Distributions – Extreme

Nowak, A.S., Collins K.R. Reliability of structures.
McGraw-Hill Higher Education 2000

Extreme type I (Gumbel distribution, Fisher-Tippett type I)

Extreme value distributions depict well probabilistic nature of the extreme values (largest or smallest) of some phenomenon over time. Consider *n* time intervals, e.g. years.There is a maximum value of some phenomenon (e.g. wind speed) during each interval (year). Determine the random model for those largest annual wind speeds.

Let $W_1, \ldots, W_n$ be the largest wind speeds in *n* years. Then

$X = \max\left(W_1, W_2, \ldots, W_n\right)$ is an extreme Type I random variable.

The CDF and PDF for this random variable are

$$F_X\left(x\right) = e^{-e^{-\alpha(x-u)}} \quad \text{for} \quad -\infty \le x \le \infty$$

$$f_X\left(x\right) = \alpha e^{-e^{-\alpha(x-u)}} e^{-\alpha(x-u)}$$

where *u* and a are distribution parameters. The basic shape of the Type I PDF function is shown in Fig. 2.18.

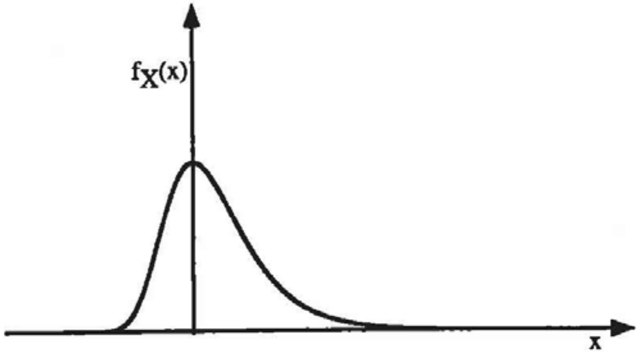


FIGURE 2.18 PDF of an extreme Type I random variable.

The mean and standard deviation for this variable may be obtained by the following approximations (Benjamin and Cornell, 1970):

$$\mu_X \approx u + \frac{0.577}{\alpha}, \qquad \sigma_X \approx \frac{1.282}{\alpha}$$

Thus if the mean and standard deviation are knownEqs. 2.60 and 2.61 can be rearranged and solved for the corresponding values of the distribution parameters as follows:

$$\alpha \approx \frac{1.282}{\sigma_X}, \qquad u \approx \mu_X - 0.45\sigma_X$$

Extreme type II(Frechet distribution, Fisher-Tippett type II)

Extreme Type II variable may model the maximum seismic load applied to a structure. The CDF and PDF are

$$F_X(x) = e^{-e^{-(u/x)^k}} \quad \text{for} \quad 0 \le x \le \infty, \qquad f_X(x) = \frac{k}{u}\left(\frac{u}{x}\right)^{k+1} e^{-(u/x)^k}$$

where u and k are distribution parameters.
The PDF for an extreme Type II variable has the general shape shown in Figure 2.19.

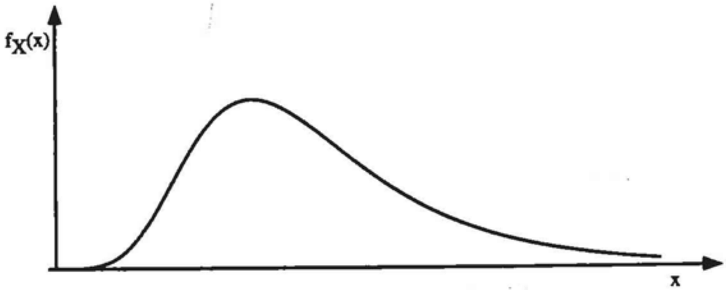


FIGURE 2.19 PDF for an extreme Type II random variable

The mean and standard deviation is given:

$$\mu_X = u\Gamma\left(1-\frac{1}{k}\right) \quad \text{for} \quad k>1$$

$$\sigma_X^2 = u^2\left[\Gamma\left(1-\frac{2}{k}\right)-\Gamma^2\left(1-\frac{1}{k}\right)\right] \quad \text{for} \quad k>2$$

The coefficient of variation, V_X is a function of *k* only. Graphs exist to calculate V_X for any *k* (see, for example, Ang and Tang, 1984)

Extreme Type III (Weibull Distribution)
The extreme Type III distribution is defined by three parameters.
Two variants, for the largest and the smallest values exist.
The CDF of the largest values is defined by

$$F_X(x) = e^{-\left(\frac{w-x}{w-u}\right)^k} \quad \text{for} \quad x \le w$$

where *w*, *u*, and *k* are parameters.
The mean and varianceare

$$\mu_X = w - (w-u)\Gamma\left(1+\frac{1}{k}\right)$$

$$\sigma_X^2 = (w-u)^2\left[\Gamma\left(1-\frac{2}{k}\right) - \Gamma^2\left(1-\frac{1}{k}\right)\right]$$

