LOAD COMBINATIONS
Time Variation

Nowak, A.S., Collins K.R. Reliability of structures.
McGraw-Hill Higher Education 2000

The time variation of loads and the possibility of simultaneous occurrence of loads are extremely important in structural reliability analysis.
We stated earlier that it is convenient to model the total load Q as a sum of individual load components Q_i such as dead load, live load, snow load, wind load, and earthquake load.
The following key questions must be answered before conducting a reliability analysis:

- What is the cumulative distribution function of the total load Q?
- What are the mean and coefficient of variation of Q?

In general, each of the load components varies in time, and hence the calculation of the CDF for the total load is very difficult.

Some sample load histories as a function of time are seen in Figure 6.17 for various load components.

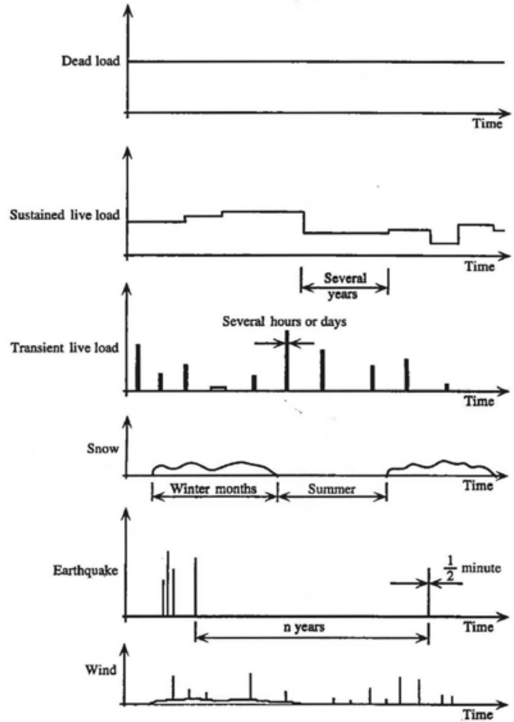

FIGURE 6.17 Time histories for various load components.

If we model the total load as the superposition of the various 10ad components, the total load would vary as illustrated in Figure 6.18

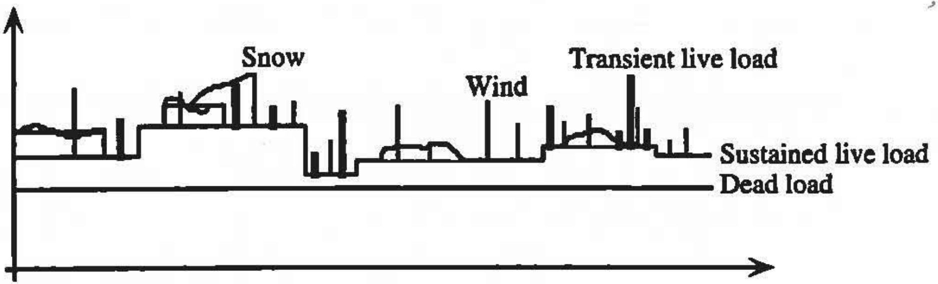

FIGURE 6.18 Superposition of loads.

Borges Model for Load Combination

In the Borges model (Borges and Castanheta, 1971), it is assumed that for each load component Q there is a basic time interval τ_i as shown in Figure 6.19.

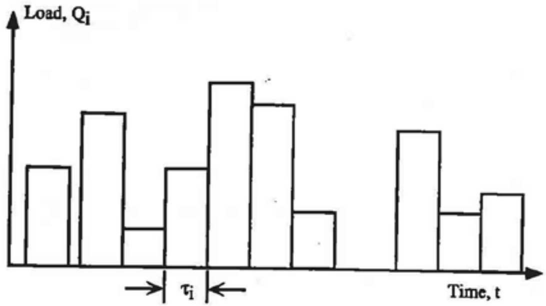

FIGURE 6.19 Time-variant load model.

