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PROBABILISTIC LIMIT STATE ANALYSIS OF MONUMENTAL STRUCTURE BY MONTE CARLO SIMULATION

Basic design variables – loads, material and geometrical parameters, imperfections, are frequently recognized as random variables. If random loads are acting on a deterministic structure, the problem is called stochastically linear, whereas the assumption of random structure features makes us classify the problem as stochastically nonlinear.

Probabilistic analysis of structural limit states belongs to the class of problems which deal with the stochastic nonlinear operator. Analytical solutions of these problems are not available at all. Numerical methods are then naturally developed in the field of random limit state analysis.

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Sensitivity of a limit state of a structure is assessed with respect to a selected random variable, in the form of the limit state probability distribution. This concept is presented in the form of a problem-oriented Monte Carlo simulation procedure, where the selected variable plays the dominant role.

Numerical example is presented serviceability limit state of the monumental structure. The dominant variable is the wind action, in the form of two correlated random variables.

These investigations can be classified into the branch of computational sciences, because the numerical procedure is the core of the presented concept. Modelling and computer simulation is nowadays regarded the third base of contemporary science, complementary to theory and experiment.

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The Monte Carlo simulation method is a numerical tool of a wide a engineering application. Three steps, due to structural design are:

- generating basic variables of the problem - random variates,
- performing deterministic operations in every simulation step,
- statistics of the set of results, interpreting the histogram.

The basic random variables, with given probability distribution functions, are represented by sets of random numbers.

The elementary event ω is assumed the structural limit state. Thus the sample space Ω consists of the limit states of the structure. The uni-dimensional random variable is defined on the sample space. This variable is the multiplier of the dominant basic variable (group of variables) of the problem. The choice of dominant variables must be completed first.

The procedure key is the performance of a single simulation step. It consists of the following operations:

- generating loads and characteristics of a particular structure in the form of random numbers - establishing a deterministic structure under deterministic loading,
- uni-parametrical increment of dominant variable (or variables), when the limit state is reached, the limit multiplier of dominant basic variables is recorded.

Consequently, various definitions of limit states may be taken.
Assumed the limit states investigated with respect to loads, one-dimensional random variable $\Lambda(\omega)$ is defined on the sample space Ω. Its values are limit load multipliers λ_i of the simulation steps, i = 1, …, N, where N is the number of realizations.

Histogram of the variable $\Lambda(\omega)$ is the estimator of the probability density function of the limit state with respect to loads. The failure probability estimator $\hat{p}_f$ can be calculated by the formula

$$\hat{p}_f = \frac{1}{N}\sum_{i=1}^{N} I(\lambda_i) \qquad (1)$$

where the indicator function is defined as follows

$$I(\lambda_i) = \begin{cases} 1 \;\; for \;\; \lambda_i \leq 1.0 \;\; (\,failure\, region) \\ 0 \;\; for \; \lambda_i > 1.0 \;\; (safe\, region) \end{cases} \qquad (2)$$

While material parameters are dominant basic variables, uni-dimensional random variable $\mathrm{M}(\omega)$ is defined on the sample space Ω. Its values are the limit material multipliers λ_i, $i = 1, \ldots, N$. Histogram of the variable $\mathrm{M}(\omega)$ represents probability distribution of the limit state with respect to material parameters. Thus the failure probability estimator $\hat{p}_f$ can be calculated by the formula

$$\hat{p}_f = \frac{1}{N} \sum_{i=1}^{N} I(\mu_i) \tag{3}$$

where the indicator function is given

$$I(\mu_i) = \begin{cases} 1 \;\; for \;\; \mu_i \geq 1.0 \;\; (failure\, region) \\ 0 \;\; for \;\; \mu_i < 1.0 \;\; (safe\, region) \end{cases} \tag{4}$$

The numerical example concerns probabilistic serviceability limit state analysis of the monumental structure of the Licheń Basilica. consecrated in 2004. The major load-carrying tower part consists of the foundation ring, four-column structure supporting the main ring, the two-storey colonnade and the dome. Both lower and upper storeys of the colonnade consist of concentric 16-column rings. A space frame model includes 224 elements. The upper deck deflection of the colonnade is investigated here.

