THE POINT ESTIMATE METHOD

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(part of the doctor thesis of P. Sorn)

The majority of the available methods, dealing with structural stability with uncertain geometrical or material parameters are numerical, among which the most popular are Stochastic Finite Element Method (SFEM) and Monte Carlo Simulation.

The only restriction of applying of the Monte Carlo methods is the time of calculations.

It allows estimating results for random variables or fields described also by a high coefficient of variations, and for linear or nonlinear engineering problems.

Few methods, for example stratified sampling, Latin hypercube sampling, and Russian roulette were formulated to reduce the time of calculations.

There are also a few analytical and valid only for relatively simple structures solutions, as First Order Second Moment (FOSM) or Second Order Second Moment (SOSM).

The expected value, standard deviation or probabilistic moments of higher order of any random function can be efficiently determined using point estimate method (PEM).

Rosenblueth (1975) proposed a more direct method for this purpose, based on numerical integrating.
This method essentially involves the use of a Gaussian quadrature for determining the probabilistic moments of random function.
The method is straightforward, easy to use, and requires little knowledge of probability theory.
PEM uses a series of point estimates – point-by-point evaluations of the response function at selected values of the input random variables – to compute the moments of the response variable.
The method applies appropriate weights to each of the point estimates of the response variable to compute moments.
It can be readily applied to response functions that are not closed-form or explicit, and to the results of existing deterministic programs.
A limitation of the Rosenblueth's point estimate method for multiple variables is that it requires calculations at 2^n points.
This significantly increases computational time and effort.
Rosenblueth (1975) also proposed a technique for reducing the number of calculation points to $2n+1$ in a case of uncorrelated variables and when skewness can be ignored.

2. THE POINT ESTIMATE METHOD

The point estimate method will be briefly presented and discussed in the paper, based on the work of Rosenblueth (1975), Christian and Baecher (1999), Baecher and Christian (2003).

In the point estimate method, a continuous random variable is generally replaced by a discrete random variable consisting of *N* pulses (Fig. 1), with a probability distribution

$$p_x(x) = \sum_{i=1}^{N} \delta(x - x_i)\, p(x_i) \tag{1}$$

where: $\delta(x - x_i)$ is Dirac delta, and $p(x_i)$ is the probability assigned to points x_i, for fixed values of random variables.

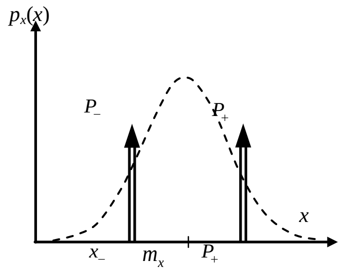


Fig.1. Distribution and probability density function

There can be infinite number of distributions described by (1).
It is most important to find a small number of points N, that have the same probabilistic moments of order k, as the probability density function $f_x(x)$.
The number of points N selected for estimation depends on the order of probabilistic moments that must be included into analysis.
Most frequently two points are chosen.
In practice, it allows for quite accurate determination of the average value and the variance of a random function.
Comparing the first three probabilistic moments, determined in accordance with the definition of density or distribution probability, one can write:

$$m_x = \int_{-\infty}^{\infty} x f_x(x)\,dx = \sum_{i=1}^{N} x_i\,p(x_i)$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - m_x)^2\, f_x(x)\,dx = \sum_{i=1}^{N} (x_i - m_x)^2\, p(x_i) \qquad (2)$$

$$\gamma_1 \sigma_x^3 = \int_{-\infty}^{\infty} (x - m_x)^3\, f_x(x)\,dx = \sum_{i=1}^{N} (x_i - m_x)^3\, p(x_i)$$

where: m_x – expected value, σ_x – standard deviation, γ_1 – asymmetry coefficient of random variable X.

While transforming the random variable X into Y using function $y = g(x)$, a probability distribution is sought $p_y(y)$, where probabilistic moments can be easily computed.
These values approximate moments of the continuous function $f_y(y)$.
The expected value of the k^{th} -power of the discrete random variable Y, which replaces the continuous function $y = g(x)$, can be approximated as follows:

$$E\left[Y^k\right] \approx \sum_{i=1}^{N} g^k\left(x_i\right) p\left(x_i\right) \tag{3}$$

In the simplest case, a continuous random variable with given values m_x, σ_x and γ_1 can be represented by two probabilities $p(x_1) = P_-$ and $p\left(x_2\right) = P_+$ (called weights) are assigned to the points $x_1 \equiv x_-$ and $x_2 \equiv x_+$ (Fig. 1).

Taking into account that the sum of the probabilities must be equal to unity, and including notations introduced in (2), one can write a following system of equations:

$$
\begin{aligned}
&P_- + P_+ = 1 \\
&m_x = x_- P_- + x_+ P_+ \\
&\sigma_x^2 = \left(x_- - m_x\right)^2 P_- + \left(x_+ - m_x\right)^2 P_+ \\
&\gamma_1 \sigma_x^3 = \left(x_- - m_x\right)^3 P_- + \left(x_+ - m_x\right)^3 P_+
\end{aligned} \tag{4}
$$

Its solution is the discretization of the points and the weights assigned to them:

$$
\begin{aligned}
&x_- = m_x + \left[\frac{\gamma_1}{2} - \sqrt{1+\left(\frac{\gamma_1}{2}\right)^2}\right]\sigma_x \;, \;\; x_+ = m_x + \left[\frac{\gamma_1}{2} + \sqrt{1+\left(\frac{\gamma_1}{2}\right)^2}\right]\sigma_x \\
&P_- = \frac{1}{2}\left[1+\frac{\gamma_1}{2}\frac{1}{\sqrt{1+\left(\frac{\gamma_1}{2}\right)^2}}\right], \;\; P_+ = 1 - P_-
\end{aligned} \tag{5}
$$

Significant simplification is achieved for zero skewness, i.e. symmetric distribution. Substituting $\gamma_1 = 0$ into (5), one obtains:

$$x_- = m_x - \sigma_x \ , \ \ x_+ = m_x + \sigma_x \ , \ \ P_- = \frac{1}{2} \ , \ \ P_+ = \frac{1}{2} \tag{6}$$

Considering the above in (3), and substituting $N = 2$, this expression takes the form:

$$E\left[Y^k\right] \approx g^k\left(x_-\right)P_- + g^k\left(x_+\right)P_+ \tag{7}$$

So, two first probabilistic moments are given by the formulas:

$$\begin{aligned} m_y &= E\left[Y^1\right] \approx y_- P_- + y_+ P_+ \\ \sigma_y^2 &= E\left[Y^2\right] - m_y^2 \approx y_-^2 P_- + y_+^2 P_+ - m_y^2 \end{aligned} \tag{8}$$

where: $y_+ = g\left(x_+\right)$, $y_- = g\left(x_-\right)$.

In the case where Y is a function of n, uncorrelated random variables $X_1, X_2, ..., X_n$, each with zero asymmetry, i.e. $y = g(x_1, x_2, ..., x_n)$, replace each continuous random variable with a discrete variable.

Probabilities are defined for the two values x_- and x_+ (smaller and higher than the expected value of the standard deviation).

Together one should select 2^n points for estimation.

For each point, a certain probability, that can be determined from the general formula below, is assigned:

$$P_{(s_1 s_2 \ldots s_n)} = \frac{1}{2^n}\left[1 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(s_i)(s_j) r_{x_i x_j}\right] \tag{9}$$

where: $s_i = \begin{cases} -1 & \text{dla } x_{i-} = m_{x_i} - \sigma_{x_i} \\ +1 & \text{dla } x_{i+} = m_{x_i} + \sigma_{x_i} \end{cases}$,

$r_{x_i x_j}$ – cross-correlation coefficient between random variables X_i and X_j.

In the given approach, probabilistic moments of random functions are written as the sum of all possible combinations multiplied by the products of probabilities. Generalizing (3) to the case of n random variables, one can write:

$$E\left[Y^k\right] \approx \sum (y_i)^k P_i \tag{10}$$

where: y_i – value of the function designated for x_i, wherein i is a suitable combination of characters + i – describing the position of the point x_i, P_i – the probability assigned to the point x_i, according to (9).

