1.1 Bayes updating (Bayes' theorem)

Nowak, A.S., Collins K.R. Reliability of structures.
McGraw-Hill Higher Education 2000

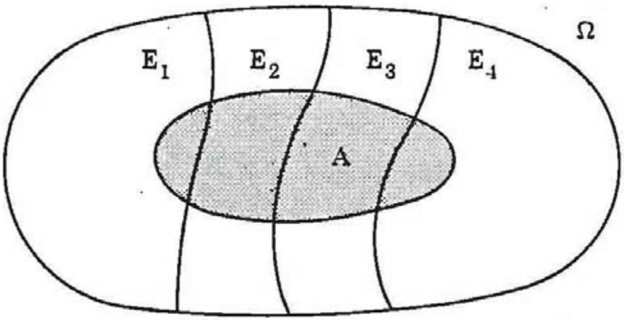
Consider a set of n events $\{A_1, A_2, ..., A_n\}$ of all components A_i mutually exclusive and collectively exhaustive.

These conditions require that

$$A_1 \cup A_2 \cup ... \cup A_n = \Omega \tag{0.1}$$

$$P\left(A_1 \cup A_2 \cup ... \cup A_n\right) = P\left(\Omega\right) = 1 \tag{0.2}$$

where Ω is the sample space.



Consider another event E defined in the sample space Ω.
Obviously, E must "overlap" with one or more of the events A_i,
orif E occurs then one or more of the A_i events must also occur.
The probability of E is determined (Total probability – high schools)

$$P(E) = \sum_{i=1}^{n} P(E|A_i) P(A_i) \tag{0.3}$$

Consider the following problem: what is the probability of occurrence of a particular event A_i: if event E occurs?
According to Bayes' theorem the probability that A_i occurs is

$$P(A_i|E) = \frac{P(E|A_i) P(A_i)}{\sum_{j=1}^{n} P(E|A_j) P(A_j)}$$

Bayes' theorem is useful for judgemental information
and for updating probabilities based on observed outcomes.

Determination of the strength of a structural component.

Define a random variable A - the strength of the component.

For simplicity of explanation, assume that the strength can only assume one ofdiscrete values $\{a_1, a_2, ..., a_n\}$.

The probability of each value a_i is estimated $P(A = a_i) = p_i$.

These probabilitiesare the prior probabilities, they are based on past experience and judgement.

Assume that some field tests are conducted, so we wish to update the probabilities in the light of this additional information.

Define each p_i' as the posterior or updated probability.

Let event E represent a possible test result.

The test result must give one of the possible values $\{a_1', a_2', ..., a_n'\}$ The updated probability can be found using Bayes' theorem

Using the presented notation we get the following equation

$$P\left(A=a_i \middle| E=a'_j\right)=\frac{P\left(E=a'_j \middle| A=a_i\right)P\left(A=a_i\right)}{\sum_{i=1}^{n}P\left(E=a'_j \middle| A=a_i\right)P\left(A=a_i\right)}$$

or

$$p'_i=\frac{P\left(E=a'_j \middle| A=a_i\right)p_i}{\sum_{i=1}^{n}P\left(E=a'_j \middle| A=a_i\right)p_i}$$

The conditional probabilities reflect any uncertainty in the tests themselves.

The probability $P\left(E=a'_j \middle| A=a_i\right)$ is the probability that the test result will indicate a'_j given that the true value is a_i.

In other words it gives some idea of the test confidence.

EXAMPLE

Consider a steel beam where corrosion is observed.
Determine the actual shear strength of the web.
Sufficiently accurate is to assume one of the following five values: R_v, $0.9R_v$, $0.8R_v$, $0.7R_v$, and $0.6R_v$ for the shear strength
Past experience with corrosion estimates probabilities of these values to 0,0.15,0.30,0.40, and 0.15, respectively.
A field test is conducted, leading to the strength equal to $0.8R_v$.
Reliability of the test is reasonably good but not perfect.
Table 2.6 is a conditional probability matrix indicating confidence in the test.

Update the probabilities in light of this new information.

Test information for Example 2.13

	Actual beam strength				
Test values	R_V	$0.9R_V$	$0.8R_V$	$0.7R_V$	$0.6R_V$
$0.6R_V$	0	0	0.05	0.15	0.70
$0.7R_V$	0	0.05	0.20	0.75	0.25
$0.8R_V$	0.1	0.25	0.70	0.10	0.05
$0.9R_V$	0.3	0.65	0.05	0	0
R_V	0.6	0.05	0	0	0
Column sum	1.0	1.0	1.0	1.0	1.0

Solution. The updated probabilities are calculated using Eq. 2.104. First of all we interpret the matrix. Each entry in Table 2.6 is a probability $P\left(E = a'_j \middle| A = a_i\right)$.

Each column is its own sample space in which the actual beam strength is known. Values of each column are probabilities of a mutually exclusive and collectively exhaustive set of experiments for a particular beam strength.

The sum of probabilities in each column must add up to 1.

Observe that the same property does not apply to each row.

The updated posterior probabilities are calculated using Eq. 2.104:

$$p_i' = \frac{P\left(E = a_j' \,\middle|\, A = a_i\right) p_i}{\sum_{i=1}^{n} P\left(E = a_j' \,\middle|\, A = a_i\right) p_i}$$

We need conditional probabilities contained in the highlighted row. Each of the updated probabilities requires the same denominator in the preceding formula, to be calculated first:

$$\sum_{i=1}^{n} P\left(E = 0.8R_v \,\middle|\, A = a_i\right) p_i = (0.10)(0) + (0.25)(0.15) + (0.70)(0.30) +$$

$$+ (0.10)(0.40) + (0.05)(0.15) = 0.295$$

$$P\left(A=0.6R_v \,\middle|\, E=0.8R_v\right)=\frac{P\left(E=0.8R_v \,\middle|\, A=0.6R_v\right)P\left(0.6R_v\right)}{\sum_{i=1}^{n}P\left(E=a_j' \,\middle|\, A=a_i\right)p_i}=$$

$$=\frac{(0.05)(0.15)}{0.295}=0.025$$

$$P\left(A=0.8R_v \,\middle|\, E=0.8R_v\right)=\frac{P\left(E=0.8R_v \,\middle|\, A=0.8R_v\right)P\left(0.8R_v\right)}{\sum_{i=1}^{n}P\left(E=a_j' \,\middle|\, A=a_i\right)p_i}=$$

$$=\frac{(0.10)(0.40)}{0.295}=0.136$$

$$P\left(A=0.9R_v \middle| E=0.8R_v\right)=\frac{P\left(E=0.8R_v \middle| A=0.9R_v\right)P\left(0.9R_v\right)}{\sum_{i=1}^{n} P\left(E=a'_j \middle| A=a_i\right)p_i}=$$

$$=\frac{(0.25)(0.15)}{0.295}=0.127$$

$$P\left(A=R_v \middle| E=0.8R_v\right)=\frac{P\left(E=0.8R_v \middle| A=R_v\right)P\left(R_v\right)}{\sum_{i=1}^{n} P\left(E=a'_j \middle| A=a_i\right)p_i}=$$

$$=\frac{(0.10)(0)}{0.295}=0.0$$

Comparing, the prior and posterior probabilities are listed in the Table:

Comparison of prior and posterior probabilities

	Posterior probability	Prior probability
$P(0.6R_V)$	0.025	0.15
$P(0.7R_V)$	0.136	0.40
$P(0.8R_V)$	0.712	0.30
$P(0.9R_V)$	0.127	0.15
$P(R_V)$	0	0
Sum	1	1