The time interval is defined so that:

1. The magnitude of Q_i can be considered as constant during this time period.
2. The occurrence or nonoccurrence of Q_i in each time interval corresponds to repeated independent trials with probability of occurrence p.

Given the occurrence of a particular load, the probability distribution of its amplitude is $F_i(q)$.

This can be interpreted as a conditional CDF for Q_i.

We assume based on physical grounds that the load must be nonnegative, so $F_i(q)$ is zero for $q \leq 0$.

For the basic time interval τ_i, we can obtain the unconditional CDF of Q (in an interval τ_i) as

$$\begin{aligned} F(q) = P(Q_i \leq q \text{ in interval } \tau_i) &= P(\text{load occurs})P(Q_i \leq q| \text{ load occurs}) \\ &\quad + P(\text{no load occurs})P(Q_i \leq q| \text{ no load occurs}) \\ &= pF_i(q) + (1-p)1 \qquad (6.12) \\ &= 1 - p\left[1 - F_i(q)\right] \end{aligned}$$

The logic behind the last term in Eq. 6.12 is as follows.

The load can either occur or not occur.

If p is the probability of occurrence, then (1 – p) is the probability of nonoccurrence.

If the load does not occur, then Q must be zero by default since positive values of Q imply that the load occurs.

If Q = 0, then the probability of the load being less than or equal to q must be equal to 1 (i.e., a certain event) since q must be a positive number.

Now, we consider two basic time intervals.

For two intervals. the unconditional probability can be expressed as

$$\begin{aligned} F(q) &= P(Q_i \leq q \text{ in interval } 2\tau_i) \\ &= P(Q_i \leq q \text{ in first interval} \cap Q_i \leq q \text{ in second interval}) \end{aligned} \tag{6.13}$$

where $\cap$ denotes the intersection of the two events.

If we assume that the values of Q_i are independent from interval to interval, the probability of the intersection is simply the product of the probabilities, or

$$F(q) = \{1 - p[1 - F_i(q)]\}^2 \tag{6.14}$$

If we extend this basic idea to n basic time intervals, we can conclude that

$$F(q) = \{1 - p[1 - F_i(q)]\}^n \tag{6.15}$$

Now. consider the total load Q as being composed of two components Q_1 and Q_2, that is, $Q = Q_1 + Q_2$.
The CDF of Q is difficult to determine, but the parameters (mean and variance) are relatively straightforward.
Let τ_1 be the basic time interval for load component Q_1 and let τ_2 be the basic time interval for Q_2.
The probabilities of occurrence in the basic time interval for each load component are p_1 and p_2, respectively.
If we define k as the ratio τ_1 / τ_2 where k is an integer, then the CDF of Q_l corresponding to the time period τ_2, is

$$F(q_1) = \{1 - p_1[1 - F_1(q_1)]\}^k \qquad (6.16)$$

and for Q_2 it is

$$F(q_2) = \{1 - p_2[1 - F_2(q_2)]\} \qquad (6.17)$$

Since the variables defined by the CDFs in Eqs. 6.16 and 6.17 are defined for the same time period (τ_2), the sum of the variables is consistently defined.

Therefore, the mean and variance can be calculated using the formulas presented in Chapter 3 for the sum of random variables. The results are

$$\mu_Q = \mu_1 + \mu_2 \qquad (6.18)$$

where μ_1 and σ_1 are calculated for the CDF in Eq. 6.16 and μ_2 and σ_2 are calculated for the CDF in Eq. 6.17.

$$\sigma_Q = \sqrt{\sigma_1^2 + \sigma_2^2} \qquad (6.19)$$

EXAMPLE 6.2. Consider the load component, X, with CDF $F_x(x)$, corresponding to the basic time interval, τ, shown in Figure 6.20. If p = 1 and k = 4, then the CDF for the time interva14τ is as shown in Figure 6.20 by the curve labeled $F_X^4(x)$.

