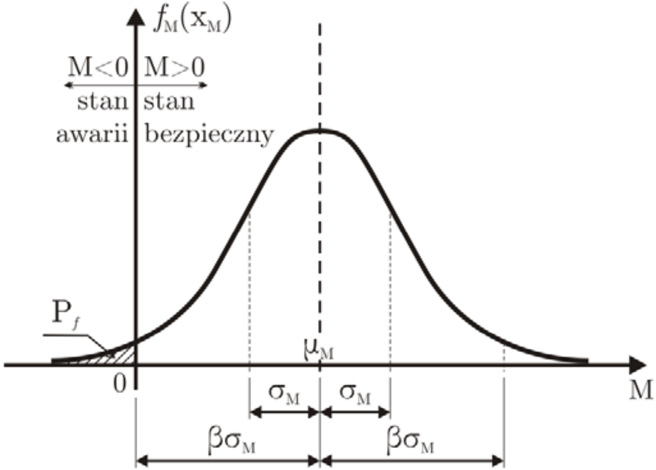
Assuming each contribution in a random variable form **the safety margin** $M$ is

$$M = g(R,S) = R - S$$

Assuming these contributions **Gaussian random variables**, it reads

$$P_f = P(R - S \le 0) = P(M \le 0) = \Phi\left(\frac{0 - \mu_M}{\sigma_M}\right)$$

here $\Phi(\bullet)$ is the CDF of a standard Gaussian variable $N(0, 1)$




**Structural reliability measure – reliability index** (Cornell,1969)

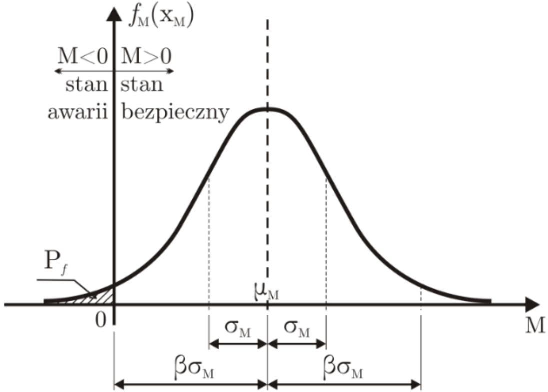
$$\beta = \frac{1}{v_M} = \frac{\mu_M}{\sigma_M}$$

here: $v_M$ – coefficient of variation of the safety margin,

$\mu_M$ – mean value of the safety margin,

$\sigma_M$ – its standard deviation.

**The Cornell reliability index measures the distance** from the considered point to the limit state surface boundary $\mu_M$ in the units of its uncertainty scale parameter $\sigma_M$




The reliability index $\beta$ may be linked directly with **failure probability** $P_f$ by an inverse relation

$$\beta = -\Phi^{-1}(P_f)$$

here $\Phi^{-1}(\bullet)$ is an inverse function of the Gaussian CDF .

In the case of uncorrelated random variables $R$ i $S$ the reliability index $\beta$ (called Cornell reliability index) takes the form

$$\beta = \beta_C = \frac{\mu_M}{\sigma_M} = \frac{\mu_R - \mu_s}{\sqrt{\sigma_R^2 + \sigma_S^2}}$$

while the random variables $R$ i $S$ are **correlated**

$$\beta_C = \frac{\mu_M}{\sigma_M} = \frac{\mu_R - \mu_s}{\sqrt{\sigma_R^2 - 2\rho_{RS}\sigma_R\sigma_S + \sigma_S^2}}$$

here: $\mu_R$ – mean value of a random variable *R* (resistance),
$\mu_S$ – mean value of a random variable *S* (action),
$\sigma_R$, $\sigma_S$ – standard deviations of variables $R$ and $S$ ,
$\rho_{RS}$ – correlation coefficient of both variables.
The Cornell reliability index may be estimated in a relatively simple way when the **limit surface** $g(\mathbf{x}) = 0$ **is a plane** (safety margin is a linear function).