For example, for two ($n = 2$) random variables X_1 i X_2, formula (9) takes the form:

$$P_{(s_1 s_2)} = \frac{1}{4}\left[1 + (s_1)(s_2) r_{x_1 x_2}\right] \tag{11}$$

The probabilities can be denoted with a series of + or − indicators, in a way that the first sign refers to the variable X_1, the second to X_2, the third to X_3, etc. Probabilities $P_{--}, P_{+-}, P_{-+}, P_{++}$ should be determined at four points (Fig. 2), with the coordinates: $(m_{x_1} - \sigma_{x_1}, m_{x_2} - \sigma_{x_2})$, $(m_{x_1} + \sigma_{x_1}, m_{x_2} - \sigma_{x_2})$, $(m_{x_1} - \sigma_{x_1}, m_{x_2} + \sigma_{x_2})$, $(m_{x_1} + \sigma_{x_1}, m_{x_2} + \sigma_{x_2})$, respectively.

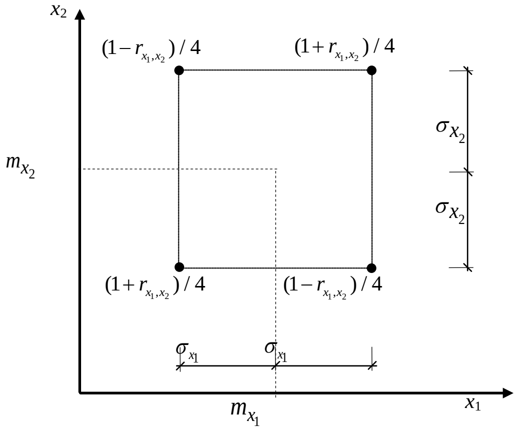


Fig. 2. The probabilities for the two correlated random variables

According to the formula (11) and the introduced indicators, the probabilities are equal to:

$$P_{++} = P_{--} = \frac{1}{4}\left[1 + r_{x_1 x_2}\right] \quad , \quad P_{+-} = P_{-+} = \frac{1}{4}\left[1 - r_{x_1 x_2}\right] \tag{12}$$

In the case that X_1 and X_2 are uncorrelated ($r_{x_1,x_2} = 0$), the probabilities at all the points are the same and equal to 0.25.

The expected value and variance of random value Y according to (10), are given by the formulas:

$$\begin{aligned} m_y &\approx y_{++}P_{++} + y_{+-}P_{+-} + y_{-+}P_{-+} + y_{--}P_{--} \\ \sigma_y^2 &\approx y_{++}^2 P_{++} + y_{+-}^2 P_{+-} + y_{-+}^2 P_{-+} + y_{--}^2 P_{--} - m_y^2 \end{aligned} \tag{13}$$

where: $y_{++} = g(x_{1+}, x_{2+})$, $y_{+-} = g(x_{1+}, x_{2-})$, $y_{-+} = g(x_{1-}, x_{2+})$, $y_{--} = g(x_{1-}, x_{2-})$

In the above notation $g_{++}(x_{1+}, x_{2+})$ means that a value of the function y is calculated for $x_{1+} = m_{x_1} + \sigma_{x_1}$, $x_{2+} = m_{x_2} + \sigma_{x_2}$, etc.

ROSENBLUETH'S 2*k*+ 1 POINT ESTIMATE METHOD

Complex problems may also be solved by using a point estimate method. Although there are many such methods, the 2K + 1 method proposed by Rosenblueth (1975) is one of the easiest to implement.

Loosely speaking, this method may be thought of as a simulation technique in which the number of simulations is N = 2K + 1 where K is the number of input random variables.

The basic idea is to evaluate a function of random variables at 2K + 1 key points and then to use this information to estimate the mean and variance (or coefficient of variation) of the function.

However, the CDF of the function cannot be obtained by this method.

Consider a function *Y* described by

$$Y = f\left(X_1, X_2, ..., X_k\right) \tag{14}$$

where $f\left(\bullet\right)$ is some deterministic function (but possibly not known in closed form) and the X_i (i = 1, 2, ... , k) are the random input variables. The steps in Rosenblueth's 2k+1 method are as follows:

1. Determine the mean value (μ_{X_i}) and standard deviation (σ_{X_i}) for each of the k input random variables.

2. Define y_0 as the value of (14) when all input variables are equal to their mean values, that is,

$$y_0 = f\left(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_k}\right) \tag{15}$$

3. Evaluate the function Y at $2K$ additional points as follows. For each random variable X_i evaluate the function at two values of Xi which are shifted from the mean value μ_{X_i} by $\pm\sigma_{X_i}$ while all other variables are assumed to be equal to their mean values. These values of the function will be referred to as y_i^+ and y_i^- ·The subscript denotes the variable which is shifted, and the superscript indicates the direction of the shift. In mathematical notation,

$$y_i^+ = f\left(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_i} + \sigma_{X_i}, \ldots, \mu_{X_k}\right) \tag{16a}$$

$$y_i^- = f\left(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_i} - \sigma_{X_i}, \ldots, \mu_{X_k}\right) \tag{16b}$$

4. For each random variable, calculate the following two quantities based on y_i^+ and y_i^-:

$$\bar{y}_i = \frac{y_i^+ + y_i^-}{2} \tag{17a}$$

$$v_{y_i} = \frac{y_i^+ - y_i^-}{y_i^+ + y_i^-} \tag{17b}$$

5. Calculate the estimated mean and coefficient of variation of Y as follows:

$$\bar{Y} = y_0 \prod_{i=1}^{k} \left(\frac{y_i}{y_0} \right) \tag{18a}$$

$$v_Y = \sqrt{\left\{ \prod_{i=1}^{k} \left(1 + v_{y_i}^2 \right) \right\} - 1} \tag{18a}$$

There are two distinct advantages to this method.
First, it is not necessary to know the distributions of the input random variables; only the first two moments are needed.
Second, the number of function evaluations (i.e., "simulations") is relatively small compared to Latin hypercube sampling or general Monte Carlo simulation.

3. RETICULATED SHELL PROBABILISTIC CALCULATIONS

A reticulated shell was chosen to check the usefulness of the point estimate method in engineering structure probability calculations. It is well known that three-dimensional shallow trusses are extremely sensitive to variations of loads and model parameters.

A typical example of a reticulated structure (Fig. 5) described among others by Rakowski and Kacprzyk (1993), was analyzed. The diameter of a three-dimensional truss was 50 m long; the truss was 8.216 m high. Tubular sections RO 647.8x20 were designed for the structural elements. All elements were made of S355 steel. The elements were connected by means of ball joints.

A simplest type of loading was assumed, i.e. a single force placed at the highest point of the structure – node no. 13 (Fig. 5). The load is not a realistic one but it was expected that it caused an extremely sensitive mechanical response of the structure. On the bases of the obtained preliminary results a reliability calculation concerning realistic loading as wind or snow can be easily performed.

Calculations were made using the MSC Nastran code (2001). The geometric and material nonlinearities were taken into account.

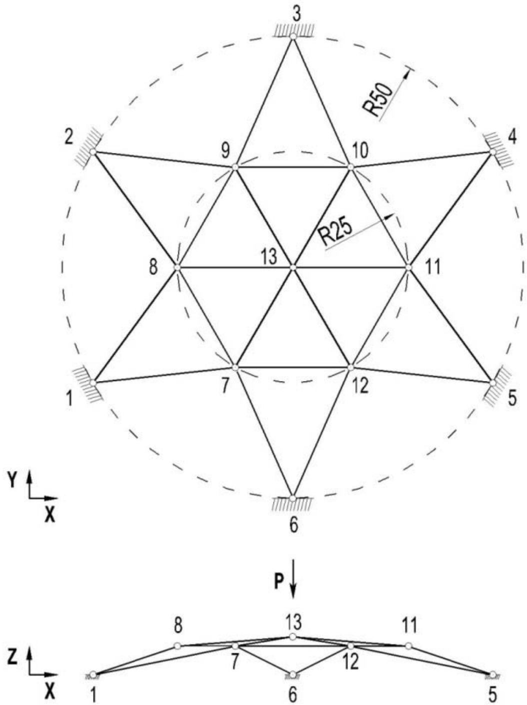


Fig.5. Three dimensional truss structure (reticulated shell)

One-dimensional random variables

First, the limit load for the ideal structure was calculated $R = 2612.3$ kN (Fig. 6).

