2.6 Univariate random variables

(*zmienne losowe jednowymiarowe)*

Nowak, A.S., Collins K.R. Reliability of structures.
McGraw-Hill Higher Education 2000

Any random variable is defined by its cumulative distribution function (CDF), $F_X(x)$.

The probability density function $f_X(x)$ of a continuous random variable is the first derivative of $F_X(x)$.

The most important continuous random variables used in structural reliability analysis are as follows: uniform, normal (Gaussian), lognormal, gamma, extreme type I (Gumbel), extreme type II (Frechet), extreme type III (Weibull). The binomial and Poisson distributions of discrete random variables are distinguished too.

Uniform distribution

A *uniform random var*iable or *uniform distribution (rozkładrówno-mierny)* denotes a constant PDF function in the interval [a, b]. Thus all numbers in this interval are equally likely to appear.

Mathematically the uniform PDF function is defined as follows:

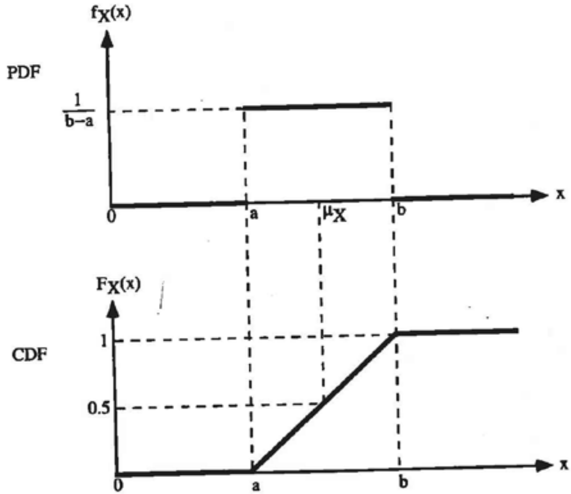
$$f_X(x) = \begin{cases} 0 & x < a \\ \dfrac{1}{b-a} & a \le x \le b \\ 0 & x > b \end{cases}$$

a and b define the lower and upper bounds of the random variable.

The cumulative distribution function (CDF) for a uniform random variable is

$$F_X(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a < x \le b \\ 1 & x > b \end{cases}$$

The PDF and CDF for a uniform random variable are shown in Figure 2.9.



PDF and CDF of a uniform random variable

The mean and variance are as follows:

$$\mu_X = \frac{a+b}{2} \qquad \sigma_X^2 = \frac{(b-a)^2}{12}$$

$$\sigma_X^2 = E\left(X^2\right) - \left[E\left(X\right)\right]^2$$

$$E\left(X\right) = \int_{-\infty}^{\infty} xf\left(x\right)dx = \int_a^b x\frac{1}{b-a}dx = \frac{a+b}{2}$$

$$E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 f\left(x\right)dx = \int_a^b x^2\frac{1}{b-a}dx = \frac{b^3-a^3}{3\left(b-a\right)} = \frac{b^2+ab+a^2}{3}$$

$$\sigma_X^2 = E\left(X^2\right) - \left(E\left(X\right)\right)^2 = \frac{b^2+ab+a^2}{3} - \frac{\left(b+a\right)^2}{4} = \frac{\left(b-a\right)^2}{12}$$

Normal (Gaussian) distribution

The most important distribution is the *normal distribution (rozkład normalny)* also called the *Gaussian distribution (rozkład Gaussa).* It is a two-parameter distribution defined by the density function *(funkcja gęstości prawdopodobieństwa)*