**The safety margin function** may be presented in a form

$$g\left(x_i\right) = a_0 + \sum_{i=1}^{n} a_i x_i = a_0 + \mathbf{a}^T \mathbf{x} = g(\mathbf{x})$$

here: $a_0$ – a free term

$n$ – space dimension,

$\mathbf{a} = \{a_1, a_2, a_3, ..., a_n\}$ – vector of directional coefficients (slopes)

$\mathbf{x} = \{x_1, x_2, x_3, ..., x_n\}$ – realization vector.

The random variable of safety reserve (margin)

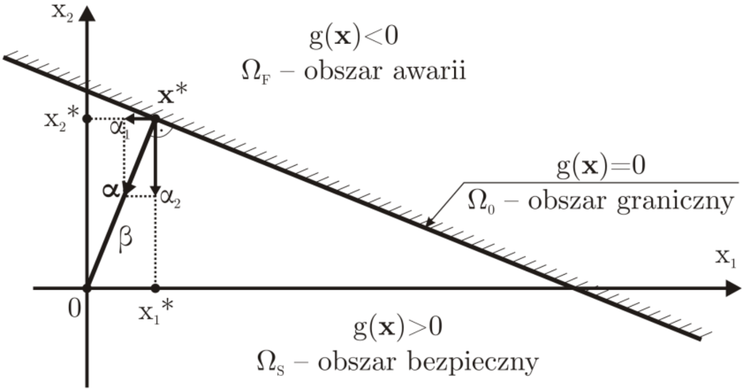
$$M = a_0 + \sum_{i=1}^{n} a_i X_i = a_0 + \mathbf{a}^T \mathbf{X}$$

The Cornell reliability index

$$\beta_C = \frac{a_0 + \sum_{i=1}^{n} a_i X_i}{\sqrt{\mathbf{a}^T C_{\mathbf{X}} \mathbf{a}}} = \frac{a_0 + \mathbf{a}^T E\left[\mathbf{X}\right]}{\sqrt{\mathbf{a}^T C_{\mathbf{X}} \mathbf{a}}}$$

here: $E[\mathbf{X}]$ - mean value vector of the random variable $\mathbf{X}$ (vector)

$C_{\mathbf{X}}$ - covariance matrix of the random variable $\mathbf{X}$ (random vector) .




Linear, $n$ - dimensional (Fig. $n=2$) limit state surface $g(\mathbf{x})=0$ in the domain of basic variable realizations .

The figure shows the Cornell reliability index $\beta_C$ and the unit normal vector to the limit surface $\boldsymbol{\alpha}$.

**The Cornell reliability index** $\beta_C$ **is invariant** in the case of any linear transformation of a basic variable $\mathbf{X}$.

If the limit surface **is not a plane** its **Taylor series expansion** is possible **with respect to the mean value point**

## The Rosenblueth – Esteva reliability index / FOSM

**The random variables** $R$ and $S$ **are often positive-valued,** here **the limit state function** $g(r,s)$ may be presented in the form

$$g(r,s) = \ln\left(\frac{r}{s}\right)$$

The safety reserve (safety margin)

$$M = \ln\left(\frac{R}{S}\right)$$

**The Rosenblueth - Esteva reliability index** (1972):

$$\beta_{RE} = \frac{E[M]}{D[M]} = \frac{E[M]}{\sqrt{V[M]}} = \frac{E\left[\ln(R/S)\right]}{\sqrt{V\left[\ln(R/S)\right]}}$$

In the case of **correlated** random variables

$$\beta_{RE} = \frac{E[M]}{D[M]} = \frac{E[R]-E[S]}{\sqrt{V[R;S]}} = \frac{E[\ln R]-E[\ln S]}{\sqrt{V[\ln R]-2Cov[\ln R;\ln S]+V[\ln S]}}$$

Here the **safety margin is a non-linear function** thus **its linearization is required**, expanding it into Taylor series with respect to the mean value point $E[R]$ and $E[S]$.

