5. MONTE CARLO METHODS

Melchers R.E., Structural reliability analysis and prediction. John Wiley & Sons 2001

The Monte Carlo method has for long been recognized as the most exact method for all the calculations that require the knowledge of the probability distribution of response of uncertain systems to uncertain inputs.
A general idea of the Monte Carlo method can be summarized as follows. Suppose that the following integral is evaluated

$$I = \int_D g(\mathbf{x})\,dx \tag{5.1}$$

where D is a region in high-dimensional space and $g(\mathbf{x})$ is the target function of interest. If independent and identically distributed random samples $\mathbf{x}_1,...,\mathbf{x}_m$, uniformly simulated from D, an approximation of I can be obtained as

$$\hat{I}_m = \frac{1}{m}\left[g(\mathbf{x}_1) + ... + g(\mathbf{x}_m) \right] \tag{5.2}$$

According to the law of large numbers the average of many independent random variables with common mean and finite variances tends to stabilize at their common mean

$$\lim_{m\to\infty} \hat{I}_m = I, \quad \text{with probability } 1 \tag{5.3}$$

Its convergence rate can be assessed by the central limit theorem

$$\sqrt{m}\left(\hat{I}_m - I\right) \to N\left(0, \sigma^2\right) \tag{5.4}$$

where $\sigma^2 = \text{var}[g(\mathbf{x})]$.

Hence, the error term of the Monte Carlo approximation is $O(m^{-1/2})$, regardless of the dimensionality of $\mathbf{x}$.

In the case of structural reliability analysis, this means, that each random variable vector $\mathbf{x}_i$ is randomly generated to obtain sample value $\hat{\mathbf{x}}_i$, and then the limit state function $G(\hat{\mathbf{x}}_i) = 0$ is checked.

If the limit state is violated, i.e. $G(\hat{\mathbf{x}}_i) \le 0$, the structure or structural element has "failed".

The experiment is repeated many times.

If N trials are conducted, the probability of failure is given approximately by

$$p_f = \frac{n\left(G(\hat{\mathbf{x}}_i) \le 0\right)}{N} \tag{5.5}$$

where $n(G(\hat{\mathbf{x}}_i) \le 0)$ denotes the number of trials n for which $G(\hat{\mathbf{x}}_i) \le 0$. The number N of trials is related to the accuracy for p_f estimation.

To apply the Monte Carlo techniques to structural reliability it is necessary (Melchers 1999):

1) to develop simulation technique for numerical sampling of the basic variables $\hat{\mathbf{x}}_i$,

2) to consider the effect of the complexity of calculating the limit state function $G(\hat{\mathbf{x}}_i)$ and the number of basic variables on the simulation techniques used,

3) to determine the amount of sampling required to obtain a reasonable estimate of the structure probability of failure p_f.

5.1. Direct Monte Carlo method

The *direct sampling* or *Simple Random Sampling* is the simplest Monte Carlo approach in solving reliability problems. It can be graphically presented as so-called *ant-hill* (see Fig 1). It does not apply any reduction method to the generated set of variates, which allows for the statistical description of the structural behaviour without scarifying the description quality. Thus, this method can be fast enough for the reliability analysis of structures with a reduced number of degree of freedom but it is too costly for any large structure analysis.

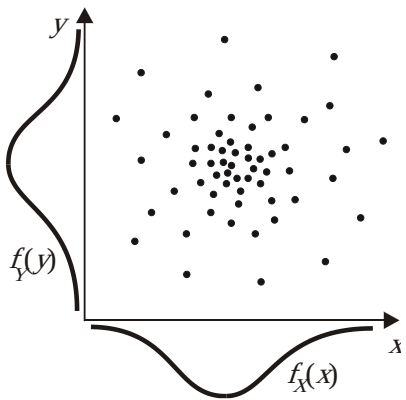

Fig. 1. Simple random sampling

In this case the probability of structure failure

$$p_f = P\left[G(\mathbf{X}) \le 0\right] = \int \ldots \int_{G(\mathbf{X}) \le 0} f_{\mathbf{X}}(\mathbf{x})\, dx \tag{5.6}$$

may be expressed as (Melchers 1999)

$$p_f = J = \int \ldots \int I\left[G(\mathbf{X}) \le 0\right] f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \tag{5.7}$$

where $I[\cdot]$ is an indicator function which equals 1 if $[\cdot]$ is "true" and 0 if $[\cdot]$ is "false". Thus, the indicator function identifies the integration domain.
The unbiased estimator of the expected value J and the estimator of standard deviation can be calculated as follows:

$$p_f \approx J_1 = \frac{1}{N} \sum_{i=1}^{N} I\left[G(\hat{\mathbf{x}}_i) \le 0\right] \tag{5.8}$$

$$\sigma_{J_1}^2 = \sum_{i=1}^{N} \frac{1}{N^2} \operatorname{var}\left[I(G \le 0)\right] = \frac{\sigma_{I(G \le 0)}^2}{N} \tag{5.9}$$

where $\hat{\mathbf{x}}_i$ represents the i-th vector of random observations from $f_{\mathbf{X}}(\cdot)$.