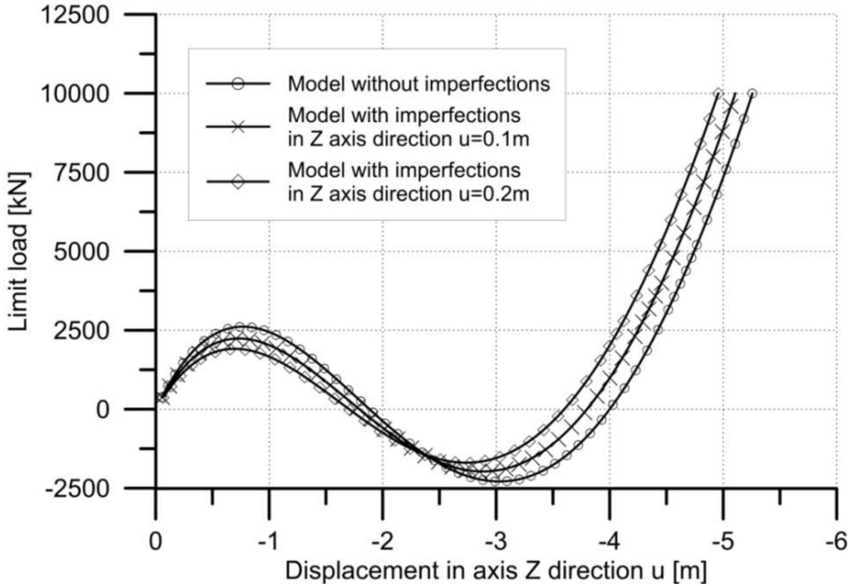


Fig.6. The equilibrium path – the ideal structure

On the basis the limit state function was defined as the value of the admissible limit load. Only geometric imperfections were considered. Several preliminary examples were made, proving that even small changes in the geometric description, i.e. the nodal vertical coordinates – resulted in considerable changes of the limit load. For example, displacement of the highest structure point (no. 13 in Fig. 5) only 0.10 m down led to a 14.3% drop of the limit load. It should be pointed out that the horizontal displacements do not influence visibly the results. Because of that they were not taken in the analysis.

Next the probabilistic analysis was performed. The geometric discrepancy – the vertical displacement – was assumed having following parameters mean value $m_u = 0.0$ m and standard deviation $\sigma_u = 0.08\,\text{m}$. This way the majority of the generated values of imperfections belong to the following interval (–0.13, 0.13) m. The assumed range of imperfections fulfills the maximum limits of discrepancies which are allowed during the production of the tubular elements. The initial vertical displacements of the nodes 7 – 12 were calculated proportionally to the discrepancies of the node no. 13 (Fig. 5). The results of the analysis can be considered realistic and engineering sound.

To obtain a result which can be assumed as a reference limit load, the direct Monte Carlo method was applied. This method does not impose any restriction to the solved random problems, and allows estimating results for random variables and random fields described by a high coefficient of variation. Any linear or nonlinear engineering problem can be analysed. Using this method, any commercial deterministic program can be implemented in the probabilistic analysis without any adjustments. The only restriction of applying the Monte Carlo methods is time of calculations.

The calculations were performed for the generated 55 samples – vertical displacements of the node 13 (Fig. 5). For each generated value, the response is calculated, and on this basis, the distribution of the outcomes can be estimated. The accuracy of the results depends on the number of the samples included in the calculations. However, nonlinear calculations for such number of initial data are impossible due to a long computation time. To determine the minimal but sufficient number of field samples (which allows estimating the results with a prescribed accuracy), a convergence analysis of the outcomes was performed. After each sample calculations, the improved expected values and standard deviations of the outcomes were estimated. Then, their variability with the growing number of samples was analysed. The calculations were finished when the scatter of the estimated values reached the assumed low level. Such analysis reduced significantly the sample numbers. It should be emphasised that during the calculation process, the random field samples were not chosen or arranged in any way; they were numerically analysed in the generation order. The results are presented in Fig.

7, 8 and 9. According to these calculations the following values of the critical load are assumed as a reference load: mean value $\mu_R = 2613.53\,\text{kN}$, standard deviation $\sigma_R = 203.27\,\text{kN}$ and skewness $\gamma_R = 0.0686$.