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right]$$

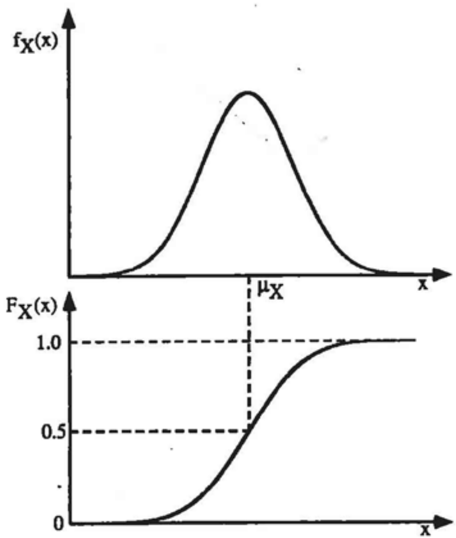
where μ and σ are equal to μ_X *(expected value, wartość oczekiwana)* and σ_X (*standard deviation, odchylenie standardowe*), respectively.This distribution will be denoted $N(\mu,\sigma)$.

The distribution function *(dystrybuanta)*, from (2.45) is given by

$$F_X(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2 \right] dt$$

This integral cannot be evaluated on a closed form.

The figure shows general shapes of both the PDF and CDF of a normal random variable.



Standard normal (Gaussian) variable

General remarks: let X be an arbitrary random variable.
The standard form of X, denoted by Z, is defined

$$Z = \frac{X - \mu_X}{\sigma_X}$$

The mathematical expectation (mean value) of an arbitrary function, $g(X)$, of the random variable X is defined

$$\mu_{g(X)} = E\left[g\left(X\right)\right] = \int_{-\infty}^{\infty} g\left(x\right) f_X\left(x\right) dx$$

The formula above, with $Z = g(X)$ and the variance property prove

$$\mu_Z = E\left[\frac{X - \mu_X}{\sigma_X}\right] = \frac{1}{\sigma_X}\left[E\left(X\right) - E\left(\mu_X\right)\right] = \frac{1}{\sigma_X}\left(\mu_X - \mu_X\right) = 0$$

$$\sigma_Z^2 = E\left(Z^2\right) - \mu_Z^2 = E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)^2\right] - 0 = \frac{1}{\sigma_X^2}\left[E\left(X - \mu_X\right)^2\right] = \frac{\sigma_X^2}{\sigma_X^2} = 1$$

Thus mean of any standard random variable is 0, its variance is 1.
Standard random variable – a “zero mean, unit variance” form.

The distribution function *(dystrybuanta)* of a normal variable

$$F_X(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right] dt$$

Substituting $s = \dfrac{t-\mu}{\sigma}, \quad dt = \sigma ds$ the equation (2.46) becomes

$$F_X(x) = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}s^2\right] ds = \Phi_X\left(\frac{x-\mu}{\sigma}\right)$$

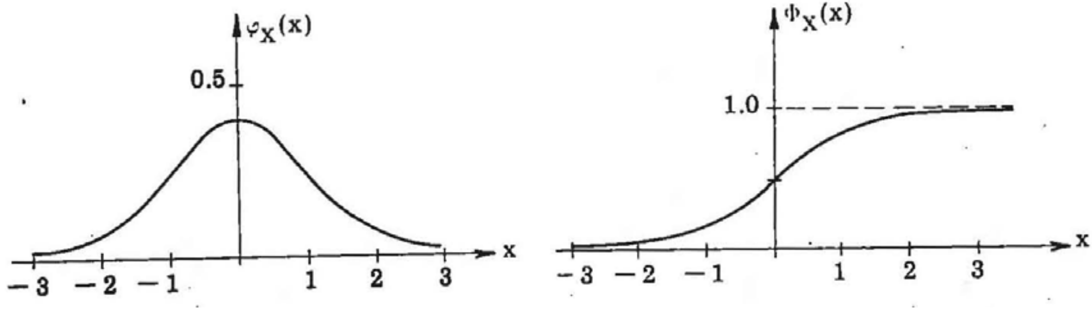
where Φ_X is a *standard normal distribution function* defined by

$$\Phi_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right] dt$$

The corresponding *standard normal density function (standardowa funkcja gęstości prawdopodobieństwa)* is

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right]$$

The functions φ_X and Φ_X are shown in figure 2.8.



