5.3. Minimum critical sets of elements conjugate with the failure mechanisms

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The limit state analysis of a structure concerns its turning into a mechanism (kinematic system). The possible mechanisms of a structure are based on its geometry, joint connections and loads. Based on an initial static analysis elements of cross-sections of a bar structure may be detected to be decisive for a structural failure. They are called **decisive (critical) elements** of a structure, usually corresponding to a maximum combination of cross-sectional forces. The limit load-carrying capacity $N(\omega)$ of each element is random.

In general, decisive elements of a bar structure are their real components (cross-section, joint, member) to be investigated separately.

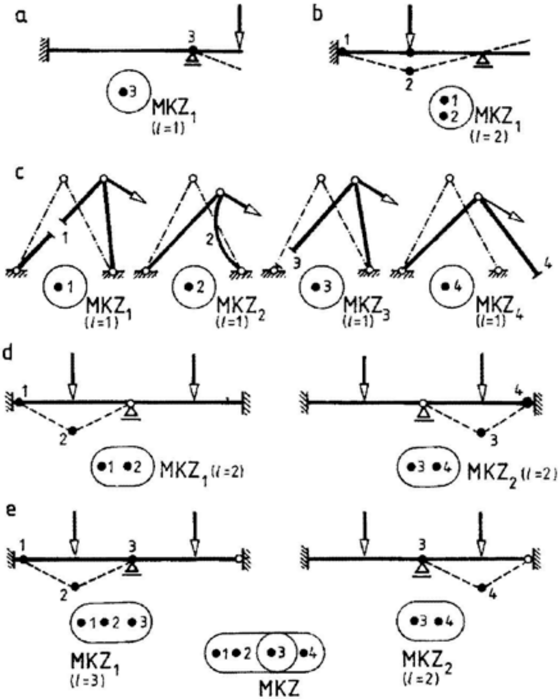
Any structure is a combination of decisive elements whose parameters are random.
The safe structure – mechanism transformation is governed by a **kinematically admissible failure mechanism** (in Polish - KDMZ) of a structure. Every such a mechanism is attached a **minimum critical set of decisive elements** (in Polish – MKZ).

A **minimum critical set of decisive elements** has at least one component proper for the structure to survive (e.g. resist its loads). If all the **critical sets** are failed the structure becomes a mechanism.

Figs. **5.11a, b** show a beam and its three decisive elements. The cantilever load case leads to a mechanism – critical set, containing one decisive element only (l = 1). The span load leads to another mechanism – critical set of two decisive elements (l = 2).

Four decisive elements make the structure in Fig. **5.11 c**: members and supports, failure occurs while breaking only one. Four mechanisms – critical sets exist, each of one decisive element only (l = 1).

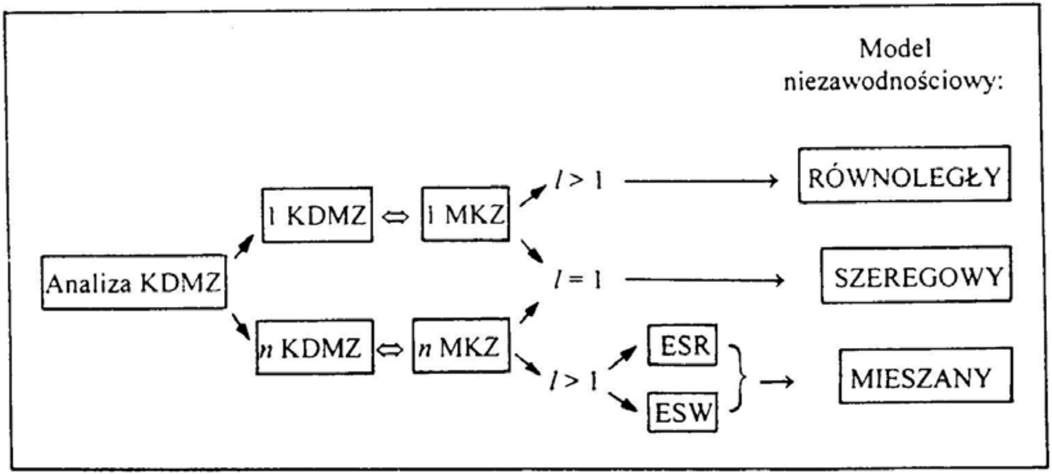
The beam in Fig. **5.11 d** involves four decisive elements (cross-sections) and two mechanisms – critical sets, each having l = 2.