The standard deviation of J_1 and hence of the Monte Carlo estimate p_f (5.8) varies inversely with $N^{1/2}$ (see also Eq. (5.4)) These observations are important in determining the number of simulations required for a particular level of confidence.

On the basis of the central limit theorem, the following confidence statement can be made concerning the number of J_1 trails in which failure are possible (see Melchers 1999)

$$P\left(-k\sigma < J_1 - \mu < +k\sigma\right) = C \tag{5.10}$$

where μ is the expected value of J_1 given by Eq. (5.8) and σ is standard deviation expressed by (5.9).

The number N of simulations for a given confidence level C in the failure probability p_f can also be obtained from (see Melchers 1999)

$$N > \frac{-\ln\left(1-C\right)}{p_f} \tag{5.11}$$

Using Eq. (5.11) for a 95% confidence level and $p_f = 10^{-3}$ the required number of simulations is more than 3000.

It is not convenient to apply the above theoretical rules to the accuracy analysis in any particular Monte Carlo calculations. According to Melchers (1999) a useful tool for this purpose is to plot progressive results of the estimate of p_f and variance $\sigma^2_{J_1}$ (Eqs. (5.8) and (5.9) respectively). Such plots (see Fig. 2) will show that these measures decline when the number of samples rises and that a degree of stability is reached at a sufficiently high number of samples.

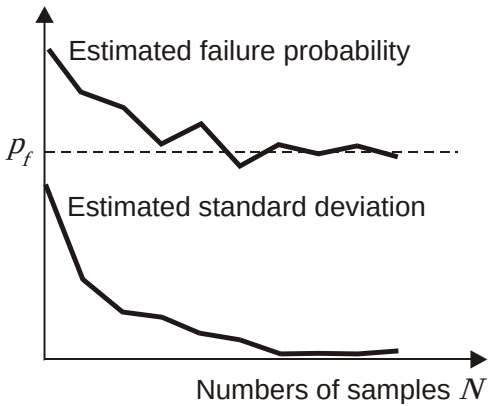

Fig. 2. Convergence of probability estimate with increasing sample size

The results may also be represented as a cumulative distribution function $F_G(g)$ (see Fig 3). The estimate of p_f in Eq. (5.8) may be improved by fitting an appropriate distribution function through the points for which $G(\cdot) \le 0$, i.e. the left-hand tail in Fig. 3 (Melchers, 1999).

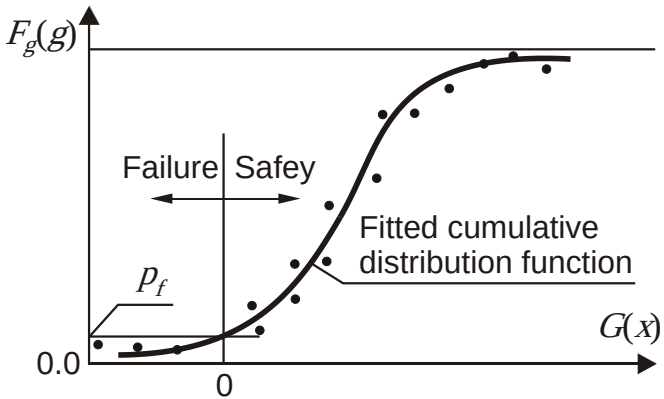

Fig. 3. Use of fitted cumulative distribution function to estimate D_1

5.2. Stratified sampling and Latin Hypercube Sampling

The *Stratified Sampling* and *Latin Hypercube Sampling* techniques have been proposed to reduce the Monte Carlo calculation.

In the case of the *Stratified Sampling* method the whole space of the variable is divided into subsets of equal probability. The acquired data are generated from each subset and an analysis is performed with corresponding sets of points (Fig. 4). A sample from inside the subset is taken either from the middle or randomly.

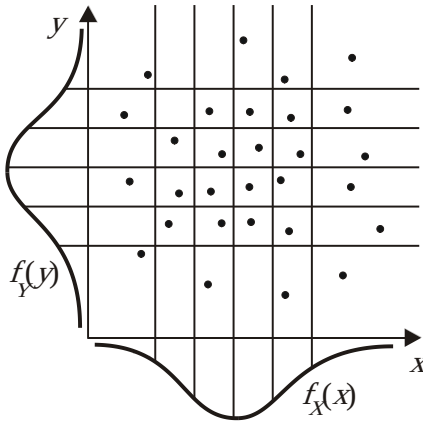

Fig. 4. Stratified Sampling

It should be stressed that the Direct Monte Carlo and the Stratified Sampling Method can also be applied to those cases in which the limit state function $G(\mathbf{X})$ is not known.

The *Latin Hypercube Sampling* method combines at random each subset number from each random variable with other subset numbers of the remaining variables only once (see Fig. 5).