Relation (2.48) proves that only a standard normal table is required. Spreadsheet packages include a standard normalCDF function.

Values of $\Phi(z)$are listed in Table1 forz ranging from 0 to –8.9 (part of the table shown, after Nowak, Collins: Reliability of structures)

Table 1. The CDF of the standard normal variable $\Phi(z)$.

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	5.00E-01	4.96E-01	4.92E-01	4.88E-01	4.84E-01	4.80E-01	4.76E-01	4.72E-01	4.68E-01	4.64E-01
−0.1	4.60E-01	4.56E-01	4.52E-01	4.48E-01	4.44E-01	4.40E-01	4.36E-01	4.33E-01	4.29E-01	4.25E-01
−0.2	4.21E-01	4.17E-01	4.13E-01	4.09E-01	4.05E-01	4.01E-01	3.97E-01	3.94E-01	3.90E-01	3.86E-01
−0.3	3.82E-01	3.78E-01	3.74E-01	3.71E-01	3.67E-01	3.63E-01	3.59E-01	3.56E-01	3.52E-01	3.48E-01
−0.4	3.45E-01	3.41E-01	3.37E-01	3.34E-01	3.30E-01	3.26E-01	3.23E-01	3.19E-01	3.16E-01	3.12E-01
−0.5	3.09E-01	3.05E-01	3.02E-01	2.98E-01	2.95E-01	2.91E-01	2.88E-01	2.84E-01	2.81E-01	2.78E-01
−0.6	2.74E-01	2.71E-01	2.68E-01	2.64E-01	2.61E-01	2.58E-01	2.55E-01	2.51E-01	2.48E-01	2.45E-01
−0.7	2.42E-01	2.39E-01	2.36E-01	2.33E-01	2.30E-01	2.27E-01	2.24E-01	2.21E-01	2.18E-01	2.15E-01
−0.8	2.12E-01	2.09E-01	2.06E-01	2.03E-01	2.00E-01	1.98E-01	1.95E-01	1.92E-01	1.89E-01	1.87E-01
−0.9	1.84E-01	1.81E-01	1.79E-01	1.76E-01	1.74E-01	1.71E-01	1.69E-01	1.66E-01	1.64E-01	1.61E-01
−1	1.59E-01	1.56E-01	1.54E-01	1.52E-01	1.49E-01	1.47E-01	1.45E-01	1.42E-01	1.40E-01	1.38E-01
−1.1	1.36E-01	1.33E-01	1.31E-01	1.29E-01	1.27E-01	1.25E-01	1.23E-01	1.21E-01	1.19E-01	1.17E-01
−1.2	1.15E-01	1.13E-01	1.11E-01	1.09E-01	1.07E-01	1.06E-01	1.04E-01	1.02E-01	1.00E-01	9.85E-02