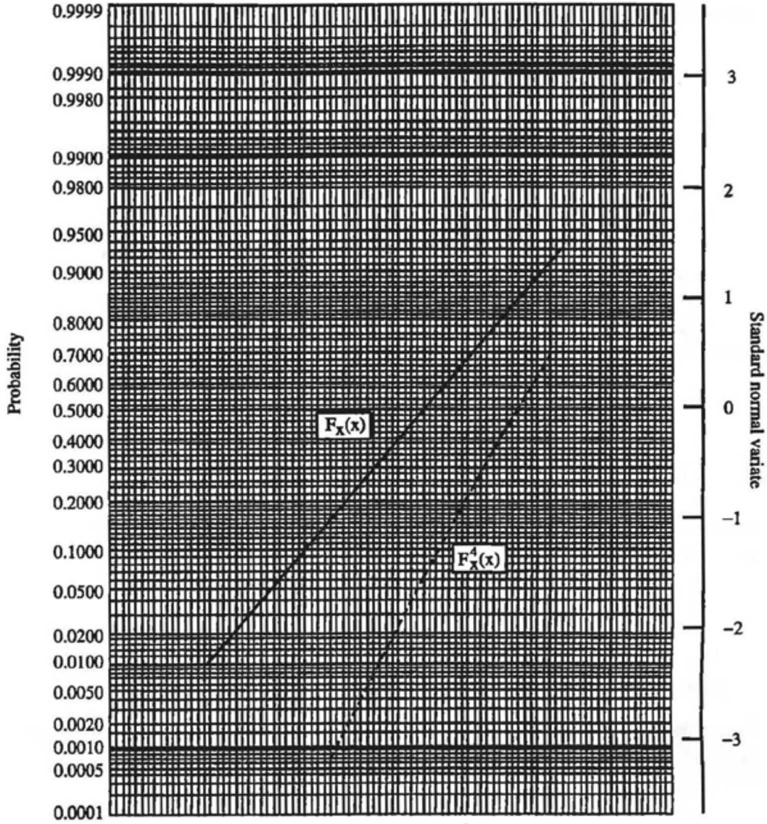

FIGURE 6.20 CDFs of load component considered in Example 6.2.

Turkstra's Role

The Borges approach for load combination can become cumbersome for several loads added together, and the definition of the basic time interval can be very difficult.

Turkstra's role (Turkstra, 1970; Turkstra and Madsen, 1980) is a practical approach to modeling load combinations.

It is based on the observation that when one load omponent reaches an extreme value, the other load components are often acting at their average values.

In other words, the possibility of two or more load components acting at their extreme values simultaneously is so remote that it is negligible.

Let X_1, X_2, ... , X_n be load components considered in defining a total load Q defined as

$$Q = X_1 + X_2 + \cdots + X_n \quad (6.20)$$

Our objective is to determine the mean and variance of the *maximum* value of Q in 50 years.

We will apply Turkstra's role to obtain these quantities.
For each load component X_i, two distributions must be considered:
CDF for the *maximum* 50-year value:

$$F_{X_{i-50}}(x) = P(\max X_i \leq x \text{ in 50 years}) \qquad (6.21)$$

CDF for the *arbitrary-point-in-time value:*

$$F_{X_i}(x) = P(X_i \leq x \text{ at any moment}) \qquad (6.22)$$

For each X_i, we can determine a mean and variance for the maximum 50-year value using Eq. 6.21 and a mean and variance for the arbitrary-point-in-time value using Eq. 6.22.
To distinguish the quantities, we will refer to the parameters for the maximum 50-year value as $\mu_{\max X_i}$; and $\sigma_{\max X_i}$.

The parameters for the arbitrary point-in-time values will be denoted $\mu_{X_i}^{apt}$ and $\sigma_{X_i}^{apt}$.

Turkstra's role states that for n load components you must consider n possible combinations of the loads to get the maximum value of the total load Q defined inEq. 6.20. Those combinations are as follows:

$$Q_{max} = \max \begin{cases} \max(X_1) + X_2^{apt} + \cdots + X_n^{apt} \\ X_1^{apt} + \max(X_2) + \cdots + X_n^{apt} \\ \vdots \\ X_1^{apt} + X_2^{apt} + \cdots + \max(X_n) \end{cases} \qquad (6.23)$$

The mean now can be calculated using the formula presented in Chapter 3.