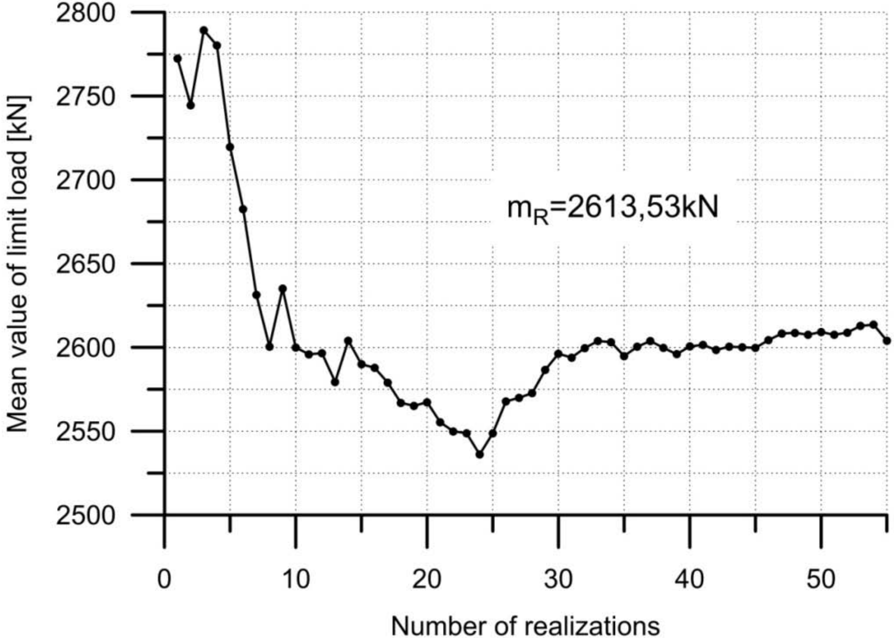


Fig. 7. The direct Monte Carlo method – expected value

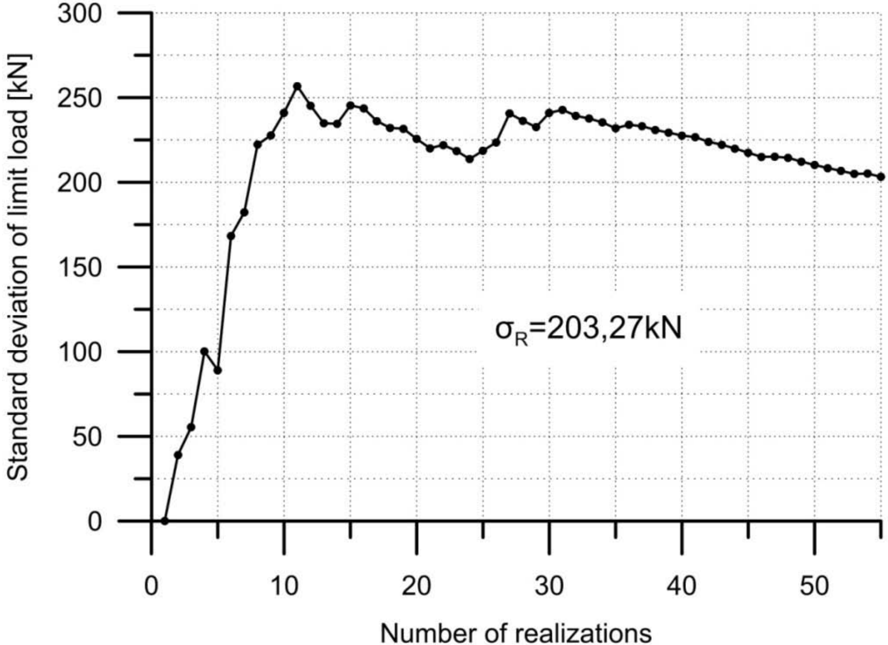


Fig. 8. The direct Monte Carlo method – standard deviations

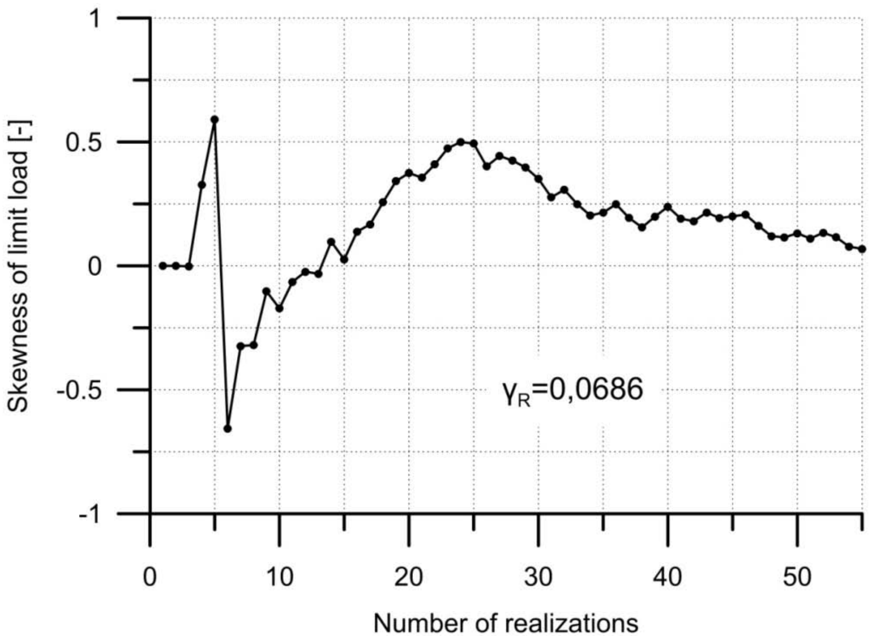


Fig. 9. The direct Monte Carlo method – skewness

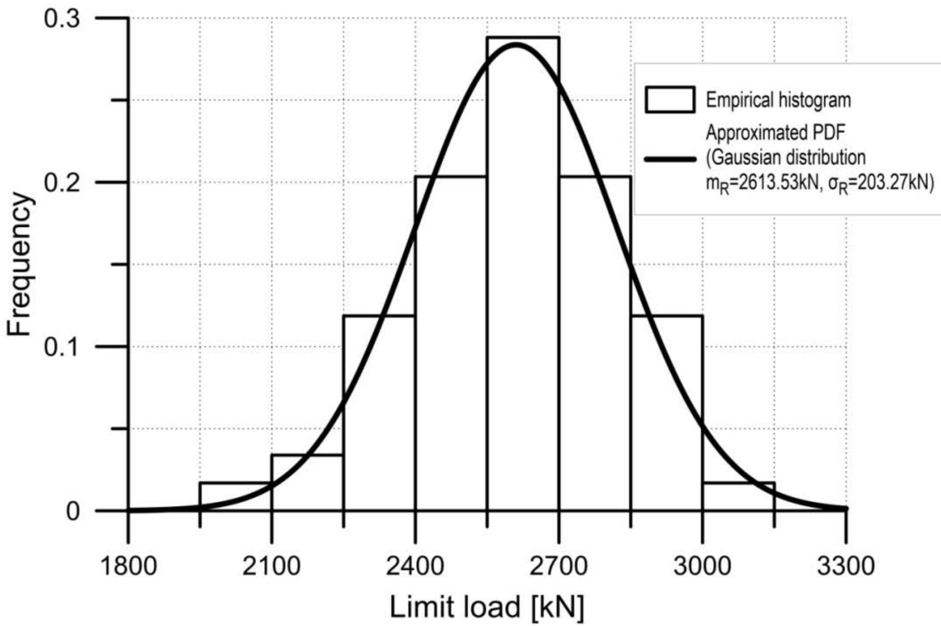


Fig. 10. The direct Monte Carlo method – PDF and histogram of limit load

A significance reductions of the number of the random samples can be achieved using the Monte Carlo reduction methods. Here, the method of stratified sampling was applied. The set of 55 random initial displacements identical to the case of the direct Monte Carlo method was used. The generated samples were classified and arranged according to the magnitude of the displacements. Two methods were applied: the whole space of the samples was divided into equal subsets or subsets of equal probability. From each subset only one sample was chosen for the analysis. The results of the calculations are presented in Figs 11 and 12. Additionally in Tab. 2 the results of similar calculations using 15 and 10 samples of the direct Monte Carlo and reduced methods are listed The comprehensive analysis proved that the stratified sampling with intervals defined by equal probabilities is the most suitable method of reliability estimations.

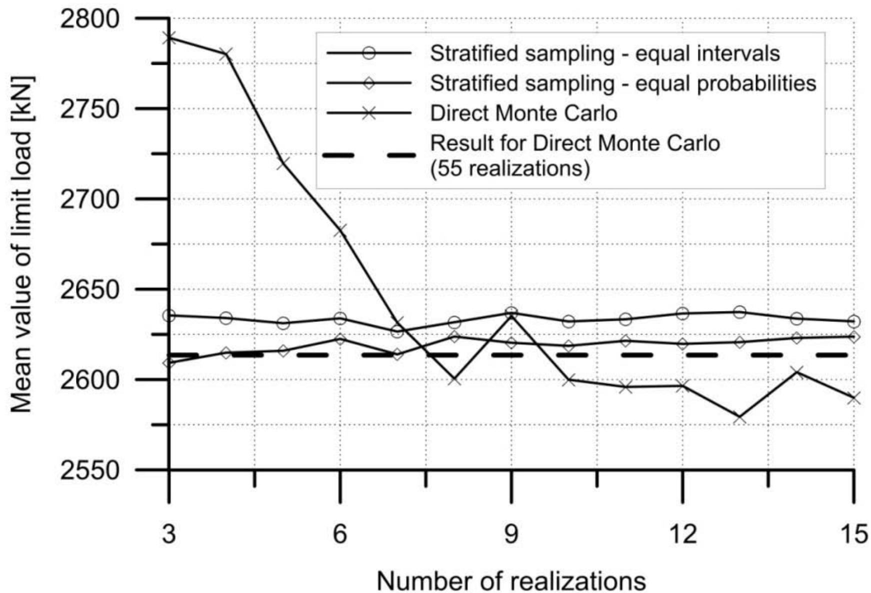


Fig. 11. Comparison of the Monte Carlo and stratified sampling – mean values

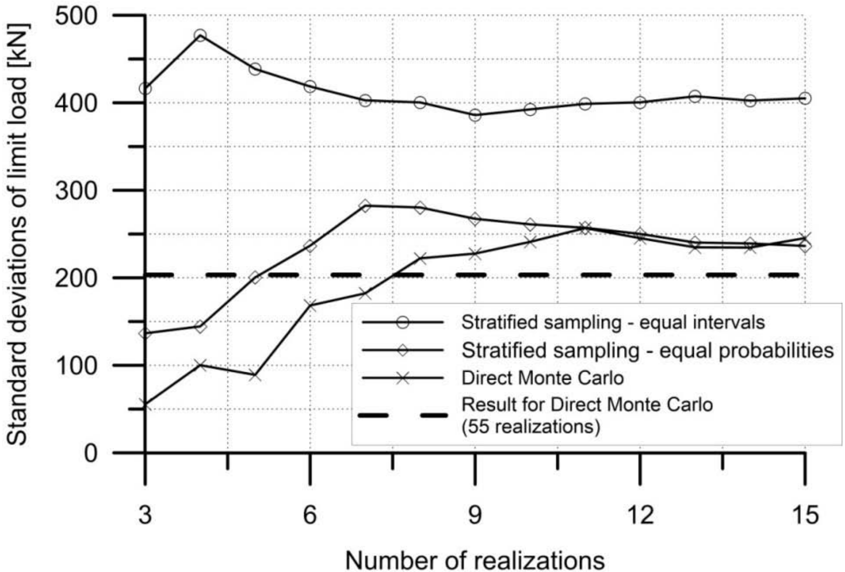


Fig. 12. Comparison of the Monte Carlo and stratified sampling – standard deviations

Tab. 2. Results of limit load analysis of the reticulated shell – one dimensional case

Method used	Number of realizations *n*	Mean value of limit load *R* [kN]	Standard deviation limit load *R* [kN
Direct Monte Carlo	59	2610.40	212.74
Direct Monte Carlo	15	2604.08	245.38
Stratified sampling – equal intervals	15	2632.22	405.12
Stratified sampling – equal probabilities	15	2623.79	236.43
Direct Monte Carlo	10	2635.18	241.06
Stratified sampling – equal intervals	10	2632.18	392.44
Stratified sampling – equal probabilities	10	2618.66	261.10
Point Estimate Method	2	2605.12	239.26