Linearization results in the following **safety margin** $M_{FO}$

$$M_{FO} = \ln \mu_R + \frac{R - \mu_R}{\sigma_R} - \ln \mu_S - \frac{S - \mu_S}{\sigma_S}$$

The **Rosenblueth – Esteva reliability index** $\beta_{RE}$

$$\beta_{RE} = \frac{\ln E\left[R\right] - \ln E\left[S\right]}{\sqrt{V_R^2 - 2Cov\left[R;S\right] + V_S^2}} = \frac{\ln \mu_R - \ln \mu_S}{\sqrt{\left(\frac{\sigma_R}{\mu_R}\right)^2 - 2\rho_{RS}\frac{\sigma_R}{\mu_R}\frac{\sigma_S}{\mu_S} + \left(\frac{\sigma_S}{\mu_S}\right)^2}}$$

$\mu_R$ – mean value of the random variable – structural resistance,

$\mu_S$ – mean value of the random variable denoting loads,

$\sigma_R$ and $\sigma_S$ – standard deviations of these variables,

$V_R$ and $V_S$ – coefficients of variation,

$\rho_{RS}$ – correlation coefficient of the variables.

While the basic variables $R$ and $S$ are **uncorrelated**

$$\beta_{RE} = \frac{\ln E[R] - \ln E[S]}{\sqrt{V_R^2 + V_S^2}} = \frac{\ln \mu_R - \ln \mu_S}{\sqrt{(\sigma_R/\mu_R)^2 + (\sigma_S/\mu_S)^2}}$$

The Rosenblueth – Esteva reliability index $\beta_{RE}$ is affected by the selection of the limit state function, to possibly make it **differ from the result of Cornell reliability index $\beta_C$ computational variant.**

Given **different linearization of a nonlinear limit state function**

$$M_{FO} = g(\mathbf{x}) + \sum_{i=1}^{n} \frac{\partial g(\mathbf{x})}{\partial x_i}(X_i - x_i)$$

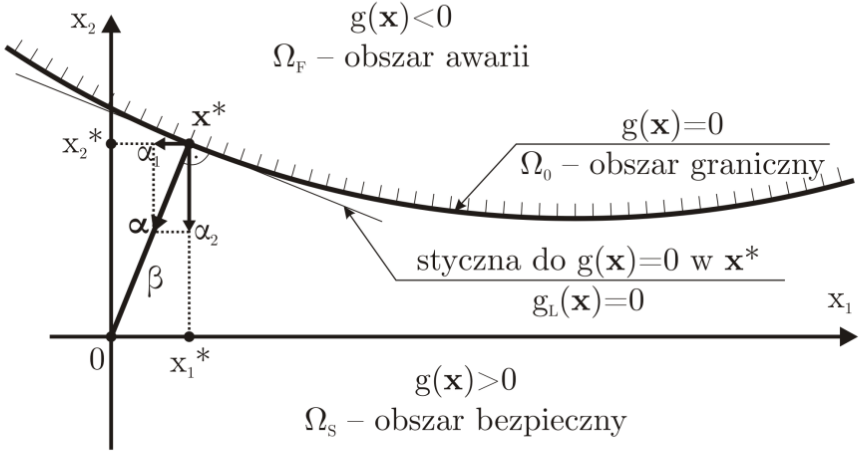
The reliability index here is the so-called **the first-order second-moment reliability index $\beta_{FOSM}$**

$$\beta_{FOSM} = \frac{g(\mathbf{x}) + \sum_{i=1}^{n} \frac{\partial g(\mathbf{x})}{\partial x_i}\left(E[X_i] - x_i\right)}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial g}{\partial x_i}(\mathbf{x}) \frac{\partial g(\mathbf{x})}{\partial x_j}\left(Cov[X_i; X_j]\right)}}$$

## The Hasofer – Lind reliability index

To avoid the non-objective computations of Rosenblueth – Esteva reliability index $\beta_{RE}$ it is suggested to introduce the appropriate **linearization point** of a safety margin variable.

The natural choice is the **closest-distance point from the realization point** of all mean values of the basic variables of the problem (Fig.)



Hasofer and Lind proposed the **non-homogeneous linear mapping** of a set of basic variables $\mathbf{X}$ into a **set of normalized and uncorrelated variables** $\mathbf{Z}$.

