4 METHODS OF STOCHASTIC MECHANICS

Probability theory is the most widespread description means of uncertainty, although not the only one. The alternatives are fuzzy and rough set theory, convex models or interval arithmetic. The format of results determines the applied approach, which may be:

- Perturbation Method – computing the first two statistic moments of the response quantities, i.e. mean, variance and correlation coefficient,
- Stochastic Finite Element Methods (SFEM) – evaluating the global response quantities in terms of random processes,
- Reliability Methods – estimating the probability of failure of the system.

Dominant approaches have been detected to estimate the structural reliability:

a) Monte Carlo methods,
b) First- and Second-order Reliability Methods (FORM and SORM methods),
c) Response Surface method,
d) Neural-network based reliability.

Most approaches to stochastic mechanics represent the spatial variability of the input parameters. Thus random field discretization is taken first.

4.1. Random fields discretization methods

Discretization of a random field $H(\cdot)$ by $\hat{H}(\cdot)$ means its approximation to a finite set of random variables $\{\chi_i, i=1,...n\}$ forming a random vector $\boldsymbol{\chi}$

$$H(\mathbf{x}) \xrightarrow{\text{Discretization}} \hat{H}(\mathbf{x}) = \mathcal{F}[\mathbf{x}, \boldsymbol{\chi}] \tag{4.1}$$

The best approximation uses the minimum number of random variables.
The discretization methods are:
– point discretization methods,
– average discretization methods,
– series expansion methods.
In the following the Finite Element Method application will be pointed out. The review work by Surdet and Der Kiureghian (2000) is a description basis.

4.2. Point discretization methods

The midpoint method approximates the random field in each element Ω_e by a single random variable – a random field value at the element centre $\mathbf{x}_c$

$$\hat{H}(\mathbf{x}) = H(\mathbf{x}_c), \qquad \mathbf{x} \in \Omega_e \tag{4.2}$$

The approximated field $\hat{H}(\cdot)$ is entirely defined by the random vector $\boldsymbol{\chi} = \{H(\mathbf{x}_c^1), \ldots H(\mathbf{x}_c^{N_e})\}$ (N_e is the number of elements in the mesh). This method over-represents the random field variability within each element.

The shape function method approximates $\hat{H}(\cdot)$ in each element using the nodal values $\mathbf{x}_i$ and the shape function as follows

$$\hat{H}(\mathbf{x}) = \sum_{i=1}^{q} N_i(\mathbf{x}) H(\mathbf{x}_i), \qquad \mathbf{x} \in \Omega_e \tag{4.3}$$

where q is the number of element nodes, $\mathbf{x}_i$ the coordinates of the i-th node and N_i polynomial shape functions associated with the element.

The approximated field $\hat{H}(\cdot)$ is $\boldsymbol{\chi} = \{H(\mathbf{x}_1), \ldots H(\mathbf{x}_N)\}$, where $\{\mathbf{x}_i, i = 1, \ldots N\}$ is the set of the nodal coordinate of the mesh. Each realization of $\hat{H}(\cdot)$ is a continuous function over Ω - this is an advantage over the midpoint method.

The integration point method discretizes the random field by associating a single random variable with each of the Gauss points. This gives accurate results for a short correlation length. However, the total number of random variables involved increases enormously with the size of the problem.

The optimal linear estimation method defines the field $\hat{H}(\cdot)$ by a linear function of nodal values $\boldsymbol{\chi} = \{H(\mathbf{x}_1), ... H(\mathbf{x}_q)\}$ in the following way

$$\hat{H}(\mathbf{x}) = a(\mathbf{x}) + b^T(\mathbf{x}) \cdot \boldsymbol{\chi} \tag{4.4}$$

where q is the number of nodes in the approximation. The functions $a(\mathbf{x})$ and $b(\mathbf{x})$ minimize the error variance $\mathrm{Var}\{H(\mathbf{x}) - \hat{H}(\mathbf{x})\}$ at each point **x.**

4.3. Average discretization methods

The spatial average method approximates the field in each element Ω_e by a constant computed as the average of the original field over the element

$$\hat{H}(\mathbf{x}) = \frac{1}{|\Omega_e|} \int_{\Omega_e} H(\mathbf{x}) d\Omega_e \equiv \hat{H}_e \tag{4.5}$$

Vector $\boldsymbol{\chi}$ consists of the random variables $\chi^T = \{\hat{H}_e, e = 1, ... N_e\}$. Fig. 2.1 shows that the randomness of the average process $\hat{H}(\mathbf{x})$ is less than for $H(\mathbf{x})$.