A significant reduction of the samples was expected using the point estimate method (Przewłócki 2007). According to (6) two displacement and their weights are calculated:

$$x_- = m_u - \sigma_u = 0.0 - 0.08 = -0.08 \qquad P_- = 0.5$$

$$x_+ = m_u + \sigma_u = 0.0 + 0.08 = 0.08 \qquad P_+ = 0.5$$

Then the reticulated shell capacities related to x_- and x_+ are calculated

$$y_- = R\left(x_-\right) = 2844.38 \text{ kN}$$

$$y_+ = R\left(x_+\right) = 2362.86 \text{ kN}$$

Substituting the above results to (8) the expected value and standard deviation were determined for the limit load:

$$m_y = E\left[Y^1\right] \approx 2844.38 \cdot 0.5 + 2365.86 \cdot 0.5 = 2605.12 \,\text{kN}$$

$$\sigma_R^2 = E\left[Y^2\right] - \mu_y^2 = 2844.38^2 \cdot 0.5 + 2365.86^2 \cdot 0.5 - 2605.12^2 = 57245.35 \,\text{kN}^2$$

$$\sigma_R = \sqrt{57245.35} = 239.26 \quad \text{kN}$$

The results were compared with the Monte Carlo solution (Tab. 2). It should be stressed that using the point estimated method only two calculations performed by the finite element program allows estimating the mean value and standard deviation of the limit load of the reticulated shell. The errors of these values with the respect to the direct Monte Carlo methods are 0.2% and 12.5% respectively. Because of the promising results the mechanical response of the reticulated shells described by two and more random variable was analysis.

3.2 Two-dimensional random variables

The next step of the analysis concerns two random variables – geometric and material parameters. The displacements of the truss nodes were the same as in the case of the one-dimensional analysis.

The second parameter – Young modulus – was assumed as normally distributed with the mean value $m_E = 210.0$ GPa and the standard deviation $\sigma_E = 4.0\,\mathrm{GPa}$.

In the case of the Monte Carlo method 60 samples – pairs of the node displacements and Young modulus were generated (see Fig, 13). The following results were obtained: the mean value $\mu_R = 2653.21\,\mathrm{kN}$, standard deviation $\sigma_R = 232.54\,\mathrm{kN}$ and skewness $\gamma_R = 0.1004$. Similar calculation were performed for 36, 24, 16 and 9 samples (see Tab. 3). The results were compared with the stratified sampling methods.

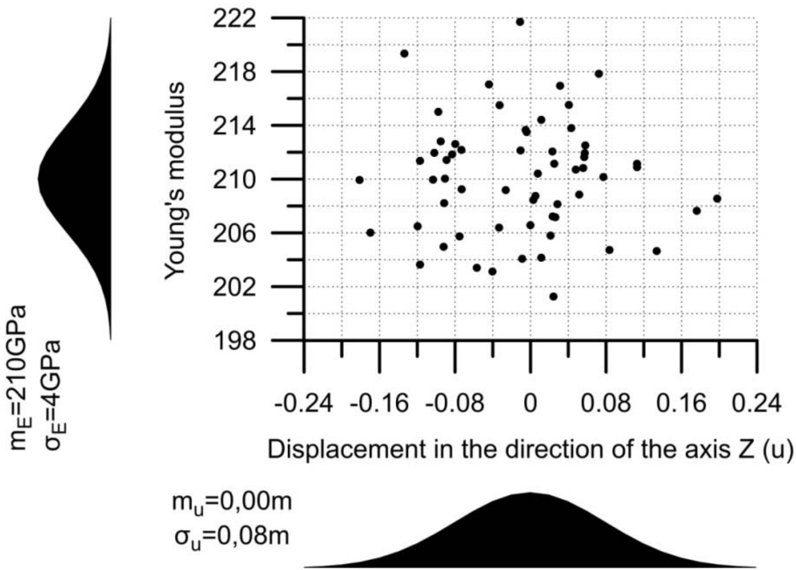


Fig. 13 Monte Carlo Method – Ant Hill

Tab. 3. Results of limit load analysis of the reticulated shell – two dimensional case

Method used	Number of realizations *n*	Mean value of resistance *R* [kN]	Standard deviation of resistance *R* [kN]
Direct Monte Carlo	60	2653,21	232,54
Direct Monte Carlo	36	2658,71	229,21
Stratified sampling – equal intervals 6x6	36	2637,43	388,28
Stratified sampling – equal probabilities 6x6	36	2604,47	255,01
Direct Monte Carlo	25	2651,48	227,16
Stratified sampling – equal intervals 5x5	25	2645,16	410,19
Stratified sampling – equal probabilities 5x5	25	2630,21	237,22
Direct Monte Carlo	16	2602,15	253,78
Stratified sampling – equal intervals 4x4	16	2647,99	389,89
Stratified sampling – equal probabilities 4x4	16	2614,67	299,57
Direct Monte Carlo	9	2728,93	309,69
Stratified sampling – equal intervals 3x3	9	2633,06	348,67
Stratified sampling – equal probabilities 3x3	9	2656,06	328,89
Point Estimate Method	2	2620,48	229,38

Point	$x_1\ [m]$	$x_2\ [GPa]$	$y\ [kN]$	$p(x_{i,j})$	$yp(x_{i,j})$	$y^2 p(x_{i,j})$
1	$x_{1-} = -0.08$	$x_{2-} = 206$	2790.18	0.25	697.545	1946276
2	$x_{1+} = 0.08$	$x_{2+} = 214$	2443.23	0.25	610.8075	1492343
3	$x_{1-} = -0.08$	$x_{2+} = 214$	2898.30	0.25	724.575	2100036
4	$x_{1+} = 0.08$	$x_{2-} = 206$	2350.19	0.25	587.548	1380848
					$\sum = 2620.475$	$\sum = 6919503$

$$E[R] = 2620.475kN\ , Var[R] \approx 6919503 - 2620.475^2 = 52614.07kN^2\ , \sigma_R \approx 229.38kN$$

3.3 PEM witch reduction of evaluation points

Dla przypadku z pojedynczą siłą skupioną przyłożoną w węźle 13 dokonano estymacji parametrów rozkładu prawdopodobieństwa siły granicznej w zależności od różnych sposobów przyjęcia zmiennych losowych. Rozważono przypadki dwu-, trzy- i czterowymiarowej zmiennej losowej. Dla każdego z przypadków dokonano obliczeń klasycznym algorytmem metody PEM oraz metodami redukcji punktów ewaluacji zaproponowanymi przez Rosenblueth'a oraz Hong'a. Przyjęte kombinacje zmiennych losowych wraz z wartościami ich parametrów stochastycznych przedstawiono w Tab. 4 natomiast wyniki obliczeń w Tab. 5.

Tab. 4. Analizowane przypadki wielowymiarowej zmiennej losowej

Set	Variable dimension	Variable type	Mean value of random variable	Standard deviation of random variable
Set "a"	2	Initial horizontal displacement of node 7-13 calculated proportionally to the displacement of the node 13	0.0 m	0.08 m
		Young modulus	210.0 GPa	4.0 GPa
Set "b"	2	Initial horizontal displacement of node 7-13 calculated proportionally to the displacement of the node 13	0.0 m	0.08 m
		Cross-section area	394.458 cm^2	2.000 cm^2
Set "c"	2	Young modulus	210.0 GPa	4.0 GPa
		Cross-section area	394.458 cm^2	2.000 cm^2
Set "d"	3	Initial horizontal displacement of node 7-13 calculated proportionally to the displacement of the node 13	0.0 m	0.08 m
		Young modulus	210.0 GPa	4.0 GPa
		Cross-section area	394.458 cm^2	2.000 cm^2
Set "e"	4	Initial horizontal displacement of node 13	0.0 m	0.08 m
		Initial horizontal displacement of node 7-12 calculated proportionally to the displacement of node 12	0.0 m	0.03 m

Young modulus	210.0 GPa	4.0 GPa
Cross-section area	394.458 cm^2	2.000 cm^2

Tab. 5. Results of limit load analysis of the reticulated shell estimated using PEM

