2.7 Random vectors

P. Thoft-Christensen, M. J. Baker
Structural reliability theory and its applications, 1982

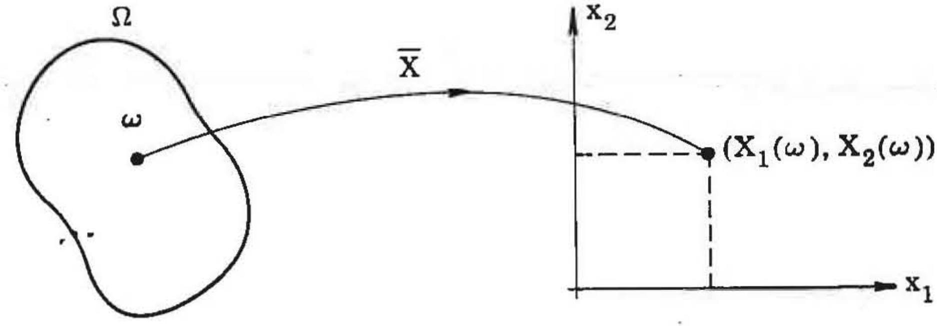
The concept of a random variable is basically used in a one-dimensional sense.

A random variable is a real-valued function $X:\Omega \to R$ mapping the sample space Ω into the real line *R*. It can easily be extended to a vector-valued random variable $\bar{X}:\Omega \to R^n$ called a *random vector (random n-tuple)*,where $R^n = R \times R \times ... \times R$.

An *n*-dimensional random vector $\bar{X}:\Omega \to R^n$ is an ordered set $\bar{X} = \left(X_1, X_2, ..., X_n\right)$of one-dimensional random variables $X_i:\Omega \to R, i=1,...,n$. All $X_1, X_2, ..., X_n$ are defined on the same sample space Ω.

Let X_1 and X_2 be two random variables. The range of the random vector $\bar{X} = \left(X_1, X_2\right)$ is then a subset of R^2 as shown in figure 2.11.

The range of an *n*-dimensional random vector is a subset of R^n.



Consider again two random variables X_1 and X_2 and their corresponding distribution functions F_{X_1} and F_{X_2}.

The latter give no information on the joint behaviour of X_1 and X_2.

Thus *joint probability distribution function (łączna dystrybuanta)*

$F_{X_1,X_2} : R^2 \to R$ is defined: $F_{X_1,X_2}(x_1, x_2) = P\left(\left(X_1 \le x_1\right) \cap \left(X_2 \le x_2\right)\right)$

We use $F_{\bar{X}}$ for F_{X_1,X_2}, where $\bar{X} = (X_1, X_2)$. The definition can be generalized to the *n*-dimensional case

$$F_{\bar{X}}(\bar{x}) = P\left(\bigcap_{i=1}^{n}(X_i \le x_i)\right) \text{where } \bar{X} = (X_1, \ldots, X_n) \text{ and } \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$$

Discrete or continuous random vectors exist, the latter of our concern only. Our analysis is restricted to two-dimensional random vectors only, to be generalized easily.

The *joint probability density function (łączna funkcja gęstości prawdopodobieństwa)* for the random vector $\bar{X} = (X_1, X_2)$ is given

$$f_{\bar{X}}(\bar{x}) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{\bar{X}}(\bar{x})$$

The inverse formula is

$$F_{\bar{X}}(\bar{x}) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_1} f_{\bar{X}}(x'_1, x'_2)\, dx'_1 dx'_2$$

The following functions exist

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{\bar{X}}(x_1, x_2)\, dx_2 \qquad f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{\bar{X}}(x_1, x_2)\, dx_1$$

They are *marginal density functions (gęstości rozkładów brzego-wych)*of a random vector X – one-dimensional functions.