The CDF of the smallest values is defined by

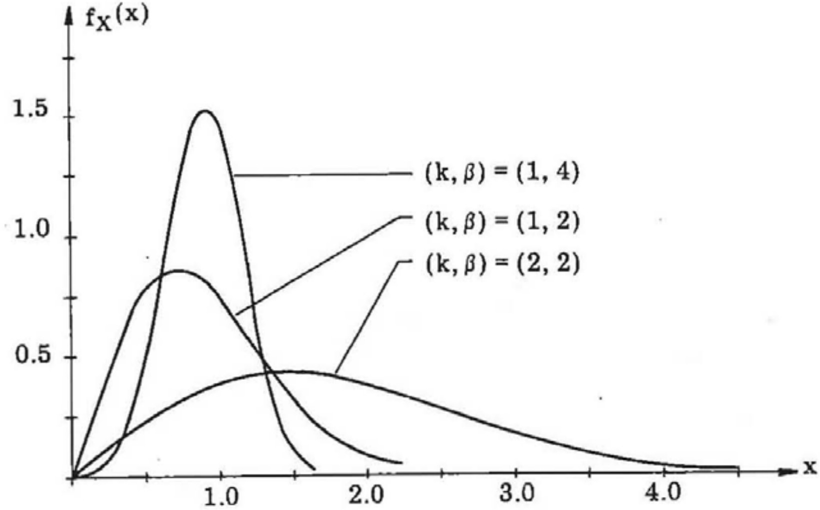
$$F_X(x) = 1 - e^{-\left(\frac{x-\varepsilon}{u-\varepsilon}\right)^k} \quad \text{for} \quad x \geq \varepsilon$$

where u, ε, and k are the parameters.

The mean and variance of the smallest values may be calculated by the following formulas:

$$\mu_X = \varepsilon - (u-\varepsilon)\Gamma\left(1+\frac{1}{k}\right)$$

$$\sigma_X^2 = (u-\varepsilon)^2\left[\Gamma\left(1-\frac{2}{k}\right) - \Gamma^2\left(1-\frac{1}{k}\right)\right]$$



$f_X(x)$
1.5
1.0
0.5
(k, β) = (1, 4)
(k, β) = (1, 2)
(k, β) = (2, 2)
1.0
2.0
3.0
4.0
x

Poisson distribution

The Poisson variable ofa discrete probability distribution may be used todetermine the PMF *(probability mass function)*for the number of occurrences of an event in a time orspace interval (0, *t*). Examples:the number of earthquakeswithin a certain time interval or the numberof defects in a certain length of rod.

The following important assumptions behind the Poisson distributionmust be checked prior to its use:

• event occurrences are independent, i.e. occurrence or nonoccurrence of an event in a prior time interval has no effect onthe occurrence of this event in the time interval considered,

• Two or more events cannot occur simultaneously.

Let a discrete random variable *N* represents the number of event occurrenceswithin a prescribed time (or space) interval (0, *t*).

Let *v* represent themean occurrence rate of the event.

This parameter is usually obtained from statistical data.

The Poisson PMF function is defined

$$P(N = n \text{ in time } t) = \frac{(\nu t)^n}{n!} e^{-\nu t} \quad n = 0, 1, 2, \ldots, \infty$$

The mean and standard deviation of the random variable N are

$$\mu_N = \nu t \qquad \sigma_N = \sqrt{\nu t}$$

Return period (or interval) τ also depicts a Poisson variable. It is simply the reciprocal of the mean occurrence rate ν: $\tau = \frac{1}{\nu}$

Return period is a deterministic average time interval between occurrences of events. The actual time interval is random.

EXAMPLE 2.6. The average occurrence rate of earthquakes (5 to 8 magnitudes) in a given region is 2.14 earthquakes/year. Determine

(a) The return period for earthquakes in this magnitude range.

(b) The probability of exactly three earthquakes (magnitude between 5 and 8) inthe next year.

(c) The annual probability of an earthquake of 5 – 8 magnitude.

Solution

(a) The return period is determined from Eq. 2.74:

$$\tau = \frac{1}{\nu} = \frac{1}{2.14} = 0.47 \text{ year}$$

One earthquake of a given rangeoccurs approx. every six months.

(b) The probability of three earthquakes preciselyin the year after a given one is determinedusing Eq. 2.72 with t = 1 and n = 3:

$$P\left(N = 3 \text{ in } 1 \text{ year}\right) = \frac{\left[\left(2.14\right)\left(1\right)^{3}\right]}{3!} e^{-(2.14)(1)} = 0.192$$

(c) We derive the annual probability of *at least one* earthquake. Therefore$P$(at least one earthquake) = 1 –P(no earthquakes), so

$$P\left(N \geq 1\right) = 1 - \frac{\left[\left(2.14\right)\left(1\right)\right]^{0}}{0!} e^{-(2.14)(1)} = 1 - e^{-(2.14)(1)} = 0.88$$