−1.3	9.68E-02	9.51E-02	9.34E-02	9.18E-02	9.01E-02	8.85E-02	8.69E-02	8.53E-02	8.38E-02	8.23E-02
−1.4	8.08E-02	7.93E-02	7.78E-02	7.64E-02	7.49E-02	7.35E-02	7.21E-02	7.08E-02	6.94E-02	6.81E-02
−1.5	6.68E-02	6.55E-02	6.43E-02	6.30E-02	6.18E-02	6.06E-02	5.94E-02	5.82E-02	5.71E-02	5.59E-02
−1.6	5.48E-02	5.37E-02	5.26E-02	5.16E-02	5.05E-02	4.95E-02	4.85E-02	4.75E-02	4.65E-02	4.55E-02
−1.7	4.46E-02	4.36E-02	4.27E-02	4.18E-02	4.09E-02	4.01E-02	3.92E-02	3.84E-02	3.75E-02	3.67E-02
−1.8	3.59E-02	3.51E-02	3.44E-02	3.36E-02	3.29E-02	3.22E-02	3.14E-02	3.07E-02	3.01E-02	2.94E-02
−1.9	2.87E-02	2.81E-02	2.74E-02	2.68E-02	2.62E-02	2.56E-02	2.50E-02	2.44E-02	2.39E-02	2.33E-02
−2	2.28E-02	2.22E-02	2.17E-02	2.12E-02	2.07E-02	2.02E-02	1.97E-02	1.92E-02	1.88E-02	1.83E-02
−2.1	1.79E-02	1.74E-02	1.70E-02	1.66E-02	1.62E-02	1.58E-02	1.54E-02	1.50E-02	1.46E-02	1.43E-02
−2.2	1.39E-02	1.36E-02	1.32E-02	1.29E-02	1.25E-02	1.22E-02	1.19E-02	1.16E-02	1.13E-02	1.10E-02
−2.3	1.07E-02	1.04E-02	1.02E-02	9.90E-03	9.64E-03	9.39E-03	9.14E-03	8.89E-03	8.66E-03	8.42E-03
−2.4	8.20E-03	7.98E-03	7.76E-03	7.55E-03	7.34E-03	7.14E-03	6.95E-03	6.76E-03	6.57E-03	6.39E-03
−2.5	6.21E-03	6.04E-03	5.87E-03	5.70E-03	5.54E-03	5.39E-03	5.23E-03	5.08E-03	4.94E-03	4.80E-03
−2.6	4.66E-03	4.53E-03	4.40E-03	4.27E-03	4.15E-03	4.02E-03	3.91E-03	3.79E-03	3.68E-03	3.57E-03
−2.7	3.47E-03	3.36E-03	3.26E-03	3.17E-03	3.07E-03	2.98E-03	2.89E-03	2.80E-03	2.72E-03	2.64E-03
−2.8	2.56E-03	2.48E-03	2.40E-03	2.33E-03	2.26E-03	2.19E-03	2.12E-03	2.05E-03	1.99E-03	1.93E-03
−2.9	1.87E-03	1.81E-03	1.75E-03	1.69E-03	1.64E-03	1.59E-03	1.54E-03	1.49E-03	1.44E-03	1.39E-03