The main loads acting on the model are: dead load, wind acting on the colonnade walls and the forces on the upper deck of the colonnade, representing the dome's weight and the wind acting on the dome. The wind load is assumed uniform on the columns of lower storey middle ring and on the columns of the upper storey outer ring (rigid plates are provided between the columns).

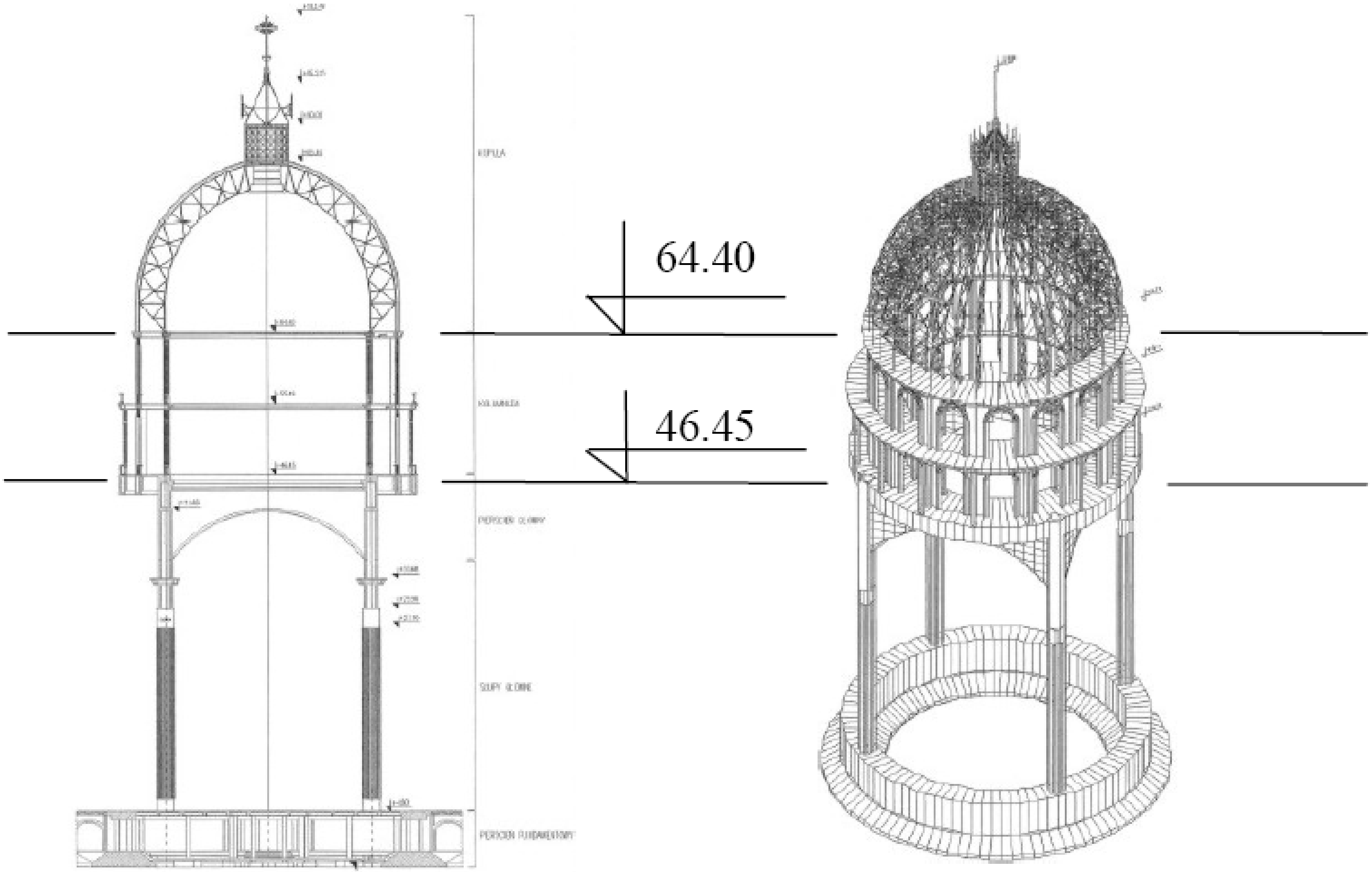

Fig. 1. General view of the tower part of the Basilica, featuring the colonnade

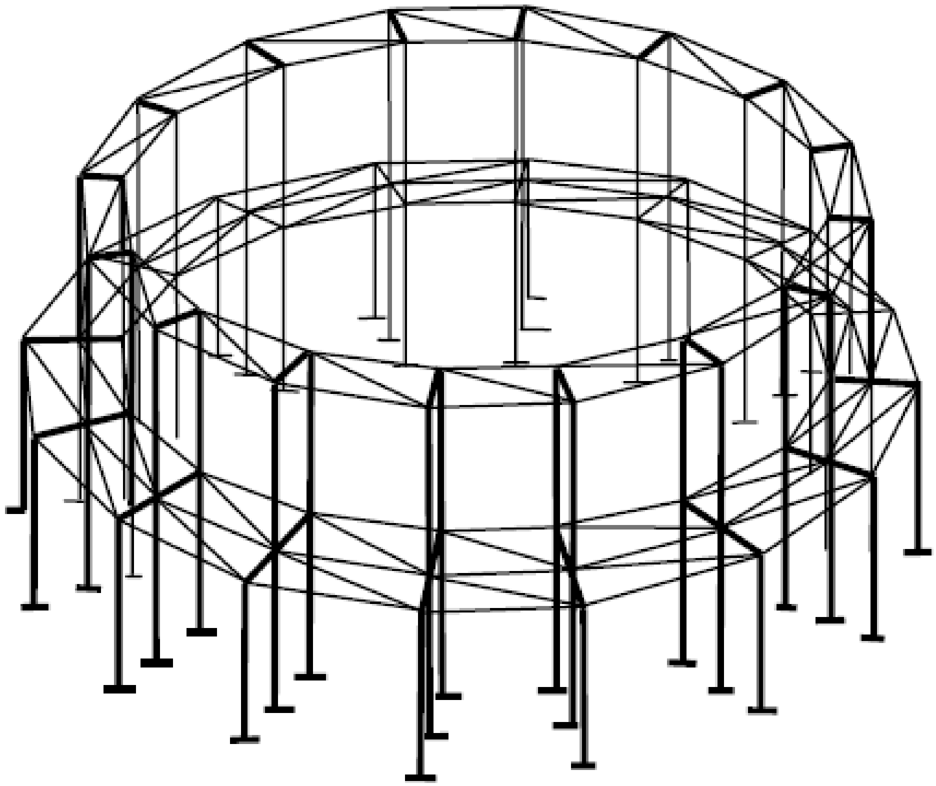

Fig. 3. Space frame model of the Basilica colonnade

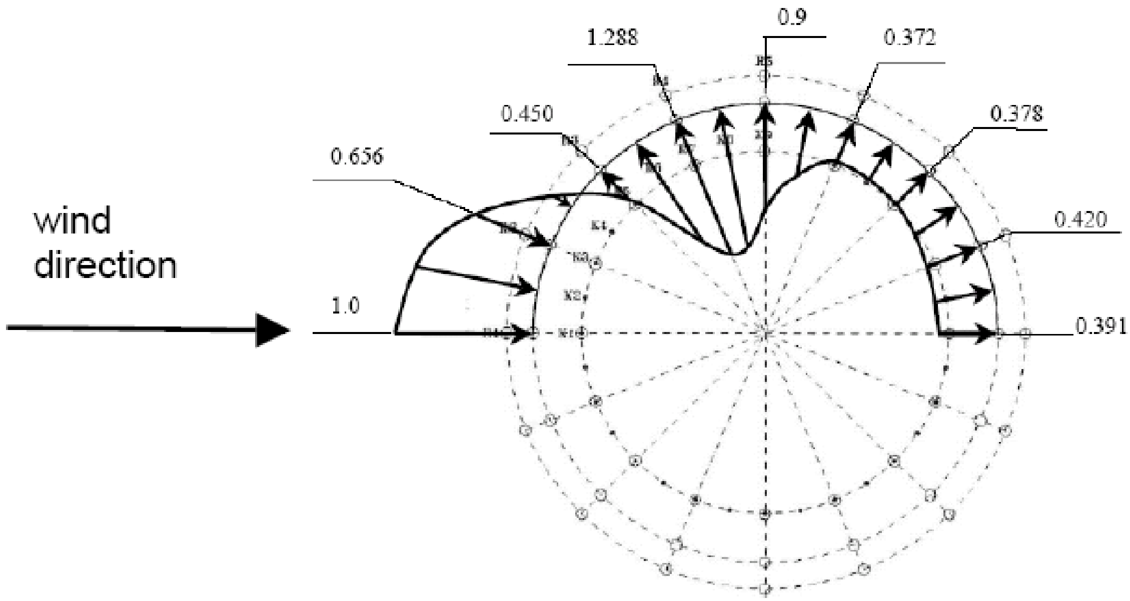

Fig. 4. Situation of the columns of the colonnade on the altitude 46.45 m (three – ring lower colonnade), relative wind load intensities, referring to the maximum value

Basic random variables of the problem were assumed as follows: probability distributions (in the form of bounded histograms):

- Dead load of colonnade and dome: Gaussian, N(1.0; 0.0333), the range (0.9, 1.1) – variable $D(\omega)$,
- Young's modulus of concrete: uniform, the range (0.8; 1.0) – variable $E(\omega)$