Rys. 5.11. Identyfikacja mechanizmów zniszczenia konstrukcji stowarzyszonych z minimalnymi krytycznymi zbiorami

The beam in Fig. **5.11 e** involves two mechanisms – critical sets, having l = 2 and l = 3, respectively. Note that both mechanisms involve a common decisive element (3) – section at the support.

Fig. 5.12 classifies bar systems due to their minimum critical sets - mechanisms and the number of decisive elements in each of them.



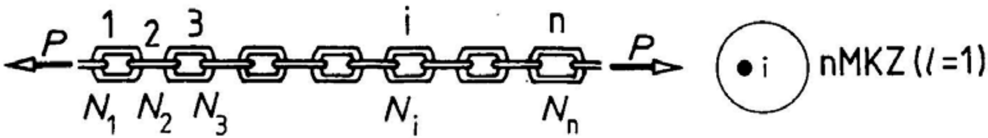
Rys. 5.12. Identyfikacja modeli niezawodnościowych konstrukcji prętowych

KDMZ – kinematically admissible mechanism of failure
MKZ – minimum critical set of decisive elements.

Each decisive element is assessed more or less important for a structure by means of a weight – a ratio of the element cross-sectional force and the total structural load.

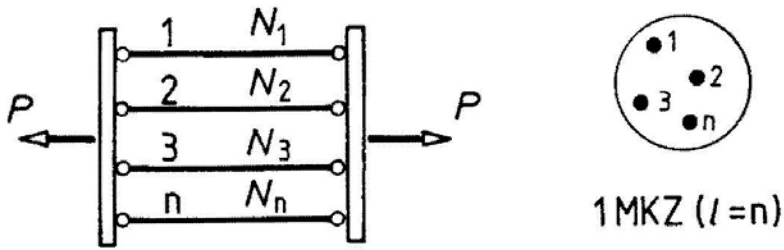
5.4. Discrete structural reliability models

Relations of decisive elements define a structure for its reliability check. Series, parallel and mixed (hybrid) systems are distinguished. A **series** system breaks if **one of** its decisive **elements** breaks only. Mechanical model is a chain of n components at tension, Fig. 5.13., breaking of its *i*-th part of a resistance N_i is a structural collapse.



Rys. 5.13. Schemat ideowy szeregowego modelu niezawodności konstrukcji

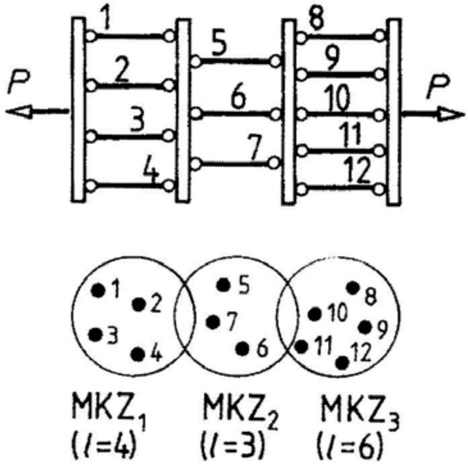
Series systems are „weakest-link" systems of n decisive elements and n mechanisms – critical sets, each having *l* = l.
A **parallel system** breaks due to a given mechanism while **all** decisive **elements** of this mechanism fail. A simple parallel model is a bundle of elements at tension (Fig. 5.14).



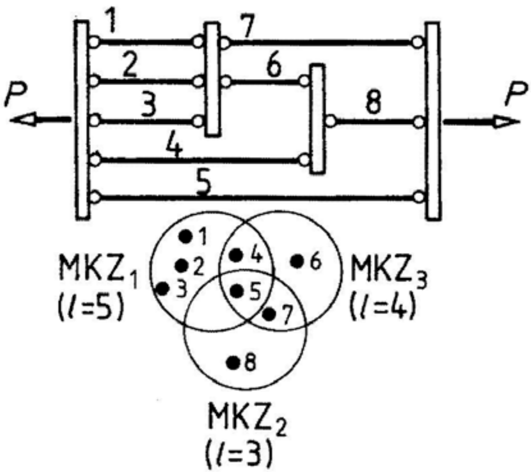
Rys. 5.14. Schemat ideowy równoległego modelu niezawodności konstrukcji

Mechanism = failure means all elements broken.
Each minimum set of elements contains *l* > 1 decisive elements.
Thus a single critical set is distinguished, of *l* = *n*.
Parallel models result in weighted summation of resistances of decisive elements.