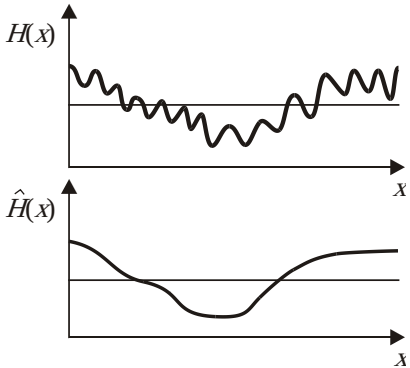

Fig. 2.1. Local averages of random field

The weighted *integral method* does not discretize the random field. In the linear elastic case the element stiffness matrices are basic random data

$$k^e = \int_{\Omega_e} \mathbf{B}^T \cdot \mathbf{D} \cdot \mathbf{B} d\Omega_e \tag{4.6}$$

where a matrix **B** relates strain components to the nodal displacements, **D** is the elasticity matrix - a product of a deterministic matrix by a univariate field

$$\mathbf{D}(\mathbf{x},\theta) = \mathbf{D}_o\left[1 + H(\mathbf{x},\theta)\right] \tag{4.7}$$

where $\mathbf{D}_o$ is the mean value and $H(\mathbf{x},\theta)$ is a zero mean process.
The weighted integral method is mesh-dependent.

4.4. Series expansion methods

The Karhunen-Loève expansion of a random field $H(\cdot)$ is based on the spectral decomposition of its autocovariance function $C_{HH}(\mathbf{x},\mathbf{x}') = \sigma(\mathbf{x})\sigma(\mathbf{x}')\rho(\mathbf{x},\mathbf{x}')$. The set of the deterministic functions over which any realization of the field $H(x,\theta_o)$ is expanded is defined by the eigenvalue problem (Fredholm integral equation)

$$\forall i = 1,... \qquad \int_{\Omega} C_{HH}(\mathbf{x},\mathbf{x}')\varphi_i(\mathbf{x}')d\Omega_{x'} = \lambda_i\varphi_i(\mathbf{x}) \tag{4.8}$$

The set of $\{\varphi_i\}$ forms a complete orthogonal basis.
Any realization of $H(\cdot)$ can be expanded over this basis as follows

$$H(\mathbf{x},\theta) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty}\sqrt{\lambda_i}\xi_i(\theta)\varphi_i(\mathbf{x}) \tag{4.9}$$

where $\{\xi_i(\theta),\ i = 1,...\}$ denotes the coordinates of the realizations of the random field with respect to the set of deterministic functions $\{\varphi_i\}$.

All possible realizations of the field $\{\xi_i, i=1,...\}$ result in a numerable set of random variables. No analytical solution for the integral eigenvalue problem (4.8) exists, so orthogonal functions $\{\varphi_i,\ i=1,...\}$ are computed numerically.

The orthogonal series expansion method avoids solving the eigenvalue problem (4.8) by selecting a complete set of orthogonal functions $\{h_i(\mathbf{x})\}_{i=1}^{\infty}$, forming an orthonormal basis

$$\int_{\Omega} h_i(\mathbf{x}) h_j(\mathbf{x}') d\Omega = \delta_{ij} \tag{4.10}$$

where δ_{ij} is a Kronecker symbol.

Any realization of the random field $H(\mathbf{x},\theta)$ is a function, which can be expanded by means of orthogonal function $\{h_i(\mathbf{x})\}_{i=1}^{\infty}$

$$H(\mathbf{x},\theta) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \chi_i(\theta) h_i(\mathbf{x}) \tag{4.11}$$

where $\mu(\mathbf{x})$ is mean function, $\chi_i(\theta)$ are zero-mean random variables. It holds

$$\chi_i(\theta) = \int_{\Omega} \left[H(\mathbf{x},\theta) - \mu(\mathbf{x}) \right] h_i(\mathbf{x}) d\Omega \tag{4.12}$$

$$E\left[\chi_k \chi_l\right] = \int_{\Omega}\int_{\Omega} C_{HH}\left(\mathbf{x},\mathbf{x}'\right) h_k\left(\mathbf{x}\right) h_l\left(\mathbf{x}'\right) d\Omega_x d\Omega_{x'} \tag{4.13}$$

where $C_{HH}(\mathbf{x},\mathbf{x}')$ is the autocovariance function of the random field $H(x,\theta)$.