Set	PEM – original method proposed by Rosenblueth			PEM with reduction of evaluation point – Rosenblueth proposition			PEM with reduction of evaluation point –Hong proposition		
	Number of realization *n*	Mean value of limit load *R* [kN]	Standard deviation of limit load *R* [kN]	Number of realization *n*	Mean value of limit load *R* [kN]	Standard deviation of limit load *R* [kN]	Number of realization *n*	Mean value of limit load *R* [kN]	Standard deviation of limit load *R* [kN]
“a”	4	2620.49	229.38	5	2605,94	246,09	4	2620.70	228.94
“b”	4	2620.24	223.64	5	2603,39	241,03	4	2620.69	223.85
“c”	4	2614.52	51.63	5	2603,39	51,00	4	2613.94	51.78
“d”	9	2620.58	265.34	7	2605,76	246,35	6	2606.38	245.94
“e”	16	2628.49	363.36	9	2658,55	341,40	8	2628.09	363.64

Analizując powyższe rezultaty można zauważyć, że różnice pomiędzy otrzymanymi wynikami dla różnych metod estymacji są nieznaczne. W przypadku metody redukcji punktów ewaluacji zaproponowanej przez Hong'a w czterech z pięciu analizowanych przypadków uzyskaliśmy prawie identyczne wyniki w porównaniu z klasyczną metodą PEM, co w połączeniu z bardzo dużą redukcją ilości obliczeń jaka jest wymagana do estymacji parametrów rozkładu prawdopodobieństwa czyni tą metodę bardzo dobrym narzędziem do rozwiązywania zagadnień stochastycznych.

W tym miejscu należy jednak zwrócić uwagę na to, że w przypadku bardzo dużej ilości zmiennych losowych punkty dla których wykonujemy obliczenia opisane są wartościami tychże zmiennych znacznie odbiegającymi od realnie występujących i mogą zafałszować wyniki. Wady tej nie posiada metoda zaproponowana przez Rosenbulth'a. Współrzędne punktów ewaluacji (zmienne losowe) zawsze przyjmują wartości z zakresu realnie występujących w przyrodzie.

Inną dodatkową zaletą podejścia zaproponowanego przez Rosenbulth'a w porównaniu z podejściem Hoong'a jest łatwość rozszerzenia analizowanego zagadnienia dla większej liczby zmiennych losowych. Dzieje się tak dlatego, że obliczenia dla n-tej zmiennej losowej są wykonywane w punktach, gdzie każda zmienna oprócz n-tej przyjmuje wartość średnią, natomiast n-ta zmienna przyjmuje wartości $m_{x_n} \pm \sigma_{x_n}$. Dzięki temu wyniki otrzymane dla n-tej zmiennej losowej mogą być wykorzystane w kombinacji z różnymi zmiennymi bez dodatkowych obliczeń.

4.0 Wnioski

1. Metoda PEM daje dużo bardziej zbliżone wyniki do wyników otrzymanych za pomocą Monte Carlo niż pozostałe metody redukcyjne, co w połączeniu z bardzo małą liczbą obliczeń oraz dostępnymi mało skomplikowanymi wzorami opisującymi tę metodę czyni ją bardzo przystępną w powszechnym użytkowaniu.
2. Możliwa jest dalsza redukcja liczby potrzebnych obliczeń do oszacowania parametrów rozkładu prawdopodobieństwa wartości, którą zainteresowany jest badacz. W tym celu możliwe jest zastosowanie kilku modyfikacji metody PEM.
3. Metoda redukcji punktów ewaluacji zaproponowana przez Hoong'a pozwala otrzymać wyniki bardzo zbliżone do wyników z klasycznego podejścia PEM – różnice nie przekraczają 0.1%.
4. Konieczne jest zbadanie przydatności metody PEM w przypadku, gdy zmienne losowe są opisane innym rozkładem niż rozkład Gaussa.

5. Konieczne jest zbadanie przydatności metody PEM w przypadku, gdy rozkład estymowany nie jest rozkładem Gaussa – w przytoczonych przykładach wykazano, że rozkład siły granicznej jest rozkładem Gaussa.

OBCIĄŻENIE SNIEGIEM - SYMETRYCZNE

CZTEROWYMIAROWA ZMIENNA LOSOWA – PEM REDUKCJA OBLICZEŃ

Point	x_1 [m]	x_2 [m]	x_3 [GPa]	x_4[cm^2]	y [kN/m^2]	$p(x_{i,j,k})$	$yp(x_{i,j,k})$	$y^2p(x_{i,j,k})$
1	$x_{1-} = -0.16$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	26.09	0.125	3.261	85.09
2	$x_{1+} = 0.16$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	26.16	0.125	3.270	85.54
3	$x_{1,mean} = 0.0$	$x_{2-} = -0.06$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	13.41	0.125	1.676	22.48
4	$x_{1,mean} = 0.0$	$x_{2+} = 0.06$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	14.69	0.125	1.836	26.97
5	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3-} = 202$	$x_{4,mean} = 394.458$	25.12	0.125	3.140	78.88
6	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3+} = 218$	$x_{4,mean} = 394.458$	27.14	0.125	3.393	92.07
7	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4-} = 354.458$	23.45	0.125	2.931	68.74
8	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4+} = 434.458$	28.79	0.125	3.599	103.61
						$\sum$	23.106	563.38

$$E[R] = 2620.475kN \, , Var[R] \approx 563.38 - 23.106^2 = 29.49kN^2 \, , \sigma_R \approx 5.43kN$$

OBCIĄŻENIE SNIEGIEM - NIESYMETRYCZNE

CZTEROWYMIAROWA ZMIENNA LOSOWA – PEM
REDUKCJA OBLICZEŃ

Point	$x_1\ [m]$	$x_2\ [m]$	$x_3\ [GPa]$	$x_4[cm^2]$	$y\ [kN/m^2]$	$p(x_{i,j,k})$	$yp(x_{i,j,k})$	$y^2p(x_{i,j,k})$
1	$x_{1-} = -0.16$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	4.90	0.125	0.6125	3.001
2	$x_{1+} = 0.16$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	5.01	0.125	0.6263	3.138
3	$x_{1,mean} = 0.0$	$x_{2-} = -0.06$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	3.62	0.125	0.4525	1.638
4	$x_{1,mean} = 0.0$	$x_{2+} = 0.06$	$x_{3,mean} = 210$	$x_{4,mean} = 394.458$	3.48	0.125	0.4350	1.514
5	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3-} = 202$	$x_{4,mean} = 394.458$	4.76	0.125	0.5950	2.832
6	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3+} = 218$	$x_{4,mean} = 394.458$	5.15	0.125	0.6438	3.315
7	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4-} = 354.458$	4.44	0.125	0.5550	2.464
8	$x_{1,mean} = 0.0$	$x_{2,mean} = 0.0$	$x_{3,mean} = 210$	$x_{4+} = 434.458$	5.47	0.125	0.6838	3.740
						$\sum$	4.6039	21.642

$$E[R] = 2620.475kN\ , Var[R] \approx 21.642 - 4.6039^2 = 0.446kN^2\ , \sigma_R \approx 0.668kN$$

Mam małą wątpliwość odnośnie algorytmu z redukcją w przypadku gdybyśmy mieli np. 20 zmiennych losowych. W takim przypadku każdy model liczymy dla punktu oddalonego o $2\sqrt{5}\sigma$, co w przypadku imperfekcji geometrycznych może czasami zmienić całkowicie geometrie konstrukcji i to w taki sposób, że całkowicie zmieni pracę konstrukcji. W rozpatrywanej kopule kratowej takim przypadkiem mógłby być przypadek gdy węzły 7-12 znajdą się wyżej niż węzeł 13. Należy jeszcze pamiętać, że w przypadku tych 20 zmiennych losowych już lepiej jest użyć metody Monte Carlo, a to z tego względu, że stosując nawet z algorytm redukujący ilość obliczeń przekraczamy granicę 30 obliczeń.