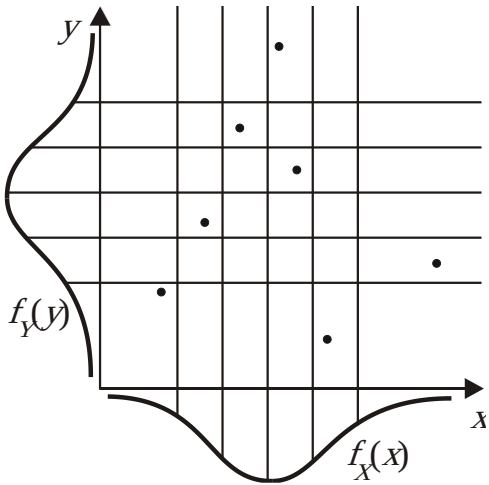

Fig. 5. Latin Hypercube Sampling

5.3. Importance sampling and search techniques

The integral (5.6) can be written using the indicator function $I[\cdot]$ as (Melchers 1999)

$$J = \int \ldots \int I\left[G(\mathbf{X}) \le 0\right] \frac{f_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{V}}(\mathbf{x})} h_{\mathbf{V}}(\mathbf{x}) d\mathbf{x} \tag{5.12}$$

where $h_{\mathbf{V}}(\mathbf{x})$ is termed the "importance-sampling" probability density function.

An unbiased estimate of J is given by (cf. (5.8))

$$p_f \approx J_2 = \frac{1}{N} \sum_{i=1}^{N} \left\{ I\left[G(\hat{\mathbf{v}}_i) \le 0\right] \frac{f_{\mathbf{X}}(\hat{\mathbf{v}}_i)}{h_{\mathbf{V}}(\hat{\mathbf{v}}_i)} \right\} \tag{5.13}$$

where $\hat{\mathbf{v}}_i$ is a vector of sample values taken from the importance sampling function $h_{\mathbf{V}}(\mathbf{v})$.

For a given level of confidence, far fewer sample points of $h_{\mathbf{V}}(\mathbf{v})$ are required than in the direct Monte Carlo method with $f_{\mathbf{X}}(\mathbf{x})$ as sampling distribution. The derivation of optimal $h_{\mathbf{V}}(\mathbf{v})$ functions is difficult and they are often selected on a priori grounds.

Sometimes it is possible to estimate the point x^*, known as the point of "maximum likelihood" or the "design point", with $f_{\mathbf{X}}(\mathbf{x})$ having the largest influence on the limit state function (see Fig. 6).

The point x^* may be found by a direct application of the numerical maximization techniques or the search algorithms. Once x^* is identified, the most common approach of choosing $h_{\mathbf{V}}(\mathbf{v})$ is to use the distribution $f_{\mathbf{X}}(\mathbf{x})$ shifted so that its mean is at x^*.

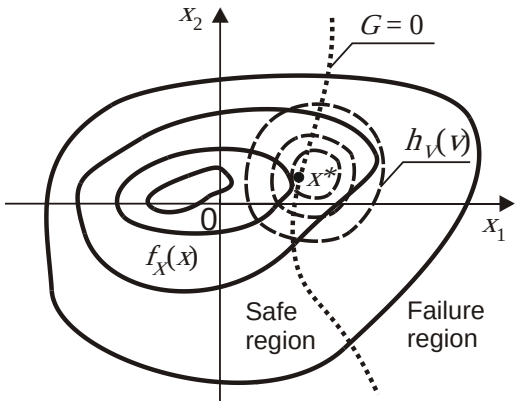

Fig. 6. Importance sampling function $I_{\mathbf{V}}[\,]$ in x space.

Adaptive sampling techniques apply modification of $h_{\mathbf{V}}(\mathbf{v})$, depending on the information being obtained from the search process (see Melchers 1999).

First the initial location of $h_{\mathbf{V}}(\mathbf{v})$, described by a mean vector and a covariance matrix is assumed.

A limited amount of sampling is then carried out. The samples which fall into the failure domain are used to relocate and change the form of $h_{\mathbf{V}}(\mathbf{v})$.

In general, it requires good physical understanding of the problem being solved.

The importance sampling method makes allowance for the estimation of the sensitivity of failure probability to changes in random variables.

Generally, if the effect of changing one or more variables on the failure probability is required to be evaluated, two Monte Carlo calculations, with or without a change should be performed.

Such an analysis is unlikely to be very helpful. If the limit state function is analytical, then the differentials $\partial G / \partial X_i$ will give the sensitivity of $G(\mathbf{X})$ to a change in X_i.

In the case of the importance sampling the probability estimate for the modified problem with a changed random variable x_i is given by (Melchers, 1999)

$$p_f + \Delta p_i = \int_D f_{\mathbf{X}+\Delta X_i}(\mathbf{x})\,d\mathbf{x} \approx \frac{1}{N}\sum_{j=1}^{N} I\left[\hat{\mathbf{x}}_j\right]\frac{f_{\mathbf{X}+\Delta X_i}\left(\hat{\mathbf{x}}_j\right)}{h_{\mathbf{X}}\left(\hat{\mathbf{x}}_j\right)} \tag{5.14}$$

and the sensitivity can be estimated as follows

$$S_i = \left[\left(p_f + \Delta p_i\right) - p_i\right] \Big/ \Delta x_i \tag{5.15}$$