The result is

$$\mu_{Q_{max}} = \max \begin{cases} \mu_{\max X_1} + \mu_{X_2}^{apt} + \cdots + \mu_{X_n}^{apt} \\ \mu_{X_1}^{apt} + \mu_{\max X_2} + \cdots + \mu_{X_n}^{apt} \\ \vdots \\ \mu_{X_1}^{apt} + \mu_{X_2}^{apt} + \cdots + \mu_{\max X_n} \end{cases} \qquad (6.24)$$

For the combination that results in the largest mean value, we can also calculate the variance for that combination using the formula presented in Chapter 3.

If the kth combination is the largest, then the variance is (assuming that the various components are uncorrelated)

$$\sigma_{Q_{max}}^2 = \sigma_{\max X_k}^2 + \sum_{\text{other components}} \left(\sigma_{X_i}^{apt}\right)^2 \qquad (6.25)$$

EXAMPLE 6.3. Consider a combination of dead load, live load, and wind load.

For each load component the following information is known:

• The dead load is time invariant Therefore, $D_{\max}$ and D^{apt} are the same. Both are normally distributed with

$$\mu_D = 20; \qquad V_D = 10\% \quad \Rightarrow \quad \sigma_D = 2$$

• For the live load, L_{max} follows an extreme Type I distribution with

$$\mu_{L_{max}} = 30; \qquad V_{L_{max}} = 12\% \quad \Rightarrow \quad \sigma_{L_{max}} = 3.6$$

The arbitrary-point-in-time value, L^{apt}, follows a gamma distribution with

$$\mu_L = 9; \qquad V_L = 31\% \quad \Rightarrow \quad \sigma_L = 2.8$$

• For the wind load, W_{max} follows an extreme Type I distribution with

$$\mu_{W_{max}} = 24; \qquad V_{W_{max}} = 20\% \quad \Rightarrow \quad \sigma_{W_{max}} = 4.8$$

The arbitrary-point-in-time value is lognormally distributed with

$$\mu_W = 1; \qquad V_W = 60\% \quad \Rightarrow \quad \sigma_W = 0.6$$

Calculate the parameters of the combined effect of these loads using Turkstra's rule.

Solution. The totalload, Q, is D + L + W.

By Turkstra's rule, we must consider three possible combinations to find Q_{max}:

$$Q_{max} = \max \begin{cases} D_{max} + L^{apt} + W^{apt} \\ D^{apt} + L_{max} + W^{apt} \\ D^{apt} + L^{apt} + W_{max} \end{cases}$$

However, in this case, since D_{max} and D^{apt} are equal, we need to consider only two combinations:

$$Q_{max} = D^{apt} + \max \begin{cases} L_{max} + W^{apt} \\ L^{apt} + W_{max} \end{cases}$$

To find the mean value of Q_{max}, we must apply Eq. 6.24.

$$\mu_{Q_{max}} = 20 + \max \begin{cases} 30 + 1 \\ 9 + 24 \end{cases} = 20 + \max \begin{cases} 31 \\ 33 \end{cases} = 53$$

Since the second load combination controls, we calculate the variance based on that combination. Applying Eq. 6.25, we find

$$\sigma^2_{Q_{max}} = \left(\sigma_D^{apt}\right)^2 + \left(\sigma_L^{apt}\right)^2 + \left(\sigma_{W_{max}}\right)^2$$

$$= (2)^2 + (2.8)^2 + (4.8)^2 = 34.9 \quad \Rightarrow \quad \sigma_{Q_{max}} = 5.9$$

Load Coincidence Method

Poisson pulse processes

Before considering the load coincidence method, it is necessary to introduce the concept of a Poisson pulse process.

A Poisson pulse process is a mathematical model that can be used to represent the temporal variation and intensity of loadings that act on structures.

The basic idea behind the Poisson pulse process is to represent a load Q(t) as a sequence of pulses that occur during some reference time interval T.