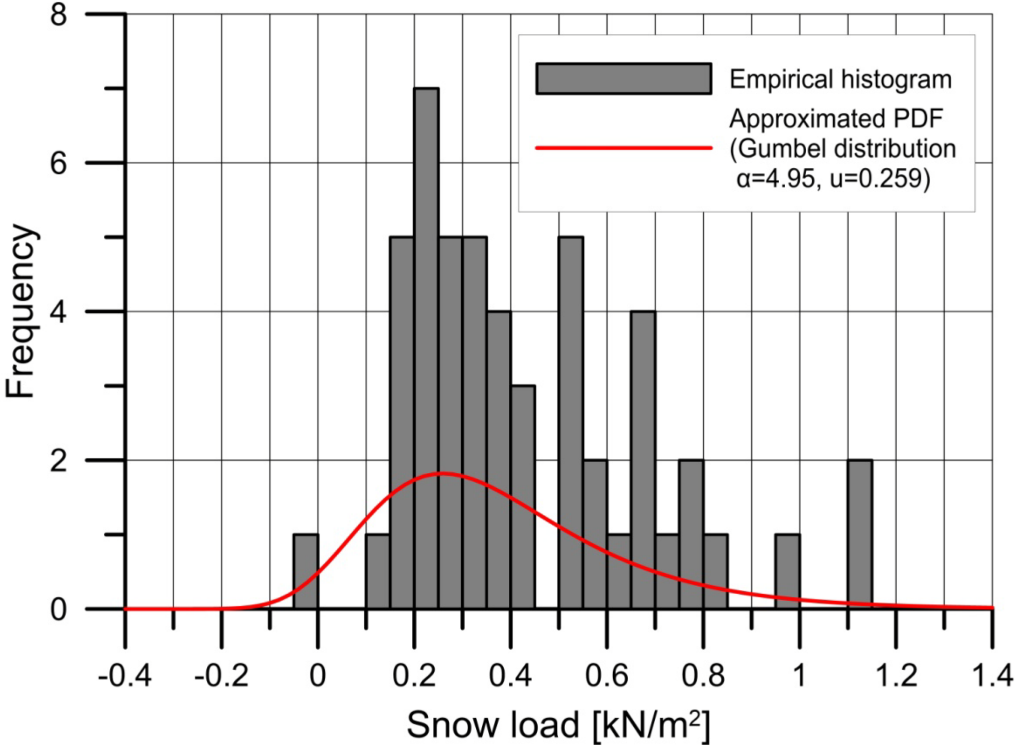
Load type	Variable dimension	Variable type	Number of realization *n*	Mean value of resistance *R* [kN]	Standard deviati of resistance *R* [kN]
Snow symmetric load	4	Initial horizontal displacement of node 13 Initial horizontal displacement of node 7-14 Young modulus Cross-section area	8	23,106	5,43
Snow asymmetric load	4	Initial horizontal displacement of node 13 Initial horizontal displacement of node 7-14 Young modulus Cross-section area	8	4,6039	0,668
Snow symmetric load	7	Initial horizontal displacement for node 13 Initial horizontal displacement for 7node 7-12 (each independently)	4	6,024	3,599

Można wykorzystać metodę PEM do obliczeń niezawodności konstrukcji inżynierskich. Stosując algorytm redukujący liczbę obliczeń można przeprowadzać metodą PEM obliczenia dla większej ilości zmiennych losowych. Przy jednoczesnym uwzględnieniu symetrii układu i obciążenia ilość obliczeń może ulegać dalszej redukcji (tabela powyżej). W przypadku rzeczywistych oddziaływań jakim jest śnieg w symetrycznym oraz niesymetrycznym układzie obciążenia dla rozpatrywanej konstrukcji przemieszczenie węzła 13 nie jest tak niebezpieczne jak w przypadku obciążenia siłą skupioną w wierzchołku. Konstrukcja wykazuje znacznie większą wrażliwość w przypadku wstępnych przemieszczeń węzłów na pierścieniu pośrednim.

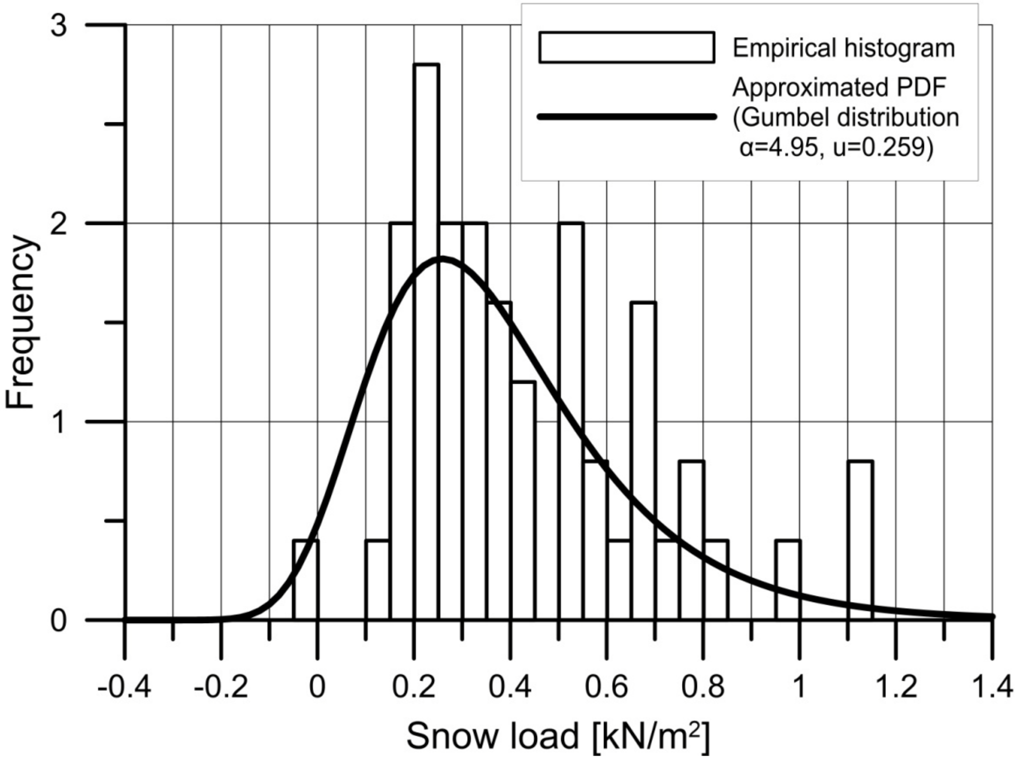
Trzeba się zastanowić czy policzymy to dla większej liczby parametrów:

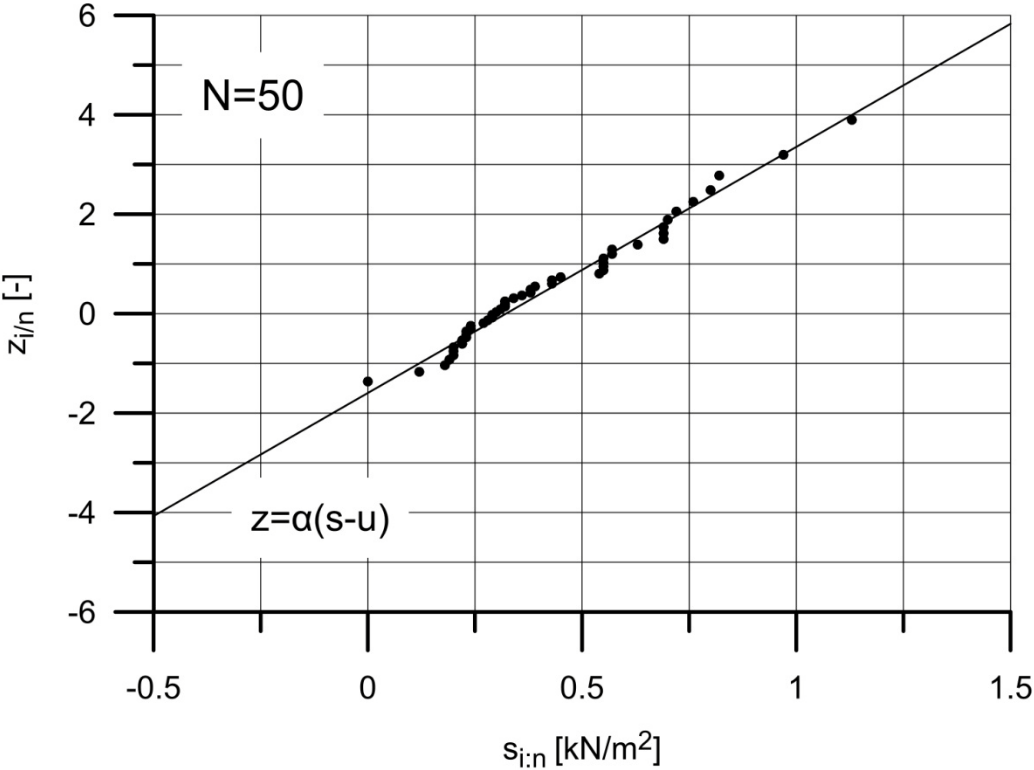
- pole losowe imperfekcji w węzłach – 7 zmiennych?
- dwie zmienne jako przemieszczenia (na różnych wysokościach) + materiał lub przekrój?
- inne propozycje?