Values $\Phi(z)$ for $z > 0$ can also be obtained from Table 1 by applying the symmetry property of the normal distribution:

$\Phi(z) = 1 - \Phi(-z)$

The probability information for the standard normal random variable allows for the CDF and PDF values for any normal random variable by a simple coordinate transformation.

Let *X* be anynormal random variable and *Z* be a standard form of *X*. Rearranging Eq. 2.27 we can show that $X = \mu_X + Z\sigma_X$.

The definition of CDF implies

$$F_X(x) = P(X \le x) = P(\mu_X + Z\sigma_X \le x) = P\left(Z \le \frac{x - \mu_X}{\sigma_X}\right)$$

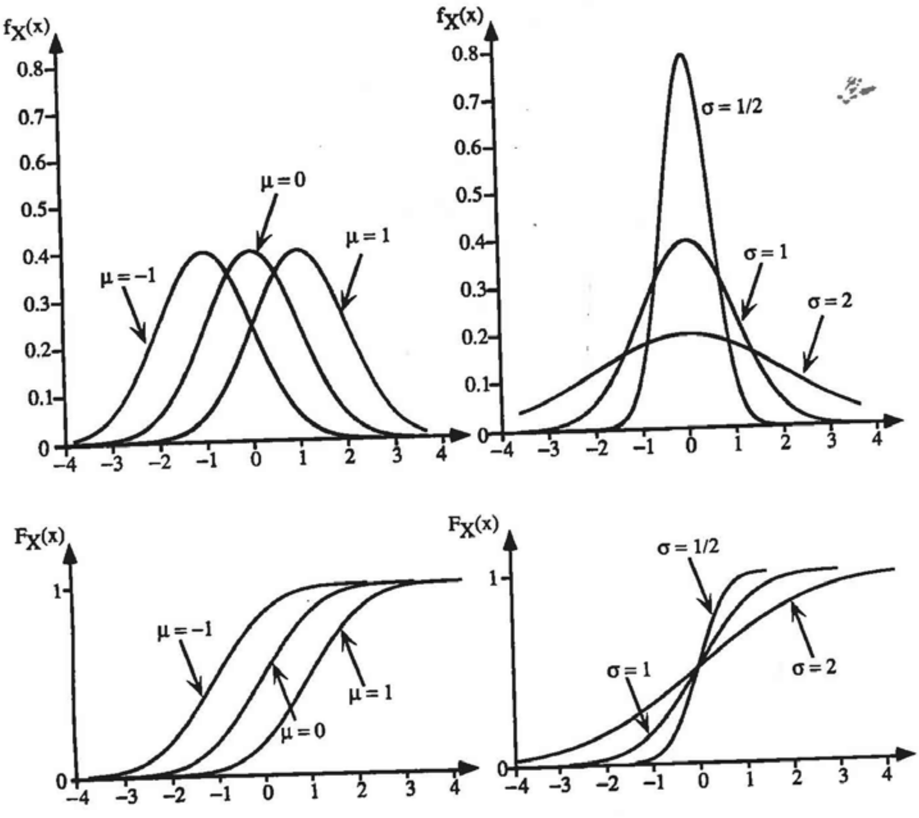
$$F_X(x) = \Phi\left(\frac{x - \mu_X}{\sigma_X}\right) = F_Z(z)$$

The relationship between the PDF of any normal random variable, $f_X(x)$, with the PDF of the standard normal variable, $\phi(x)$:

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \Phi\left(\frac{x - \mu_X}{\sigma_X} \right) = \frac{1}{\sigma_X} \phi\left(\frac{x - \mu_X}{\sigma_X} \right)$$

Using above eqs. the distribution functions for an arbitrary normal random variable (given μ_X and σ_X) may be derived, using information in Table 1.

CDFs and PDFs for normal random variables are shown in Fig

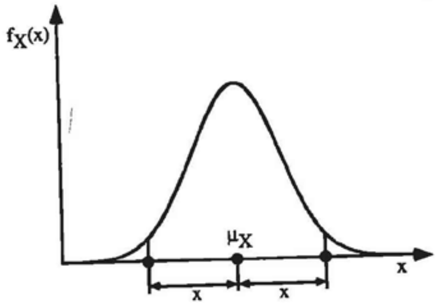


Important propertiesof the PDF and CDF functions for normal random variables are summarized as follows:

1. The PDF $f_X(x)$ is symmetric about the mean μ_X

$$f_X(\mu_X + x) = f_X(\mu_X - x)$$

The illustration isshown in Figure 2.14.



2. The symmetry property of 1. yields $F_X(\mu_X + x) + F_X(\mu_X - x) = 1$

It is a generalized form of Eq. 2.36 - a property expressed for $\Phi(z)$

EXAMPLE 2.3.

(a) Assume Z a standard normal random variable and $z = -2.16$, what are the PDF and CDF values?

Solution. From Table 1, $\Phi(z = -2.16) = 0.0154$.

−2	2.28E-02	2.22E-02	2.17E-02	2.12E-02	2.07E-02	2.02E-02	1.97E-02	1.92E-02	1.88E-02	1.83E-02
−2.1	1.79E-02	1.74E-02	1.70E-02	1.66E-02	1.62E-02	1.58E-02	1.54E-02	1.50E-02	1.46E-02	1.43E-02
−2.2	1.39E-02	1.36E-02	1.32E-02	1.29E-02	1.25E-02	1.22E-02	1.19E-02	1.16E-02	1.13E-02	1.10E-02

From Eq. 2.35

$$\phi(z = -2.16) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(-2.16)^2\right] = f_Z(z) = 0.0387$$

(b) If $z = +1.51$ what is $\Phi(1.51)$?