Mixed (hybrid) models combine two former ones, Figs. 5.15, 5.16.



Rys. 5.15. Schemat ideowy szeregowego sprzężenia zbiorów o równoległych połączeniach elementów sprawczych



Rys. 5.16. Schemat ideowy sprzężenia zbiorów o równoległych połączeniach elementów sprawczych z elementami wspólnymi

Fig. 5.15 shows a series hyper-chain of elements - parallel bundles of decisive elements. Fig. 5.16 shows a mixed system, having common elements in critical sets, not denoted in the previous case.

Identification of a system is denoting its decisive elements and kinematic mechanisms, then relations between elements and mechanisms. General classification of systems may be proposed:

l) *n* decisive elements of a **series** connection – the number of mechanisms is equal to the number of elements, each critical set is one-element only, $l = 1$;

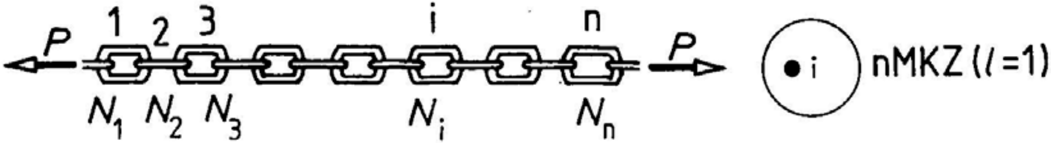
2) having $l > 1$ decisive elements of a **parallel** connection – a single mechanism linked to a single critical set of elements, $l > 1$;

3) series system of a number of critical sets of $l > 1$ (Fig. 5.15);

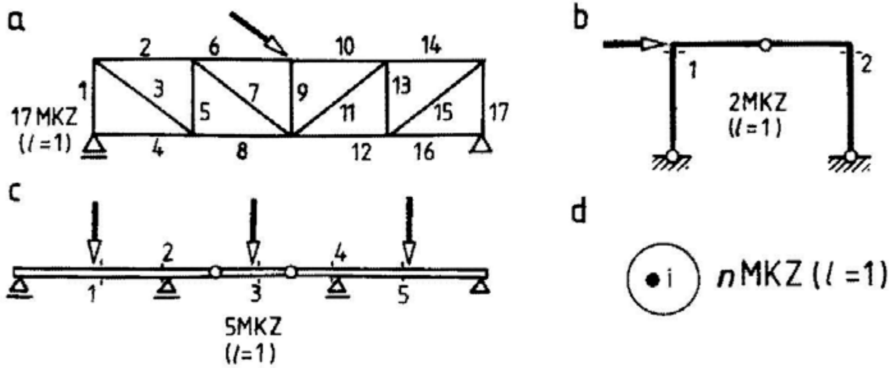
4) a number of critical sets of decisive elements of $l > 1$, decisive elements may be common for various mechanisms (Fig. 5.16).

5.5. Series connections of decisive elements

Statically determinate structures take one failed element for a total collapse. Thus the weakest link (cross-section, joint, member) defines the system safety, its resistance is crucial for the system.
A simple model is a chain (Fig. 5.13), examples shown in Fig. 5.17.



Rys. 5.13. Schemat ideowy szeregowego modelu niezawodności konstrukcji



Rys. 5.17. Przykłady schematów konstrukcji o szeregowym modelu niezawodnościowym

Fig. 5.13 shows a chain at tension, made of *n* links of elastic-plastic Prandtl material – a series system. Decisive elements are links of the chain at their yield failure. A single link at yield breaks the chain.

Fig. 5.17 a is a determinate truss girder, loaded at joints. Decisive elements are axially loaded members. Buckling of any compressed member or yield of any tensile member disables the system.

Fig. 5.17 b shows a statically determinate frame. Any plastic hinge makes the system collapse.

Fig. 5.17c shows a Gerber statically determinate. Any additional hinge (plastic) turns the system into a mechanism.
Each of the presented structures exhibits *l*= 1, the number of critical sets (mechanisms) is equal to the number of elements *n* (Fig. 5.17d). Thus series model is an attribute of statically determinate systems.

Random resistance of a series system $N(\omega)$ is stated by a weight resistance $a_i N_i(\omega)$ of a weakest link (weakest decisive element).

$$N(\omega) = \min_{i=1}^{n} a_i N_i(\omega) \tag{1}$$

$N_i(\omega)$ – random capacity of an *i*-th decisive element,
a_i – weight of an *i*- th decisive element,
n – the total number of decisive elements.