- Wind load – the variables: $W_1(\omega)$ for the lower storey and $W_2(\omega)$ for the upper storey and the dome. Both variables are quadratically transformed variables V_1 and V_2 - wind velocities in both intervals. The variables V_1 and V_2 are correlated, Gumbel distributed.

Assumption is made that the dominant variables are both the wind actions W_1 and W_2, thus structural sensitivity to wind actions is examined throughout the example.

Three variants of calculations are performed. They differ in the correlation coefficients of the variables $W_1(\omega)$ and $W_2(\omega)$.

The technique to generate correlated random variables of a given covariance matrix is based on the following theorem, formed by Devroye:

Theorem. Let $\mathbf{X} \equiv \left\{ X_i \right\}, \; i = 1, 2, ..., d$ be the random vector composed of the i.i.d. random variables of zero mean and unit variance. There exists a nonsingular matrix **H**, that fulfils the equation

$$\mathbf{Y} = \mathbf{HX} \qquad (5)$$

where **Y** is the random vector of a given covariance matrix **C**. The matrix **H** may be derived from the equation:

$$\mathbf{HH}^{\mathrm{T}} = \mathbf{C} \qquad (6)$$

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Indeed, the statistical moments of Y satisfy the assumptions:

$$\begin{aligned} &E(\mathbf{Y}) = \mathbf{H}E(\mathbf{X}) = 0 \\ &E\left(\mathbf{Y}\mathbf{Y}^T\right) = \mathbf{H}E\left(\mathbf{X}\mathbf{X}^T\right)\mathbf{H}^T = \mathbf{H}\mathbf{H}^T = \mathbf{C} \end{aligned} . \qquad (7)$$

where $E(\cdot)$ is the expectation operator. No restrictions are introduced on variable types. We will find the matrix **H** from (6), given the matrix **C**. It is possible to build a lower triangular matrix **H** satisfying (7).

In two-dimensional cases, given the correlation coefficient *h*, the matrix **H** may be derived as:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \\ h & \sqrt{1-h^2} \end{bmatrix}. \qquad (8)$$

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The following steps are distinguished in the algorithm:

* Generation of the vector **X** consisting of two uncorrelated random variables, uniformly distributed in the range <0, 1>. All the histograms in the paper show relative frequencies on their ordinates.