From Table 1, $\Phi(z = -1.51) = 0.0655$.

−1.4	8.08E-02	7.93E-02	7.78E-02	7.64E-02	7.49E-02	7.35E-02	7.21E-02	7.08E-02	6.94E-02	6.81E-02
−1.5	6.68E-02	6.55E-02	6.43E-02	6.30E-02	6.18E-02	6.06E-02	5.94E-02	5.82E-02	5.71E-02	5.59E-02
−1.6	5.48E-02	5.37E-02	5.26E-02	5.16E-02	5.05E-02	4.95E-02	4.85E-02	4.75E-02	4.65E-02	4.55E-02

Using Eq. 2.36 $\Phi(z) = 1 - \Phi(-z)$

$$\Phi(1.51) = 1 - 0.0655 = 0.9345$$

(c) Given $\Phi(z) = 0.80 \times 10^{-4}$, what is the corresponding value of z?

From Table 1, we note that this $\Phi(z)$ value is not tabulated.

−3.7	1.08E-04	1.04E-04	9.96E-05	9.57E-05	9.20E-05	8.84E-05	8.50E-05	8.16E-05	7.84E-05	7.53E-05
−3.8	7.23E-05	6.95E-05	6.67E-05	6.41E-05	6.15E-05	5.91E-05	5.67E-05	5.44E-05	[illegible]	[illegible]
−3.9	4.81E-05	4.61E-05	4.43E-05	4.25E-05	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

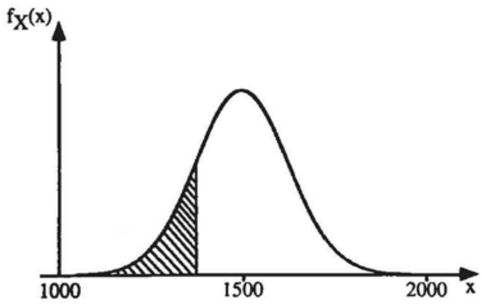
Linear interpolation will be applied for the problem.

Take two values: $z = -3.77$ gives $\Phi = 0.816 \times 10 - 4$

and $z = -3.78$ produces $\Phi = 0.784 \times 10 - 4$.

Interpolation leads to the approximate value of $z = -3.775$.

EXAMPLE 2.4. Assume X a normal random variable of $\mu_X = 1500$ and $\sigma_X = 200$. The PDF for the variable is shown in Fig. 2.15.



(a) Compute $F_X(1300)$

From Eq. 2.39

$$F_X(x) = \Phi\left(\frac{x-\mu_X}{\sigma_X}\right) \Rightarrow F_X(1300) = \Phi\left(\frac{1300-1500}{200}\right) = \Phi(-1)$$

From Table 1 we get $\Phi(-1) = 0.159$.

(b) Calculate $F_X\left(1900\right)$

$$F_X\left(x\right)=\Phi\left(\frac{x-\mu_X}{\sigma_X}\right)\Rightarrow F_X\left(1900\right)=\Phi\left(\frac{1900-1500}{200}\right)=\Phi\left(2\right)$$

From Eq. 2.36 $\Phi\left(2\right)=1-\Phi\left(-2\right)$

From Table 1 $\Phi\left(-2\right)=0.228\times10^{-1}$

Therefore, $\Phi\left(2\right)=1-\left(0.228\times10^{-1}\right)=0.977$

(c) Calculate $F_X\left(1700\right)$

Observe that x = 1700 is 200 units away from the mean value of 1500.