Two examples are shown in Figure 6.21.
Both the intensity X and the duration t' of each pulse are treated as random variables, and their values are independent from occurrence to occurrence.
(In other words, the intensity and duration do not depend on previous intensities and durations.)
In general, the pulses can have arbitrary shapes, but a rectangular shape is considered here for simplicity.
In Figure 6.21, the load $Q_1(t)$ is representative of a *sparse* process, which means that there are significant periods of time during which the load is zero.

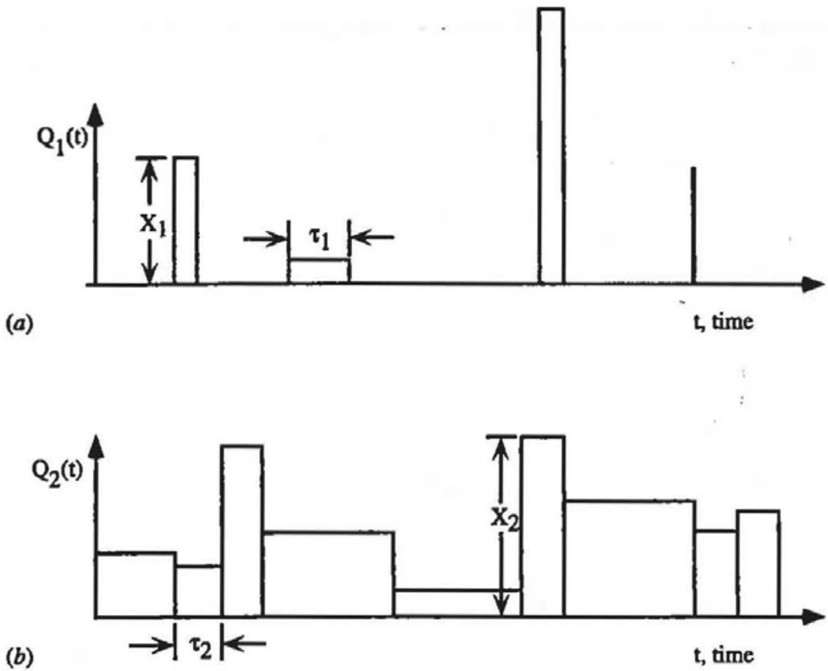

FIGURE 6.21 Examples of rectangular Poisson pulse processes. (*a*) A sparse process. (*b*) A full process.

On the other hand, load $Q_2(t)$ is representative of f*ull* process, which means that the load is never zero, but the intensity can vary in a stepwise manner as shown.

Suppose that a load Q(t) can be represented as a Poisson pulse process with an occurrence rate λ and a mean duration μ_t.

Furthermore, assume that the intensity X of each pulse follows a CDF $F_x(x)$.

For design purposes, we may be interested in the distribution of the maximum lifetime value Q_{max} of the load for a particular time period (0, T).

It can be shown (Melchers, 1987 Wen, 1977, 1990) that theCDF for Q_{max} is of the form

$$F_{Q_{max}}(q, T) = \exp\left\{-\lambda T\left[1 - F_X(q)\right]\right\} \tag{6.26}$$

Combinations of Poisson pulse processes

In its simplest form, the load coincidence model is a load combination model in which Poisson pulse processes are combined. For example, suppose that we are interested in determining the distribution of a combined total load (or load effect) Q(t) defined as

$$Q(t) = Q_1(t) + Q_2(t) \qquad (6.27)$$

in which $Q_1(t)$ and $Q_2(t)$ are independent loads that are modeled as Poisson pulse processes.

The relevant parameters for the load models are the mean durations $\mu_{\tau 1}$ and $\mu_{\tau 2}$ the occurrence rates λ_1 and λ_2, and the intensity distributions F_{X1} and F_{X2} of the pulses.

Figure 6.22 schematically illustrates this combination problem. The maximum values of Q_1 and Q_2 in the reference interval (0, T) are identified as $Q_{1,max}$ and $Q_{2,max}$.