This random variable set (random vector) shows a zero eigenvalue vector $(E[\mathbf{Z}]=\mathbf{0})$ and a unit covariance matrix $(\mathbf{C}_{\mathbf{Z}}=Cov[\mathbf{Z};\mathbf{Z}^T]=\mathbf{I})$.

Transformation of the basic random variable set $\mathbf{X}$ into a normalized random variable set $\mathbf{Z}$

$$\mathbf{Z}=\mathbf{A}\left(\mathbf{X}-E\left[\mathbf{X}\right]\right)$$

here $\mathbf{A}$ is a transformation matrix following the standard linear algebra techniques.

The covariance matrix of the basic random variable set $\mathbf{X}$

$$\mathbf{C}_{\mathbf{X}}=Cov\left[\mathbf{X};\mathbf{X}^T\right]=E\left[\left(\mathbf{X}-E\left[\mathbf{X}\right]\right)\left(\mathbf{X}-E\left[\mathbf{X}\right]\right)^T\right]$$

The covariance matrix of a set of normalized random variables

$$\mathbf{C}_{\mathbf{Z}}=Cov\left[\mathbf{Z};\mathbf{Z}^T\right]=\mathbf{A}\mathbf{C}_{\mathbf{X}}\mathbf{A}^T=\mathbf{A}\,Cov\left[\mathbf{X};\mathbf{X}^T\right]\mathbf{A}^T=\mathbf{I}$$

**The limit condition**

$$g(\mathbf{Z}) = g\left(\mathbf{X}(\mathbf{Z})\right) = g\left(\mathbf{A}^{-1}\mathbf{Z} + E[\mathbf{X}]\right) = 0$$

may be presented in the form

$$\mathbf{z} = \mathbf{A}\left(\mathbf{x} - E[\mathbf{X}]\right)$$

**The single realization point,** including mean values in a realization domain $\mathbf{x}$ is mapped into an origin of a system - realization space $\mathbf{z}$.

**The limit state surface** $\Omega_0$ of a realization vector $\mathbf{x}$ is mapped into coordinates of a limit state surface $\Omega_0$ of a realization vector $\mathbf{z}$.

**Geometric distance** from the origin of the realization space $\mathbf{z}$ to an arbitrary limit surface $\Omega_0$ point of a realization vector $\mathbf{z}$ **is the measure of standard deviation** from the realization point $\mathbf{x}$ of mean values to the corresponding limit surface point $\Omega_0$.

The distance from the origin of the realization space (domain) $\mathbf{z}$ – **the initial point of reliability analysis**, to the limit surface may be measured by the so-called **Veneziano function (reliability index function)** (1974) defined in the domain

$$\beta(\mathbf{z}) = \sqrt{\mathbf{z}^T \mathbf{z}} \;\; ; \;\; \mathbf{z} \in \Omega_0^{(\mathbf{z})}$$

or in the realization space

$$\beta(\mathbf{x}) = \sqrt{\left(\mathbf{x} - E[\mathbf{X}]\right)^T \mathbf{C}_{\mathbf{X}}^{-1} \left(\mathbf{x} - E[\mathbf{X}]\right)} \;\; ; \;\; \mathbf{x} \in \Omega_0^{(\mathbf{x})}$$

**The shortest distance** from the initial point of reliability analysis to the chosen limit surface point is **the measure of the Hasofer – Lind reliability index** $\beta_{HL}$

$$\beta_{HL} = \min_{g(\mathbf{z})=0} \beta(\mathbf{z}) = \min_{g(\mathbf{z})=0} \sqrt{\mathbf{z}^T \mathbf{z}} \;\; ; \;\; \mathbf{z} \in \Omega_0^{(\mathbf{z})}$$

or equivalently

$$\beta_{HL} = \min_{g(\mathbf{x})=0} \beta(\mathbf{x}) = \min_{g(\mathbf{x})=0} \sqrt{\left(\mathbf{x} - E[\mathbf{X}]\right)^T \mathbf{C}_{\mathbf{X}}^{-1} \left(\mathbf{x} - E[\mathbf{X}]\right)} \;\; ; \;\; \mathbf{x} \in \Omega_0^{(\mathbf{x})}$$

The realization point $\mathbf{x}$ in the searching process for the reliability index $\beta_{HL}$ is denoted by $\mathbf{x}^*$ named a **design point of reliability analysis.**

**While the limit surface is a** *n*-th dimensional **hyperplane**, numerical values of Corlell $\beta_C$ and Hasofer – Lind $\beta_{HL}$ reliability indices are equal.