Reliability of a series system is based on known distributions of element resistances N_i and element load effects X_i .

The simplest **chain model** (Fig. 5.13) **assumes equal element forces**, so all the weights equal a_i = l.
Statically determinate structures, shown in Figs. 5.17a - c, detect **various element forces** X_j , thus various weights a_i.

Deterministic models may assume identical resistance of chain links and various capacities of truss members or frame cross-sections.

Random approach investigates reliability of decisive elements, due to their random resistances $N_i(\omega)$ and random load effects $X_i(\omega)$.
Note that load effects X_i are not external actions – they may be bar element cross-sectional forces produced by the loads.

Probability of a service of an *i*-th decisive element – its reliability

$$p_i = \Pr\{N_i > X_i\} \qquad (2)$$

Reliability of a series model is the probability of a proper function of all its elements, is equal to

$$R = \prod_{i=1}^{n} p_i \qquad (3)$$

n – the number of decisive elements,
p_i – reliability of an *i*-th element.

In practice the so-called design resistance (capacity) is investigated, due to structural loading at a given significance level.
Design load-carrying capacity N_{0i} of a decisive element (or a structure) corresponding to a standard Gaussian t is a fractile

$$N_{0i} = \bar{N}_i - t s_{Ni} \qquad (4)$$

where $\bar{N}_i$ is a mean capacity of an *i*-th element (or a structure), s_{Ni} - standard deviation of a capacity of an *i*-th element (or a structure).

The t value may be derived from

$$t = \frac{\bar{N}_i - N_{0i}}{s_{Ni}} \qquad (5)$$

Design capacity of a series model on a given significance level is the N_0 value corresponding to resistance R equal to the predicted, target value R_0 (e.g. R_0 = 0,99865). The equation yields

$$R_0 = \prod_{i=1}^{n} p_i \tag{6}$$

The trial-and-error solution may be used: the load X is imposed, resulting in element cross-sectional forces X_i. Then partial safety of elements $p_i = \Pr\{N_i > X_i\}$ is obtained, finally, equation (6).

The N_0 value making the equation (6) fulfil is a design capacity of a structure on a given significance level (target reliability R_0).

Reliability of a series system is a probability of load X resisting for all decisive elements, determined from (4).

Random load X produces load effects $X_i(\omega)$ at elements i = 1, 2, ...

Resistances of decisive elements $N(\omega)$ are also assumed random.

Random safety margin $Z_i(\omega)$ of a decisive element of random resistance $N_i(\omega)$ and random load $X_i(\omega)$ equals

$$Z_i(\omega) = N_i(\omega) - X_i(\omega) \qquad (7)$$

The mean value $\bar{Z}_i$ and standard deviation S_{Zi} of a random safety margin of a decisive element are computed by

$$\bar{Z}_i = \bar{N}_i - \bar{X}_i \qquad (8)$$

$$s_{Zi} = \sqrt{s_{Ni}^2 + s_{Xi}^2} \qquad (9)$$

Reliability index t_i of a decisive element i of a resistance denoted by $\bar{N}_i$, S_{Ni} and load denoted by $\bar{X}_i$, S_{Xi} equals

$$t_i = \frac{\bar{N}_i - \bar{X}_i}{\sqrt{s_{Ni}^2 + s_{Xi}^2}} \qquad (10)$$

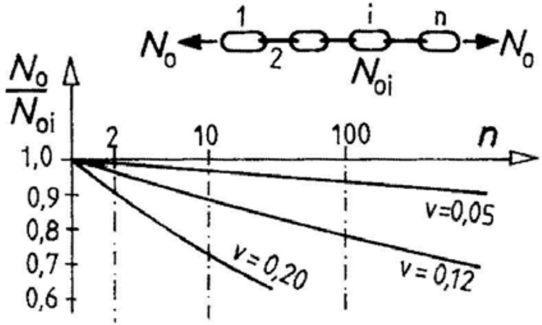
Standard Gaussian tables numerically estimate reliability of an *i*-th $R_i = P_i(t_i)$ in terms of t_i. **Formula (3) points out safety reduction of a series system with the increasing number of elements.**

Given a single decisive element reliability $R_1 = p_1 = 0{,}9$ a structure composed of three identical elements detects $R_3 = 0{,}9^3 = 0{,}729$. **Statistical weakening** is an important feature of series systems.