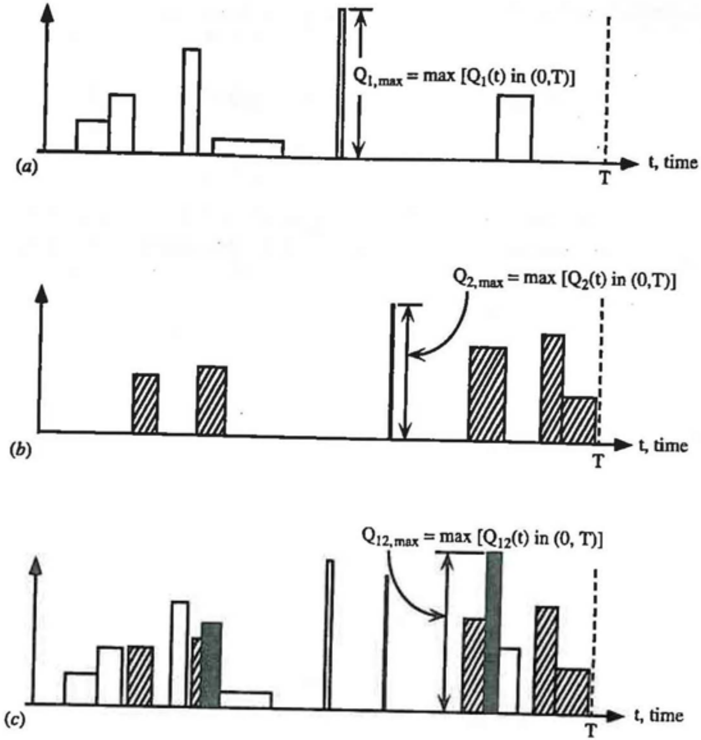

FIGURE 6.22 Illustration of the sum of two rectangular pulse processes. (*a*) Pulse process $Q_1(t)$ acting alone. (*b*) Pulse process $Q_2(t)$ acting alone. (*c*) Combined process $Q(t) = Q_1(t) + Q_2(t)$.

For a linear combination of pulse processes such as Eq. 6.27, the possibility of simultaneous occurrence (overlap) of pulses exists as shown by the shaded regions in Figure 6.22c.
The occurrence of these overlapping regions is referred to as the *coincidence process* (Wen, 1990) and is identified as $Q_{12}(t)$.
The maximum of the coincidence process is identified as $Q_{12,max}$.
Wen (1977, 1990) has shown that the coincidence process is also a Poisson pulse process with occurrence rate λ_{12} mean duration $\mu_{\tau 12}$.
Approximate relationships for λ_{12} and $\mu_{\tau 12}$ (Wen, 1977,1990) are

$$\lambda_{12} = \lambda_1 \lambda_2 (\mu_{\tau_1} + \mu_{\tau_2}) \qquad (6.28a)$$

$$\mu_{\tau_{12}} = \frac{\mu_{\tau_1} \mu_{\tau_2}}{\mu_{\tau_1} + \mu_{\tau_2}} \qquad (6.28b)$$

Now consider the determination of the CDF of the maximum value of Q(t) as defined in Eq. 6.27.

From Figure 6.22c, we know the CDF of Q_{max} can be expressed

$$F_{Q_{max}}(q) = P\left[\left(Q_{1,max} \le q\right) \cap \left(Q_{2,max} \le q\right) \cap \left(Q_{12,max} \le q\right)\right] \qquad (6.29)$$

According to Wen (1990), a conservative approximation to Eq. 6.29 is

$$F_{Q_{max}}(q) \approx P\left(Q_{1,max} \le q\right) P\left(Q_{2,max} \le q\right) P\left(Q_{12,max} \le q\right) \qquad (6.30)$$

Since $Q_{1,max}$., $Q_{2,max}$., and $Q_{12,max}$. are random variables representing maximum values of Poisson pulse processes, Eq. 6.26 can be used to model each one.