The expansion optimal linear estimation method is an extension of the optimal linear estimation using a spectral representation of the random nodal variables $\boldsymbol{\chi}$. Assuming that $H(\cdot)$ is Gaussian, the spectral decomposition of the covariance matrix $\boldsymbol{\Sigma}_{\chi\chi}$ of $\boldsymbol{\chi} = \{H(\mathbf{x}_1),...H(\mathbf{x}_N)\}$ is

$$\boldsymbol{\chi}(\theta) = \boldsymbol{\mu}_{\chi} + \sum_{i=1}^{N} \sqrt{\lambda_i}\,\xi_i(\theta)\,\boldsymbol{\phi}_i \tag{4.14}$$

where $\{\xi_i, i = 1,...N\}$ are independent standard normal variables, $(\lambda_i, \boldsymbol{\phi}_i)$ are eigenvalues and eigenvectors of the covariance matrix $\boldsymbol{\Sigma}_{\chi\chi}$ fulfilling $\boldsymbol{\Sigma}_{\chi\chi}\boldsymbol{\phi}_i = \lambda_i \boldsymbol{\phi}_i$. The (4.14), (4.4) and the optimal linear estimation lead to

$$\hat{H}\left(\mathbf{x},\theta\right) = \mu\left(\mathbf{x}\right) + \sum_{i=1}^{N} \frac{\xi_i\left(\theta\right)}{\sqrt{\lambda_i}} \boldsymbol{\phi}_i^T \boldsymbol{\Sigma}_{H(\mathbf{x})\chi} \tag{4.15}$$

4.5. Selection of the random field mesh

The finite element mesh and the random field mesh have to be stated properly

– the design of the FE mesh is governed by stress gradients of the response,
– the typical element size L_{RF} in random field mesh is related to the correlation length of the autocorrelation function.

Depending on the discretization method different recommendations relating to the element size and mesh construction can be found in the literature.

For example:

$$L_{RF} \approx \frac{a}{4} \text{ to } \frac{a}{2} \tag{4.16}$$

– if L_{RF} is too small the random variables appearing in the discretization are highly correlated leading to numerical instabilities.
– As the correlation length is usually constant over Ω the associated mesh can be constructed on a regular pattern.
– Several elements of the FE mesh grouped in a single one may produce the random field mesh. Thus the random vector χ may reduce its size.
– Reliability analysis shows that the refinement of the random field mesh should be linked with the gradient of the limit state function.

4.6. Generation of random variates

Generation of random numbers is operated by computer programs. In fact the "pseudo numbers" are produced this way, repeating after a long cycle interval, but the practical problems do not distinguish this.

Basic variables are seldom uniformly distributed and only such set can be obtained by the "pseudo-generators". A sample value for a basic variable with a given nonuniform distribution is called a "random variate". Two techniques for generating random variables exist: inversion and rejection method.

The *inversion method* starts with generation of uniformly distributed random numbers r_i $(0 \le r_i \le 1)$, next come the corresponding variables by inversion of the cumulative distribution function $F_{X_i}(x_i)$ (see Fig. 2.2).

$$F_{X_i}(x_i) = r_i \qquad \Rightarrow \qquad x_i = F_{X_i}^{-1}(r_i) \tag{4.17}$$

This uniquely fixes the sample value $x_i = \hat{x}_i$ when an analytic expression for the inverse $F_{X_i}^{-1}(r_i)$ exists (for example, the normal, Weibull, exponential, Gumbel and other distributions). The technique can also be applied to basic variables with CDF taken from observation.