3.3 Reliability caluclations



Frequency
Snow load [kN/m²]
Empirical histogram
Approximated PDF
(Gumbel distribution
α=4.95, u=0.259)
0
2
4
6
8
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4


Empirical histogram
Approximated PDF
(Gumbel distribution
α=4.95, u=0.259)
Frequency
3
2
1
0
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
Snow load [kN/m²]

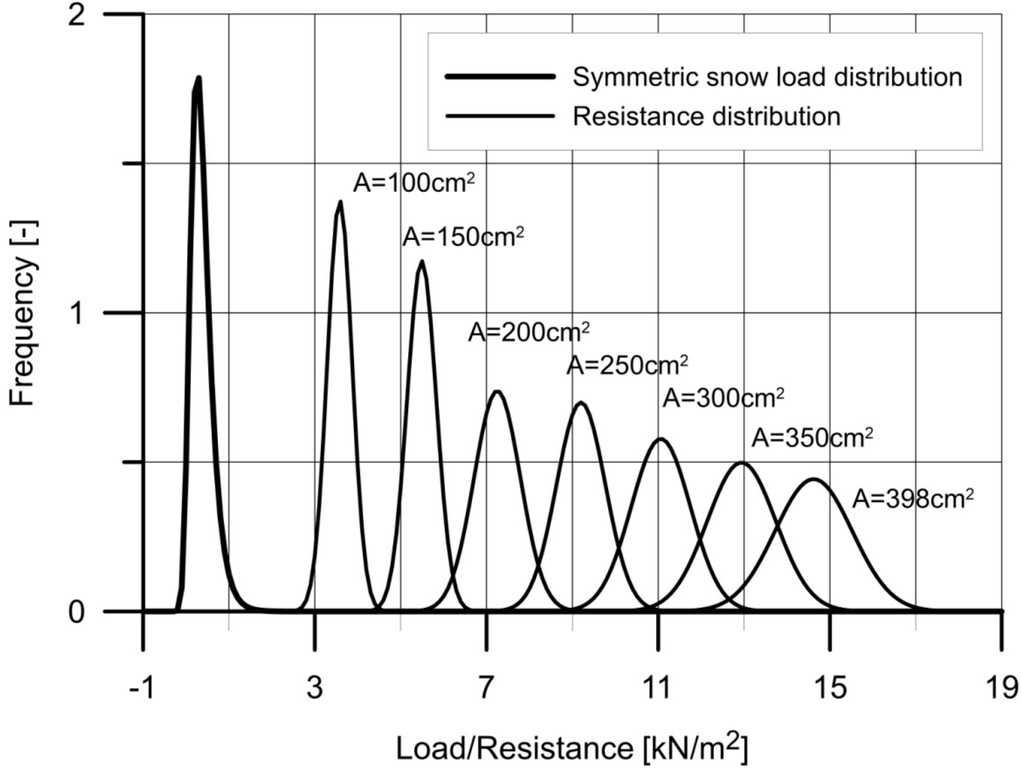


N=50
z=α(s-u)
6
4
2
0
-2
-4
-6
-0.5
0
0.5
1
1.5
z_i/n [-]
s_i:n [kN/m2]

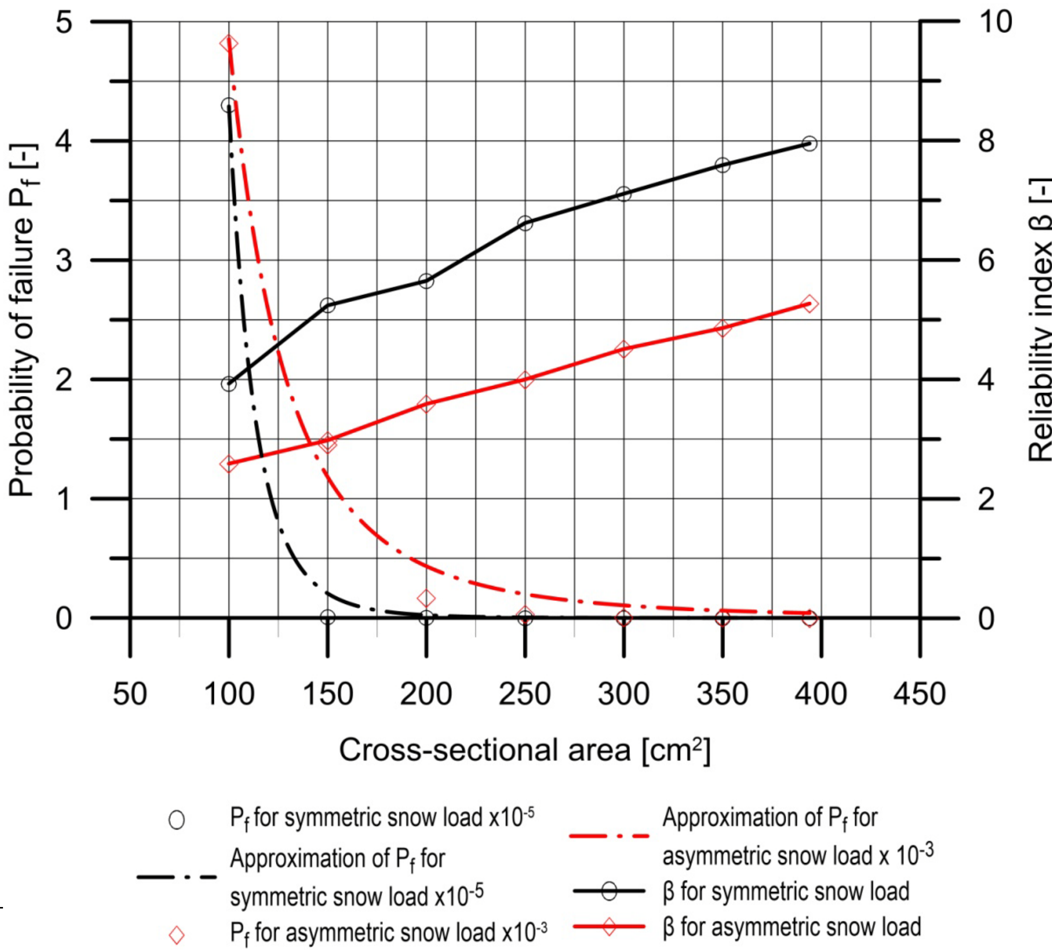
Rozkład wyjścia nośności jest CHYBA (?) normalny. Obciążenie śniegiem jest wg Gumbela więc musimy policzyć dokładna całkę. Zrobię to i opiszę.

Przemek, może narysować (sprawdzić) aproksymację rozkładem Gumbella tych wyników z książki?

Poniżej wykres pokazujący jak zmienia się położenie histogramów obciążenia śniegiem od nośności w zależności od pola przekroju kształtownika.



Symmetric snow load distribution
Resistance distribution
A=100cm²
A=150cm²
A=200cm²
A=250cm²
A=300cm²
A=350cm²
A=398cm²
Frequency [-]
2
1
0
-1
3
7
11
15
19
Load/Resistance [kN/m2]



Probability of failure P_f [-]
Reliability index β [-]
Cross-sectional area [cm²]
0
1
2
3
4
5
0
2
4
6
8
10
50
100
150
200
250
300
350
400
450
P_f for symmetric snow load x10^{-5}
Approximation of P_f for symmetric snow load x10^{-5}
P_f for asymmetric snow load x10^{-3}
Approximation of P_f for asymmetric snow load x 10^{-3}
β for symmetric snow load
β for asymmetric snow load

Funkcje aproksymujące dane są wzorem $y = \alpha x^{\beta}$. Poniżej w tabeli zestawiłem wartości współczynników α i β.

Load type	α	β	Coef of determination
Symmetric snow load	4,2965x1015	-7,5	0,9975
Asymmetric snow load	4,5319x107	-3,4851	0,99