UsingEq. 2.42 we get

$$F_X\left(1500+200\right)=1-F_X\left(1500-200\right)=1-F\left(1300\right)=1-0.159=0.841$$

(d) Calculate f_X for $x = 1300$

From Eq. 2.40,

$$f_X(x) = \frac{1}{\sigma_X}\phi\left(\frac{x-\mu_X}{\sigma_X}\right) \Rightarrow f_X(x) = \frac{1}{200}\phi\left(\frac{1300-1500}{200}\right) = \frac{1}{200}\phi(-1)$$

Using Eq. 2.35

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(z)^2\right] \Rightarrow \phi(-1) = 0.242$$

Therefore, $f_X(1300) = 0.00121$

(e) Calculate $f_X(1500)$.

Using Eq. 2.40,

$$f_X(x) = \frac{1}{\sigma_X}\phi\left(\frac{x-\mu_X}{\sigma_X}\right) \Rightarrow f_X(x) = \frac{1}{200}\phi\left(\frac{1500-1500}{200}\right) = \frac{1}{200}\phi(0)$$

$$\phi(0) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(0)^2\right] = 0.399$$

$$f_X(1500) = 0.00199$$

The inverse of the CDF of the standard normal distribution function may be used. Not existing in closed form, it has an approximate formula useful for a wide range of probability values.

Let $p = \Phi(z)$. The inverse problem would be to find $z = \Phi^{-1}(p)$.

The following formula can be used if p is less than or equal to 0.5:

$$z = \Phi^{-1}(p) = -t + \frac{c_0 + c_1 t + c_2 t^2}{1 + d_1 t + d_2 t^2 + d_3 t^3} \qquad \text{for} \qquad p \le 0.5$$

where $c_0 = 2.515517$, $c_1 = 0.802853$, $c_2 = 0.0010328$,

$d_1 = 1.432788$, $d_2 = 0.189269$, $d_3 = 0.001308$ and $t = \sqrt{-\ln\left(p^2\right)}$.

For $p > 0.5$, Φ^{-1} is obtained for $p^* = (1 - p)$, then the following relationship is used:

$$z = \Phi^{-1}(p) = -\Phi^{-1}\left(p^*\right)$$

Logarithmic normal distribution

Let the random variable $Y = \ln X$ be normally distributed $N(\mu_Y, \sigma_Y)$.The random variable X follows the *logarithmic normal distribution (rozkład lognormalny)*, with parameters $\mu_Y \in R$, $\sigma_Y > 0$

Thelog-normal density function is stated, for $x > 0$

$$f_X(x) = \frac{1}{\sigma_Y \sqrt{2\pi}} \frac{1}{x} \exp\left[-\frac{1}{2} \left(\frac{\ln x - \mu_Y}{\sigma_Y} \right)^2 \right]$$

Let X be log-normally distributed with the parameters μ_Y and σ_Y.

Note that μ_Y and σ_Y are not equal to μ_X and σ_X.

It can be shown that

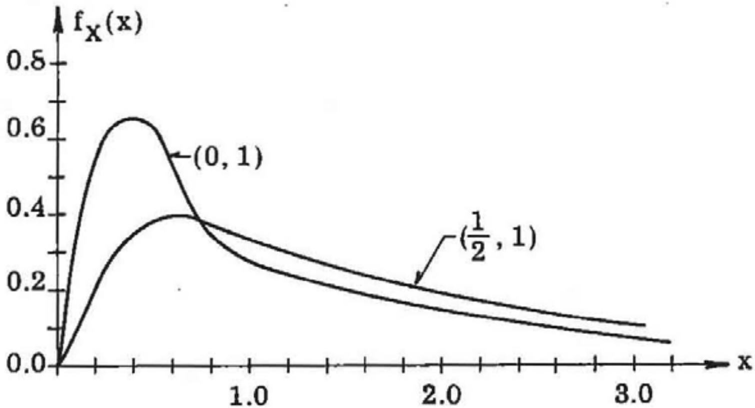
$$\mu_X(x) = \exp\left(\mu_Y + \frac{1}{2} \sigma_Y^2 \right)$$

$$\sigma_X = \sqrt{\mu_X^2 \left(e^{\sigma_Y^2} - 1 \right)}$$

$$F_X(x) = P(X \le x) = \Phi\left(\frac{\ln x - \mu_Y}{\sigma_Y}\right)$$

The log-normal density functions with the parameters $(\mu_Y, \sigma_Y) = (0,1)$ and (1/2, 1) are presented in Fig. 2.9.