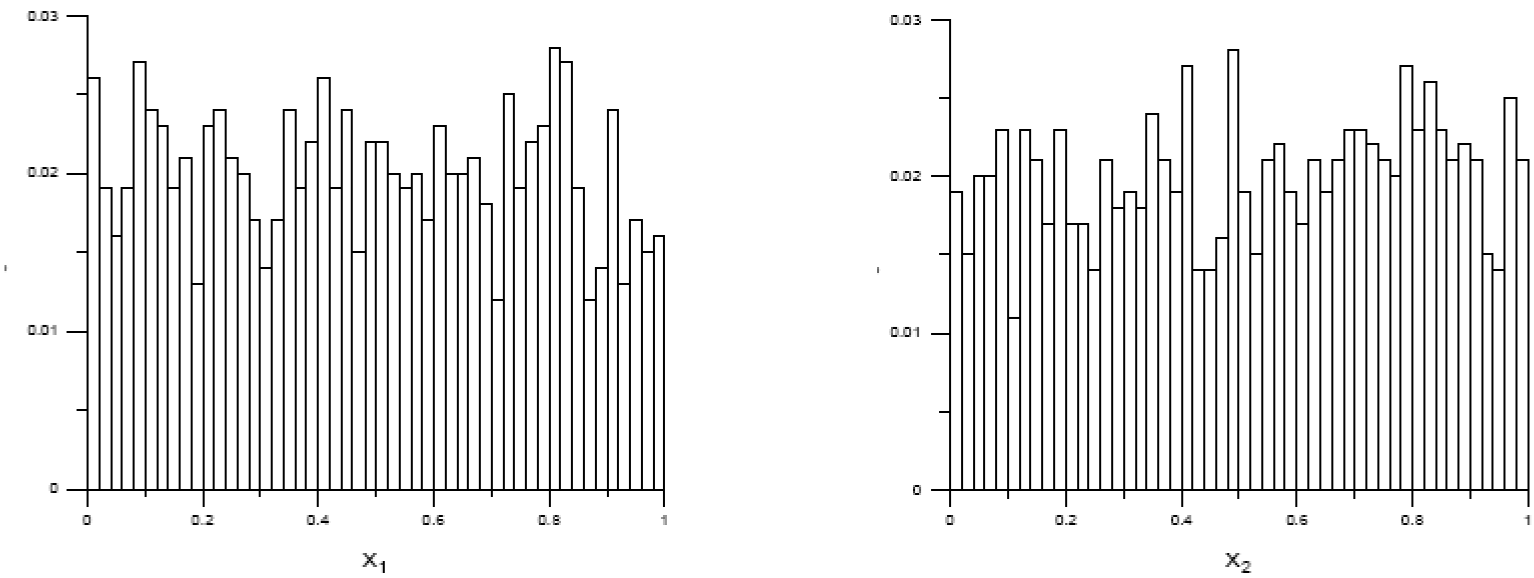


Fig. 5. Histograms of variables X_1 and X_2, uncorrelated, uniformly distributed, the range <0, 1>

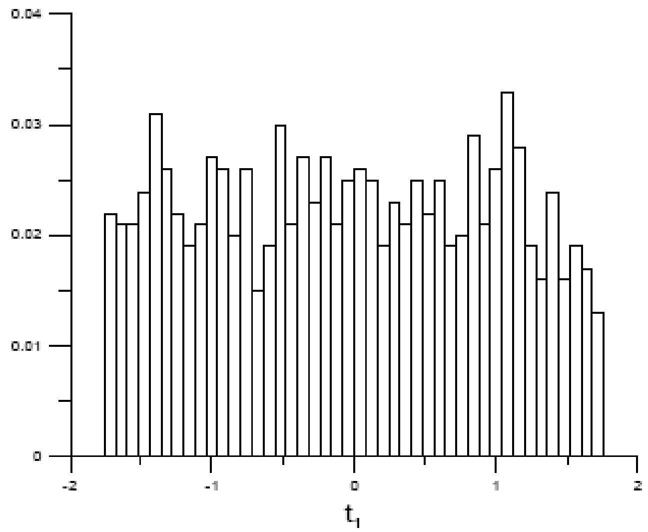
* Linear mapping $\mathbf{X} \rightarrow \mathbf{Y}$; $\mathbf{Y}$ is the vector of uncorrelated uniformly distributed variables of zero mean and unit variance (standardized), according to the formula:

$$Y_i = \left(2X_i - 1\right)\sqrt{3}, \qquad i = 1,2 \qquad (9)$$

* Matrix operation $\mathbf{T} = \mathbf{HY}$, resulting in the vector $\mathbf{T}$ of a covariance matrix $\mathbf{C}$. Taking the matrix $\mathbf{H}$ in the form (8), we get the following relations:

$$\begin{cases} T_1 = Y_1 \\ T_2 = hY_1 + Y_2\sqrt{1-h^2} \end{cases} \qquad (10)$$