**Thus the Hasofer – Lind reliability index generalizes the Cornell reliability index in the case of nonlinear limit surfaces**

**If the design point** of Hasofer – Lind reliability analysis **is a linearization point** of a basic random variable, the Hasofer – Lind coincides with the first-order second-moment reliability index $\beta_{FOSM}$

**Hasofer – Lind reliability index $\beta_{HL}$ is a solution of a non-linear optimization problem of a single objective function**

The solution of the problem may be taken from a multitude of iterative algorithms, **optimization is aimed at finding the relevant design point $\mathbf{x}^*$.**

**Searching algorithm of a design point $\mathbf{x}^*$** in the space $\mathbf{z}$.
The design point is investigated $\mathbf{z}^*$, corresponding to the point $\mathbf{x}^*$ and the unit vector normal to the limit surface $\boldsymbol{\alpha}^*$ at the point $\mathbf{z}^*$.

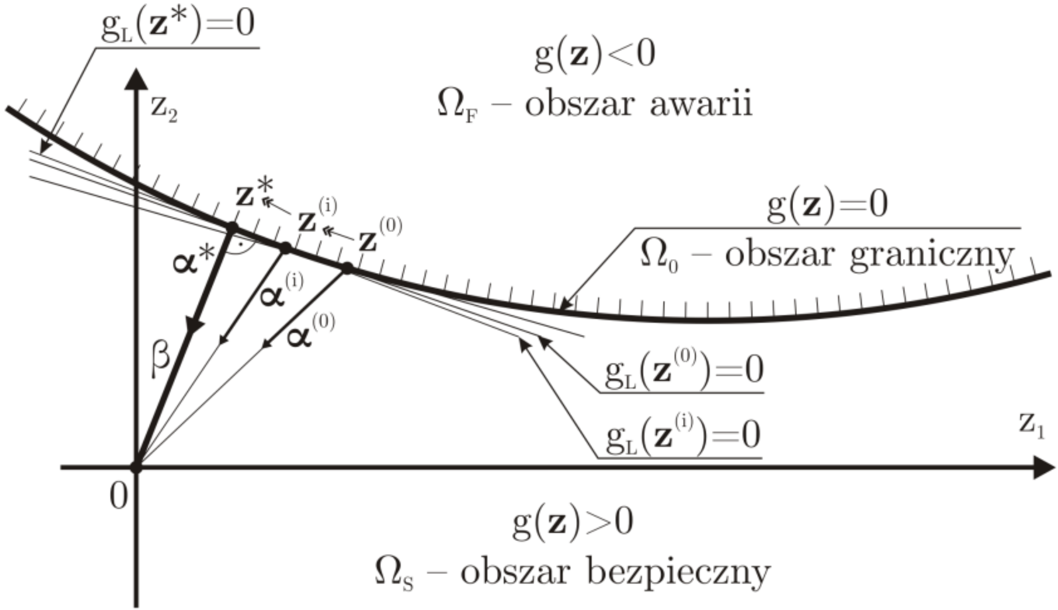
The Hasofer – Lind reliability index $\beta_{HL}$ is a proportional coefficient between the realization vector and the vector normal to the limit surface: $\mathbf{z}^* = \boldsymbol{\alpha}^* \beta_{HL}$
The point $\mathbf{z}^*$ is a limit of the algorithm sequence
$\mathbf{z}^{(0)} \gg \mathbf{z}^{(1)} \gg \ldots \gg \mathbf{z}^{(m)} \gg \ldots \gg \mathbf{z}^*$.