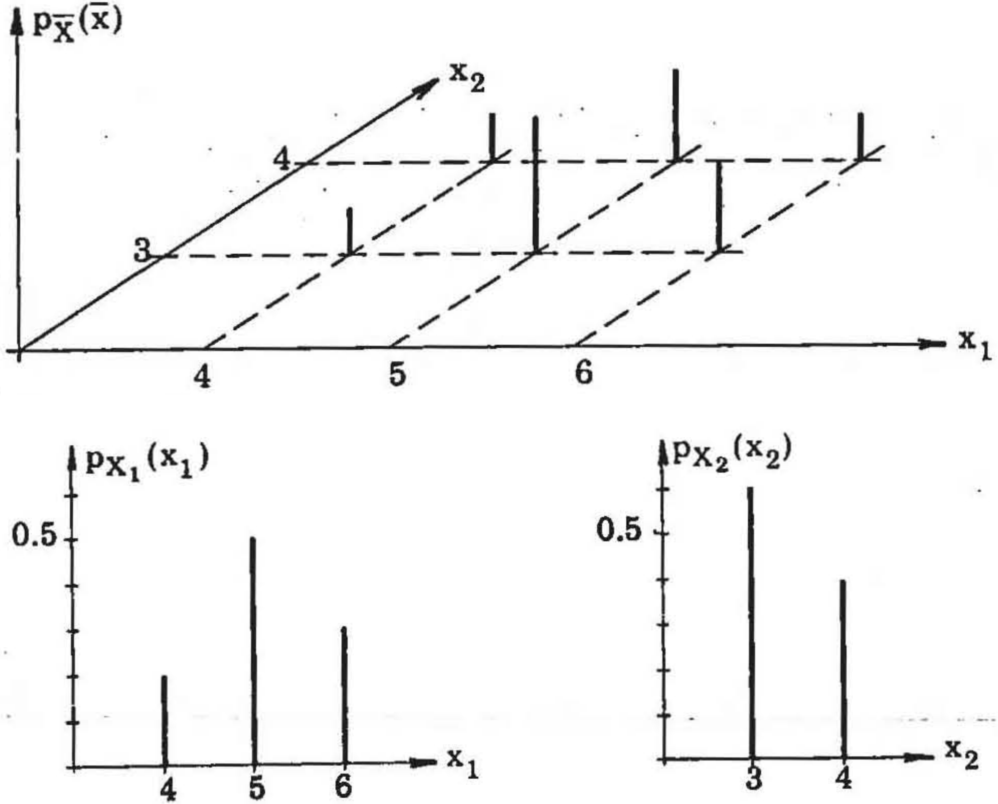
Example 2.13.

Consider again example 2.1 and let a 2-dimensional discrete random vector $\bar{X} = (X_1, X_2)$ be defined on Ω by

P(4, 3) = 0.1
P(4, 4) = 0.1
P(5, 3) = 0.3
P(5, 4) = 0.2
P(6, 3) = 0.2
P(6, 4) = 0.1

The joint mass function $p_{\bar{X}}$, and the marginal mass functions p_{X_1} and p_{X_2} are illustrated in Figures below.

Note that $p_{\bar{X}}(x_1, x_2) \neq p_{X_1}(x_1) \cdot p_{X_2}(x_2)$.



$p_{\overline{X}}(\overline{x})$
x_2
4
3
x_1
4
5
6
$p_{X_1}(x_1)$
0.5
x_1
4
5
6
$p_{X_2}(x_2)$
0.5
x_2
3
4

2.8 CONDITIONAL DISTRIBUTIONS

Probability of occurrence of event E_1 conditional upon the occurrence of event E_2 was defined by $P\left(E_1 \middle| E_2\right) = \dfrac{P\left(E_1 \cap E_2\right)}{P\left(E_2\right)}$

The *conditional probability mass function* for two jointly distributed discrete random variables X_1 and X_2 is defined

$$p_{X_1|X_2}\left(x_1 \middle| x_2\right) = \frac{p_{X_1,X_2}\left(x_1, x_2\right)}{p_{X_2}\left(x_2\right)}$$

Continuous cases define the *conditional probability density function*

$$f_{X_1|X_2}\left(x_1 \middle| x_2\right) = \frac{f_{X_1,X_2}\left(x_1, x_2\right)}{f_{X_2}\left(x_2\right)}$$

where $f_{X_2}\left(x_2\right) > 0$ and where f_{X_2} is a marginal PDF.

Mind the discrete / continuous diversity: $p_{X_1|X_2}$ is a conditional mass function, $f_{X_1|X_2}$ a conditional density function.

Two random variables X_1 and X_2 are *independent* if

$$f_{X_1|X_2}\left(x_1|x_2\right)=f_{X_1}\left(x_1\right)$$

Chich implies

$$f_{X_1|X_2}\left(x_1|x_2\right)=f_{X_1}\left(x_1\right)\cdot f_{X_2}\left(x_2\right)$$

Integrating with respect to x_1 gives *conditional distribution function*

$F_{X_1}\left(x_1\right)=\int_{-\infty}^{\infty}F_{X_1|X_2}\left(x_1|x_2\right)f_{x_2}\left(x_2\right)dx_2$, similarly the x_2 case.

Example 2.14.

Consider two jointly distributed discrete random variables X_1 and X_2 again. Note that

$$p_{X_1,X_2}\left(5,3\right)=p_{X_1}\left(5\right)p_{X_2}\left(3\right)$$

but for example

$$p_{X_1,X_2}\left(6,4\right)\neq p_{X_1}\left(6\right)p_{X_2}\left(4\right)$$

Therefore, X_1 and X_2 aredependent.

2.9 Functions of random variables

A continuous random variable Y which is a function $f(X)$ of a continuous random variable X is defined, the density function f_Y may determined given the density function f_X as follows

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| \qquad \text{where } x = f^{-1}(y).$$

Expanding the problem we have a random vector $\bar{Y} = (Y_1, Y_1, \ldots, Y_n)$ - function $\bar{f} = (f_1, f_1, \ldots, f_n)$ of a random vector $\bar{X} = (X_1, X_2, \ldots, X_n)$, that is $Y_i = f_i(X_1, \ldots, X_n)$, where $i = 1, 2, \ldots, n$.