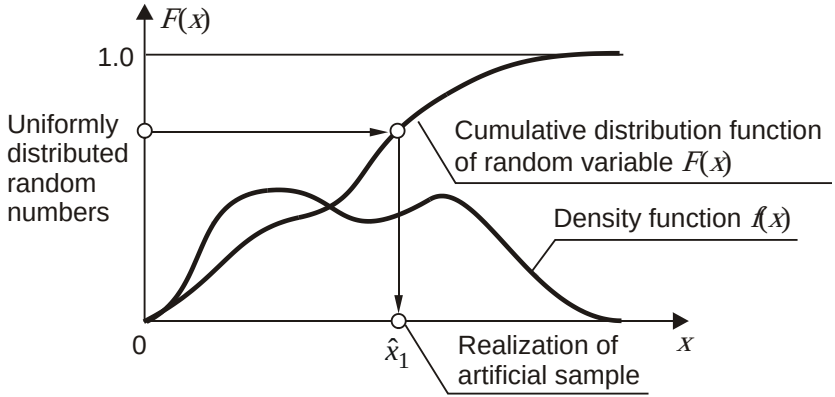

Fig. 2.2. Inversion method of generating random variates

The *rejection method* is illustrated in Fig. 2.3. A random variate x_i is generated using a fictitious probability density $f(x_i)$ that envelopes the target one $p(x_i)$. The generated variable x_i is accepted with a specified probability that depends on the ratio of the true and fictitious densities.

A useful technique to estimate the required joint density is the *Nataf's method* making use of the multidimensional Gaussian distributions whose correlation coefficient is modified by a nonlinear transformation of given marginal and Gaussian densities. The corresponding samples of the correlated variables can

then be generated by means of the approximated distributions. The joint density function of all variables is usually ignored in stochastic analysis, so marginal distribution and covariance functions are enough.

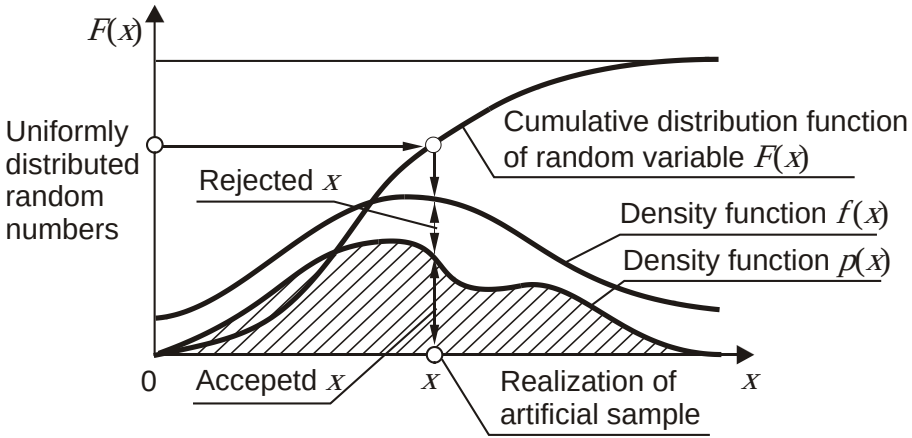

Fig. 2.3. Inversion and rejection methods of generating random variables

4.7. Stochastic finite element methods

The most common analytical method to assess statistical measures of responses to uncertainties of structures and loads is the *Perturbation Technique*. This method uses the Taylor expansion of a mathematical operator that relates the input and output variables. Truncation of the series to the first two terms implies that the accuracy of the method is limited to the cases of low coefficients of variation of input variables, not more than 0.15. This condition is more restrictive in the case of nonlinear systems.

The *Hierarchical Closure Approximation* assumes that higher order moments of the system and the output are considered functions of lower order.

Various problems are solved by means of the S*tochastic Finite Element* Modelling (SFEM).

The powerful analytical method recently developed is the *Spectral Approach*. This technique consists of the following steps:

a) description of a random field by truncated infinite series using the Karhunen-Loeve decomposition,

b) projection of the decomposed random field of the solution on a class of polynomials on nonlinear systems known as Homogeneous Chaos,

c) solution of the resulting system of equations.

In a one-dimensional application the spectral density function of a harmonic process can be presented as follows

$$x(s) = \sum_{i=1}^{N} A_i(\kappa_i)\cos(\kappa_i s + \phi_i) \tag{4.18}$$

in which the ϕ_i are random phases.

The harmonic process is convergent to the Gaussian one with a single-sided spectral density (i.e. defined only over positive wave number or frequencies) $G(\kappa) = 2S(\kappa)$. This case the wave amplitudes $A_i(\kappa_i)$ are estimated as functions of discretized spectral density intervals of the length $\Delta\kappa$

$$A_i(\kappa_i) = \sqrt{2G(\kappa_i)\Delta\kappa} \tag{4.19}$$

due to the fact that the variance of the process is equal to the area under the curve of the spectral density.