The unit vector $\boldsymbol{\alpha}^{(m)}$ is parallel to the gradient vector of the path defined at a point $\mathbf{z}^{(m)}$, directed onto the failure region $\Omega_F$

$$\boldsymbol{\alpha}^{(m)} = -\frac{\nabla g\left(\mathbf{z}^{(m)}\right)}{\left|\nabla g\left(\mathbf{z}^{(m)}\right)\right|}$$

The parameter $\nabla g(\mathbf{z}^{(m)})$ is the **gradient of the limit function**

$$\nabla g\left(\mathbf{z}^{(m)}\right) = \left(\frac{\partial g}{\partial z_1}\left(\mathbf{z}^{(m)}\right); \frac{\partial g}{\partial z_2}\left(\mathbf{z}^{(m)}\right); \ldots; \frac{\partial g}{\partial z_n}\left(\mathbf{z}^{(m)}\right)\right)$$




**The iteration method is based on linearization.**

The starting point of this algorithm step is $\mathbf{z}^{(m)}$ the surface $z_{n+1} = g(\mathbf{z})$ is replaced by a $n$–dimensional tangent plane at the point $\mathbf{z}^{(m)}$.

The starting point of the algorithm (initial point) may be stated **at the origin of the coordinate system** of the realization space $\mathbf{z}$, or the so-called **supporting point** of the following coordinates

$$\mathbf{x}^{(p)} = \left\{ \frac{1}{\sqrt{n}}; \frac{1}{\sqrt{n}}; \frac{1}{\sqrt{n}};\ldots; \frac{1}{\sqrt{n}} \right\}$$

**The intersection** between this – dimensional plane and the plane $z_{n+1} = 0$

$$g(\mathbf{z}) = g\left(\mathbf{z}^{(m)}\right) + \sum_{i=1}^{n} \frac{\partial g}{\partial z_1}\left(\mathbf{z}^{(m)}\right)\left(z_i - z_i^{(m)}\right) = 0$$

The next point in the algorithm sequence $\mathbf{z}^{(m+1)}$ is the closest one to the initial step point

$$\mathbf{z}^{(m+1)} = \left(\mathbf{z}^{(m)^T} \boldsymbol{\alpha}^{(m)}\right)\boldsymbol{\alpha}^{(m)} + \frac{g\left(\mathbf{z}^{(m)}\right)}{\left|\nabla g\left(\mathbf{z}^{(m)}\right)\right|}\boldsymbol{\alpha}^{(m)}$$

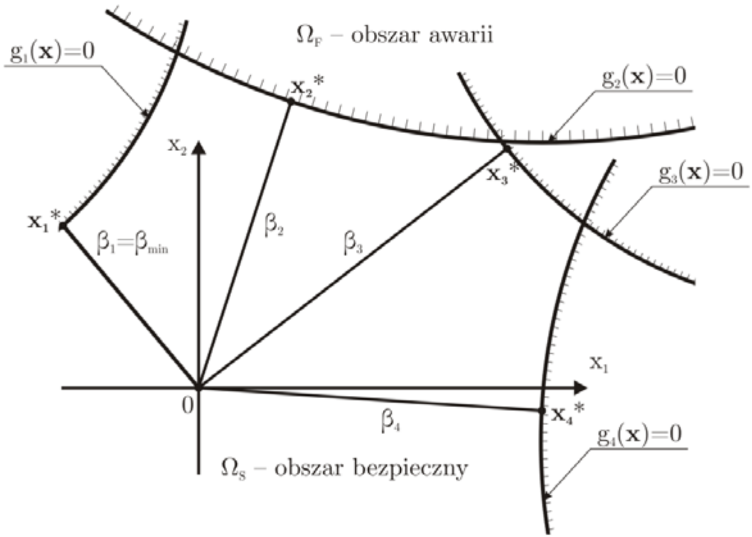
**While the algorithm sequence converges to the point $\mathbf{z}^*$, it refers to the Hasofer – Lind reliability index value $\beta_{HL}$**

**The limit surface may include a number of design points**

In this case it is necessary to find all point and their corresponding, distinct reliability indices