Each function f_i $i = 1, 2, \ldots, n$ is a one-to-one mapping, so inverse relations exist: $X_i = g_i(Y_1, \ldots, Y_n)$

It can then be shown that

$$f_{\bar{Y}}(\bar{y}) = f_{\bar{X}}(\bar{x})|J|$$

where $\bar{x} = (x_1, x_2, \ldots, x_n)$ and $\bar{y} = (y_1, y_2, \ldots, y_n)$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \dots & \dots & \dots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$ is the Jacobian determinant.

Let the random variable, Y be a function f of the random vector $\bar{X} = \left(X_1, X_2, ..., X_n \right)$

It can be shown that

$$E\left(Y\right) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f\left(\bar{x}\right) f_{\bar{X}}\left(\bar{x}\right) dx_1 \, ... dx_n$$

where $\bar{x} = \left(x_1, x_2, ..., x_n\right)$ and $f_{\bar{X}}\left(\bar{x}\right)$ is the probability density function for the random vector $\bar{X}$.

Let X_1 and X_2 be two random variables with the expected values $E\left[X_1\right] = \mu_{X_1}$ and $E\left[X_2\right] = \mu_{X_2}$.

The *mixed central moment*defined by

$$\mathrm{Cov}\left[X_1, X_2\right] = E\left[\left(X_1 - \mu_{X_1}\right)\left(X_2 - \mu_{X_2}\right)\right]$$

is called the *covariance* of X_1 and X_2.

The ratio

$$\rho_{X_1 X_2} = \frac{Cov\left[X_1, X_2\right]}{\sigma_{X_1}\sigma_{X_2}}$$

is called the *correlation coefficient (współczynnikkorelacji)*, where σ_{X_1} and σ_{X_2} are the standard deviations of random variables X_1 and X_2.

It measures a linear dependence between a pair of random variables.

The inequalities hold $-1 \le \rho_{X_1 X_2} \le 1$.

Two random variables X_1 and X_2 are *uncorrelated* if $\rho_{X_1 X_2} = 0$.

The following identity

$$\mathrm{Cov}\left[X_1, X_2\right] = E\left[\left(X_1 - \mu_{X_1}\right)\left(X_2 - \mu_{X_2}\right)\right] = E\left[X_1 \cdot X_2\right] - E\left[X_1\right]E\left[X_2\right]$$

is specified in the case of uncorrelated random variables X_1 and X_2

$$E\left[X_1 \cdot X_2\right] = E\left[X_1\right]E\left[X_2\right]$$

It isimportant that independent random variables are uncorrelated, but uncorrelated variables may be dependent.

Note that $\mathrm{Cov}\left[X_i, X_i\right] = \mathrm{Var}\left[X_i\right]$. Total correlation between random variables $X_1, X_2, ..., X_n$ may have the *covariance matrix* $\overline{\overline{C}}$ form

$$\overline{\overline{C}} = \begin{vmatrix} \mathrm{Var}\left[X_1\right] & \mathrm{Cov}\left[X_1, X_2\right] & ... & \mathrm{Cov}\left[X_1, X_n\right] \\ \mathrm{Cov}\left[X_2, X_1\right] & \mathrm{Var}\left[X_2\right] & ... & \mathrm{Cov}\left[X_2, X_n\right] \\ ... & ... & ... & ... \\ \mathrm{Cov}\left[X_n, X_1\right] & \mathrm{Cov}\left[X_n, X_2\right] & ... & \mathrm{Var}\left[X_n\right] \end{vmatrix}$$

2.10 MULTIVARIATE DISTRIBUTIONS .

Animportant joint density function of two continuous random variables X_1 and $X_2$is the *bivariate normal density function*

$$f_{X_1,X_2}(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[\frac{-1}{2(1-\rho^2)}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right.\right.$$

$$\left.\left.-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)\right]$$

where $-\infty \le x_1 \le \infty$, $-\infty \le x_2 \le \infty$, and μ_1, μ_2 are the means σ_1, σ_2 the standard deviations and ρ the coefficient of X_1, X_2.

The *multivariate normal density function*is defined

$$f_{\overline{X}}(\overline{x})=\frac{1}{(2\pi)^{\frac{n}{2}}}\cdot\frac{1}{\left|\overline{\overline{C}}\right|^{\frac{1}{2}}}\exp\left[-\frac{1}{2}\sum_{i,j=1}^{n}(x_i-\mu_j)M_{ij}(x_j-\mu_i)\right]$$

$\overline{x}=(x_1,x_2,\ldots,x_n)$, $\overline{\overline{M}}=\overline{\overline{C}}^{-1}$, and where $\overline{\overline{C}}$ is the covariance matrix.