Physically, statistical weakening for large numbers of decisive elements means an increasing probability for a single element that the load is larger or the resistance is smaller than their respective mean values.

The number of minimum critical sets, thus the number of series-connected elements makes this effect increase.

Fig. 5.18 shows ratio of a fractile N_0 of a system capacity to a fractile N_{0i} of a single element, in terms of a number of elements *n* the coefficients of variation equal to ν_i = 0,05; 0,12; 0,20. A logarithmic scale is used for n, fractiles N_0 i N_{0i} are computed at an identical safety level $P(t_0)$.



Rys. 5.18. Zmiana parametrów bezpieczeństwa konstrukcji, o szeregowym modelu niezawodnościowym, w funkcji liczby jej elementów sprawczych

Given a chain of 10 elements, their identically distributed resistance of a mean $\bar{N}_i$ = 18,0 kN and standard deviation S_{Ni} = 1,21 kN. Task:

- design capacity for a component $p_i(t) = p(3)$;(R_i = 0,99865);
- reliability of a chain randomly loaded, $\bar{X}$ =12 kN, S_x = 1,2 kN;
- design capacity of a chain given its target reliability R =0,99865.

Design capacity of a link for its target reliability R_i = 0,99865 is

$$N_{0i} = \bar{N}_i - ts_{Ni} = 18,0 - 3 \cdot 1,21 = 14,37 \text{ kN}$$

The mean safety margin (limit state function mean) and reliability index of a single link loaded by X force: $\bar{X} = 12$ kN, S_x = 1,2 kN are

$$\bar{Z} = \bar{N}_i - \bar{X} = 18,0 - 12,0 = 6,0 \text{ kN}$$

$$t = \frac{\bar{N}_i - \bar{X}_i}{\sqrt{s_{Ni}^2 + s_{Xi}^2}} = \frac{18,0 - 12,0}{\sqrt{1,21^2 + 1,2^2}} = \frac{6,0}{1,7} = 3,53$$

Gaussian tables show p_i(3,53) = 0,9997922. Reliability of a 10-link chain, randomly loaded by an X force of $\bar{X} = 12,0$ kN, S_x = 1,2 kN is

$$R_0 = \prod_{i=1}^{10} p_i = p_j^{10} = 0,9997922^{10} = 0,997024$$

In order to assess the design capacity of a chain given its target reliability R = 0,99865 the following equation is stated

$$R_0 = \prod_{i=1}^{10} p_i = p_j^{10} = 0,999865$$

$$p_i = \sqrt[10]{0,99865} = 0,999865$$

In order to achieve a system reliability R = 0,99865 each link has to follow p_i = 0,999865 - a Gaussian CDF value for t = 3,65.

Design capacity related to a target system reliability R = 0,99865 is

$$N_0 = \bar{N} - ts_N = 18,0 - 3,65 \cdot 1,21 = 13,58 \text{ kN}$$

The ratio: system (chain) resistance) to element (link) resistance is

$$N_0 \,/\, N_{0i} = 13,58 \,/\, 14,37 = 0,945$$

This effect means statistical reduction of system resistance.

Series systems – remarks

Reliability of a series system is a product of the components reliabilities, assuming their independence.

$$Q = \prod_{i=1}^{n} Q_i$$

where Q_i denotes reliability of an i-th element.

A reliability-consistent system of Q_i = const shows $Q = Q_i^n$

Reliability of a series system increases together with the element reliabilities, decreases in the case of element reliability decrement. Reliability of large series systems is significantly reduced even in the case of high element reliability, e.g. $Q_i = 0.999$, variable n

$n = 10$ gives $Q = 0.999^{10} = 0.990$,
$n = 100$ gives $Q = 0.999^{100} = 0.905$,
$n = 1000$ gives $Q = 0.999^{1000} = 0.3677$.

While the number of elements tending to infinity ($n \to \infty$) the system resistance CDF tends to a selected extreme value distributions, regardless of the CDFs of elements:

Gumbel – infinite (unbounded) random variables

$$F(R)=1-\exp\left(-\exp\frac{R-\hat{R}}{\mu_R}\right)$$

Weibull – left-bounded random variables

$$F(R)=1-\exp\left[-\left(\frac{R-C^{1/\upsilon_R}}{\hat{R}-C}\right)\right]; \quad R>C$$

Frechet– right-bounded random variables

$$F(R)=1-\exp\left[-\left(\frac{C-\tilde{R}}{C-R}\right)^{1/\upsilon_R}\right]; \quad R<C$$