The Hasofer – Lind reliability index $\beta_{HL}$ is therefore defined

$$\beta_{HL} = \min\left(\beta_{HL}{}^{1}; \beta_{HL}{}^{2}; \beta_{HL}{}^{3}; ...; \beta_{HL}{}^{k}\right)$$



It is also possible to formulate the iterative **Hasofer - Lind algorithm in the initial realization space** $\mathbf{x}$:

$$\mathbf{x}^{(m+1)} = E[\mathbf{X}] + \mathbf{C}_{\mathbf{X}} \nabla g\left(\mathbf{x}^{(m)}\right) \frac{\left(\mathbf{x}^{(m)} - E[\mathbf{X}]\right)^T \nabla g\left(\mathbf{x}^{(m)}\right) - g\left(\mathbf{x}^{(m)}\right)}{\nabla g\left(\mathbf{x}^{(m)}\right)^T \mathbf{C}_{\mathbf{X}} \nabla g\left(\mathbf{x}^{(m)}\right)}$$

The definition of Veneziano function (reliability index function) states

$$\frac{\partial \beta(\mathbf{z})}{\partial z_i} = \frac{\partial}{\partial z_i} \sqrt{\sum_{j=1}^{n} x_j^2} = \alpha_i$$

The estimated values $\alpha_i$ at the design point $\mathbf{z}^*$, marked by $\alpha_i^*$ **measure the sensitivity of reliability index** to selected problem parameters (basic variables)

## The Hasofer – Lind – Rackwitz– Fiessler reliability index

In a general case **the basic random variables are not prescribed Gaussian**

A transformation $\mathbf{T}$ may be worked out, transforming an arbitrary random variable into a Gaussian-type variable

$$\mathbf{T}:\mathbf{X}=\left(X_1;X_2;X_3;...;X_n\right)\gg\mathbf{U}=\left(U_1;U_2;U_3;...;U_n\right)$$

here the basic variables $\mathbf{U}=\left(U_1;U_2;U_3;...;U_n\right)$ are uncorrelated, standard-Gaussian distributed

The limit surface in the realization domain $\mathbf{x}$ may be therefore mapped into the limit surface in the realization domain $\mathbf{u}$.

While the random variables are independent their probability density functions (PDFs) may be separated in the form:

$$F_{X_1};F_{X_2};F_{X_3};...;F_{X_n} \quad .$$

In this case each random variable may be transformed separately

$$\Phi(u_i) = F_{X_i}(x_i) \ ; \quad i = 1;\ldots;n$$

The transformation matrix $\mathbf{T}$

$$\mathbf{T} : u_i = \Phi^{-1}\left(F_{X_i}(x_i)\right) \ ; \quad i = 1;\ldots;n$$

Its inverse $\mathbf{T}^{-1}$

$$\mathbf{T}^{-1} : x_i = F_{X_i}^{-1}\left(\Phi(u_i)\right) \ ; \quad i = 1;\ldots;n$$

The limit state function $g_u$ in the realization domain $\mathbf{u}$ referred to the limit state function $g$ in the realization domain $\mathbf{x}$

$$g(\mathbf{x}) = g\left(\mathbf{T}^{-1}(\mathbf{u})\right) = g_u(\mathbf{u})$$

The design point of reliability assessment $\mathbf{u}^*$ in the realization domain $\mathbf{u}$ is a solution of a non-linear optimization problem of a single objective function

$$\mathbf{u}^* = \min_{g_u(\mathbf{u})=0} |\mathbf{u}| \gg \min_{g(\mathbf{x})=0} |T(\mathbf{x})| = \mathbf{x}^*$$

The relation of limit state functions in both realization domains incorporates partial derivatives

$$\frac{\partial g_u}{\partial u_i} = \frac{\partial g_u}{\partial x_i}\frac{\partial x_i}{\partial u_i} = \frac{\partial g_u}{\partial x_i}(\mathbf{x})\frac{\varphi(u_i)}{f_{X_i}(x_i)} = \frac{\partial g_u}{\partial x_i}(\mathbf{x})\frac{\varphi\left(\Phi^{-1}\left(F_{X_i}(x_i)\right)\right)}{f_{X_i}(x_i)}$$