Making the appropriate substitutions of Eq. 6.26 into Eq. 6.30, we get

$$\begin{aligned} F_{Q_{max}}(q) &\\ \approx \exp\left\{-\lambda_1 T\left[1 - F_{X_1}(q)\right]\right\} \exp\left\{-\lambda_2 T\left[1 - F_{X_2}(q)\right]\right\} \exp\left\{-\lambda_{12} T\left[1 - F_{X_{12}}(q)\right]\right\} &\\ \approx \exp\left\{-\lambda_1 T\left[1 - F_{X_1}(q)\right] - \lambda_2 T\left[1 - F_{X_2}(q)\right] - \lambda_{12} T\left[1 - F_{X_{12}}(q)\right]\right\} & \qquad (6.31) \end{aligned}$$

Note that Eq. 6.31 requires some knowledge of the CDP of the intensity, X_{12}, of the coincidence process; for rectangular pulses, this intensity is simply $X_1 + X_2$.
Further, the approximation in Eqs. 6.30 and 6 .31 is based on the assumption that $Q_{1,max}$, $Q_{2,max}$, and $Q_{l2,max}$ are mutually independent of each other.
$Q_{1,max}$ and $Q_{2,max}$ are independent since $Q_1(t)$ and $Q_2(t)$ were assumed to be independent loads.
However, $Q_{12,max}$ is positively correlated with both $Q_{2,max}$, and $Q_{l2,max}$.
The approximation is conservative because it tends to overestimate the probability of exceedance of a particular value of q (Wen, 1990).

EXAMPLE 6.4.
Consider a combination of the form shown by Eq. 6.27. Assume that the parameters describing the two processes $Q_1(t)$ and $Q_2(t)$ are as follows:

$$\lambda_1 = 2/\text{year} \qquad \mu_{\tau_1} = 1 \text{ day} = \frac{1}{365} \text{ year} \qquad X_1 \text{ is normal with } \mu = 1.2, \sigma = 0.3$$

$$\lambda_2 = 5/\text{year} \qquad \mu_{\tau_2} = 2 \text{ days} = \frac{2}{365} \text{ year} \qquad X_2 \text{ is normal with } \mu = 1.5, \sigma = 0.4$$

A time interval of 50 years (T = 50) is considered.

(a) Determine the probability of exceeding a load value of 2.7 in 50 years.

(b) Determine the value of combined load which has a probability of exceedance of 10 percent in 50 years.

Solution. First we need to determine A_{12} using Eq. 6.28a:

$$\begin{aligned}\lambda_{12} &= \lambda_1\lambda_2(\mu_{\tau_1} + \mu_{\tau_2}) \\ &= (2)(5)\left(\frac{1}{365} + \frac{2}{365}\right) \\ &= 8.219 \times 10^{-2}/\text{year}\end{aligned}$$

Next we need to determine the CDF for the coincidence process. Since the pulse processes are rectangular, the resulting intensity of the coincidence process is simply

$$X_{12} = X_1 + X_2$$

Since X_1 and X_2 are both normal random variables and since they are assumed to be independent, their sum is also a random variable. 50, using the results of Chapter 3, we can show that

$$\mu_{12} = \mu_1 + \mu_2 = 2.7$$

$$\sigma_{12} = \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{0.25} = 0.5$$

(a) We substitute the given infonnation into Eq. 6.31 and determine the value of the

CDF for a value of 2.7. The result is

$$F_{Q_{max}}(q = 2.7) = 9.142 \times 10^{-2}$$

The probability of exceedance is simply l – (9.142×10^{-2}) = 0.9086. Thus there is about a 91 percent chance that the maximum load will exceed 2.7 in 50 years.

(b) We have to find the value of q such that

$$P(Q_{max} > q) = 1 - F_{Q_{max}}(q) = 0.10$$

or

$$F_{Q_{max}}(q) = 0.90$$

By trial and error (or by plotting the function), the value of q is about 3.67.

The load coincidence model can be extended to include more than two load components, and other enhancements and modifications are also possible.