The lognormal distribution is widely used in structural reliability analysis.



Example 2.12.

Let the compressive strength X for concrete be log-normally distributed with the parameters $(\mu_Y, \sigma_Y) = (3 \text{ MPa}, 0.2 \text{ MPa})$.

Then

$$\mu_X = \exp\left(3 + \frac{1}{2} \cdot 0.04\right) = 20.49 \text{ MPa}$$

$$\sigma_X^2 = 20.49^2 (1.0408 - 1) = 17.14 \text{ (MPa)}^2$$

$$\sigma_X = 4.14 \text{ MPa}$$

and

$$P(X \le 10 \text{ MPa}) = \Phi((\ln 10 - 3)/2) = \Phi(-3.487) = 2.4 \cdot 10^{-4}$$

The PDF and CDF may be calculated using functions $\phi(z)$ and $\Phi(z)$ for a standard normal random variable Z as follows:

$$F_X(x) = P(X \le x) = P(\ln X \le \ln x) = P(Y \le y) = F_Y(y)$$

Since Y is normally distributed standard normal functions apply.

Specifically

$$F_X(x) = F_Y(y) = \Phi\left(\frac{y-\mu_Y}{\sigma_Y}\right)$$

where $y = \ln(x)$, $\mu_Y = \mu_{\ln(X)}$ = mean value of $\ln(X)$,

and $\sigma_Y = \sigma_{\ln(X)}$ = standard deviation of $\ln(X)$.

These parameters are functions of μ_X , σ_X and V_X by the following formulas:

$$\sigma^2_{\ln(X)} = \ln\left(V_X^2 + 1\right), \qquad \mu_{\ln(X)} = \ln(\mu_X) - \frac{1}{2}\sigma^2_{\ln(X)}$$

If V_X is less than 0.2, the following approximations are valid:

$$\sigma^2_{\ln(X)} \approx V_X^2, \qquad \mu_{\ln(X)} \approx \ln(\mu_X)$$

For the PDF function Eq. 2.12 gives

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}\Phi\left(\frac{\ln(x)-\mu_{\ln(X)}}{\sigma_{\ln(X)}}\right) = \frac{1}{x\sigma_X}\phi\left(\frac{\ln(x)-\mu_{\ln(X)}}{\sigma_{\ln(X)}}\right)$$

EXAMPLE 1.5. Let X be a lognormal random variable whose mean value is 250, standard deviation is 30. Find $F_X(200)$ and $f_X(200)$.

$$V_X = \frac{\sigma_X}{\mu_X} = \frac{30}{250} = 0.12$$

$$\sigma^2_{\ln(X)} = \ln\left(V_X^2 + 1\right) = 0.0143, \qquad \sigma_{\ln(X)} = 0.01196$$

$$\mu_{\ln(X)} = \ln(\mu_X) - \frac{1}{2}\sigma^2_{\ln(X)} = \ln(250) - 0.5(0.0143) = 5.51$$

$$F_X(200) = \Phi\left(\frac{\ln(x) - \mu_{\ln(X)}}{\sigma_{\ln(X)}}\right) = \Phi\left(\frac{\ln(200) - 5.51}{0.1196}\right) =$$

$$= \Phi(-1.77) = 0.0384$$

$$f_X(200) = \frac{1}{x\sigma_{\ln(X)}}\phi\left(\frac{\ln(x) - \mu_{\ln(X)}}{\sigma_{\ln(X)}}\right) = \frac{1}{200(0.1196)}\Phi(-1.77) =$$

$$= \frac{0.0833}{23.92} = 0.00384$$