Equation of a $n$-dimensional hyperplane tangent to the limit surface

$$\sum_{i=1}^{n}\frac{\partial g_u}{\partial x_i}(\mathbf{u}^*)(u_i - u_i^*) = 0$$

here $u_i^*$ - coordinates of a point on a limit surface closest to the origin of the realization coordinate system $\mathbf{u}$

The first order approximation of failure probability $P_f$

$$P_f \approx \Phi(-\beta) = \Phi\left(-\left|\mathbf{u}^*\right|\right)$$

## Improving the searching algorithm for the optimal design point of Hasofer-Lind reliability assessment (Rackwitz, Fiessler, 1978)

The design point $\mathbf{u}^*$ is assumed on the limit surface in the realization domain $\mathbf{u}$

The approximation, reversible substitution $\mathbf{u}^{(m+1)}$ to obtain $\mathbf{x}^{(m+1)}$

$$\mathbf{x}^{(m+1)} = \mathbf{x}^{(m)} + \mathbf{J}\left(\mathbf{u}^{(m+1)} - \mathbf{u}^{(m)}\right)$$

here $\mathbf{J}$ is the Jacobian determinant of the transformation defined by a matrix $\mathbf{T}$

Linking the approximation equation of Hasofera and Linda with the approximation linear reverse substitution the **approximation rule of Gaussian variable tails** is created [Ditlevsen, 1981]

$$\mathbf{z}^{(m+1)} = \left(\mathbf{z}^{(m)^T}\boldsymbol{\alpha}^{(m)}\right)\boldsymbol{\alpha}^{(m)} + \frac{g\left(\mathbf{z}^{(m)}\right)}{\left|\nabla g\left(\mathbf{z}^{(m)}\right)\right|}\boldsymbol{\alpha}^{(m)} \gg \mathbf{u}^{(m+1)} = \left(\mathbf{u}^{(m)^T}\boldsymbol{\alpha}^{(m)}\right)\boldsymbol{\alpha}^{(m)} + \frac{g_u\left(\mathbf{u}^{(m)}\right)}{\left|\nabla g_u\left(\mathbf{u}^{(m)}\right)\right|}\boldsymbol{\alpha}^{(m)}$$

The unit vector normal to the path $g_u(\mathbf{u}^{(m)})$

$$\boldsymbol{\alpha}^{(m)} = -\frac{\nabla g_u\left(\mathbf{u}^{(m)}\right)}{\left|\nabla g_u\left(\mathbf{u}^{(m)}\right)\right|}$$

The approximation, reversible substitution $\mathbf{u}^{(m+1)}$ to obtain $\mathbf{x}^{(m+1)}$ may be achieved using partial derivatives of the transformation

$$x_i^{(m+1)} = x_i^{(m)} + \frac{\varphi\left(\Phi^{-1}\left(F_{X_i}\left(x_i^{(m)}\right)\right)\right)}{f_{X_i}\left(x_i^{(m)}\right)}\left(u_i^{(m+1)} - u_i^{(m)}\right)$$

due to the approximation rule of the „tails" of Gaussian PDFs

$$F_{X_i}\left(x_i\right) = \Phi\left(\frac{x_i - \mu_i}{\sigma_i}\right)$$

$$f_{X_i}\left(x_i\right) = \frac{1}{\sigma_i}\varphi\left(\frac{x_i - \mu_i}{\sigma_i}\right)$$

or transforming

$$\sigma_i = \frac{\varphi\left(\Phi^{-1}\left(F_{X_i}\left(x_i\right)\right)\right)}{f_{X_i}\left(x_i\right)}$$

$$\mu_i = x_i - \sigma_i\, \Phi^{-1}\left(F_{X_i}\left(x_i\right)\right)$$

In this case realizations of the random variables of the algorithm may be introduced

$$z_i = \frac{x_i - \mu_i}{\sigma_i}$$

The Hasofer – Lind reliability index $\beta_{HL}$ adopted by Rackwitz and Fiessler is the so-called **Hasofer – Lind – Rackwitz – Fiessler reliability index** $\beta_{HLRF}$.