The types of $\{Y_1, Y_2\}$ and $\{T_1, T_2\}$ are not identical. Here T_1 is uniform $\left\langle -\sqrt{3}, \sqrt{3} \right\rangle$, but T_2 shows triangular (Simpson) distribution in the range $\langle -g,\ g \rangle$, where $g = \left(h + \sqrt{1-h^2}\right)\sqrt{3}$. The histograms of T_1 and T_2, assumed $h = 0.8$, are shown below.



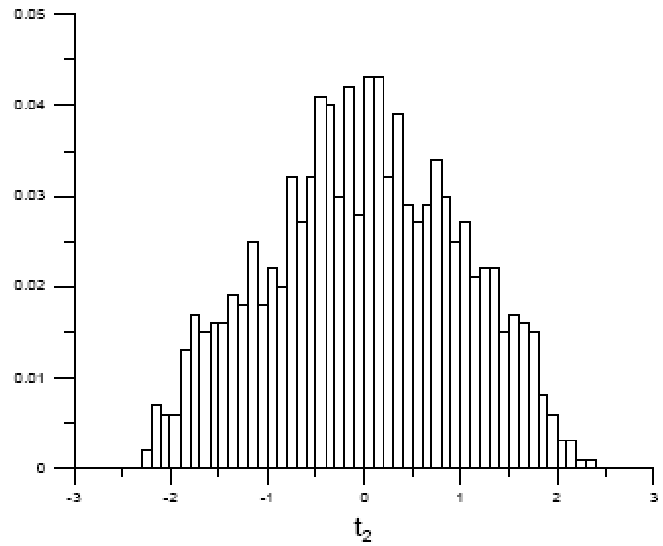


Fig. 6. Histograms of random variables T_1 and T_2, correlated, $h = 0.8$.The variable T_1 uniformly distributed in the range $\left(-\sqrt{3}, \sqrt{3}\right)$, the variable T_2 triangularly distributed

The vector transformation of **T** into **Z** occurs, it involves two uniformly distributed random variables in the range <0, 1>, with given covariance matrix **C**. The variable Z_1 is taken by the formula:

$$Z_1 = \frac{T_1}{2\sqrt{3}} + 0.5. \qquad (11)$$

while the transformation $Z_2 \rightarrow T_2$ is performed using the cumulative probability distribution function of the Simpson distribution:

$$F_T(t) = 0.5 + \frac{1}{g}\left(sign(t)\frac{t^2}{2g} + t \right). \qquad (12)$$

Transforming the vector $\mathbf{Z} = \{Z_1 \quad Z_2\}^T$ into $\mathbf{V} = \{V_1 \quad V_2\}^T$ using the inverse Gumbel distribution function:

$$V_i = F_V^{-1}\left(Z_1\right) = u - \frac{1}{\alpha}\ln\left(-\ln Z_1\right),\quad i = 1,2 \qquad (13)$$

In the above equation $F_V\left(\cdot\right)$ symbolizes the Gumbel distribution function of parameters α and u. In the worked example the following values were assumed: $u = 0.2$, $\alpha = 8.0$.

Creating the random wind load vector $\mathbf{W} = \left\{W_1 \quad W_2\right\}^T$, by the formula $W_i = V_i^2$, $i = 1, 2$

The procedure presented above results in the variables W_1 and W_2 of the correlation coefficient $r = \rho_{W_1W_2}$, which may differ from the correlation coefficient of the variables Z_1 and Z_2 (i.e. the value $h = \rho_{Z_1Z_2}$. It is possible to obtain in an iterative way, variables W_1 and W_2 of a correlation coefficient approximately equal to the arbitrary value r.

The following operations make up the single simulation step:

* Space frame analysis, calculating the initial value of the horizontal upper deck deflection $u(\omega)$ in the wind direction(Fig. 3),

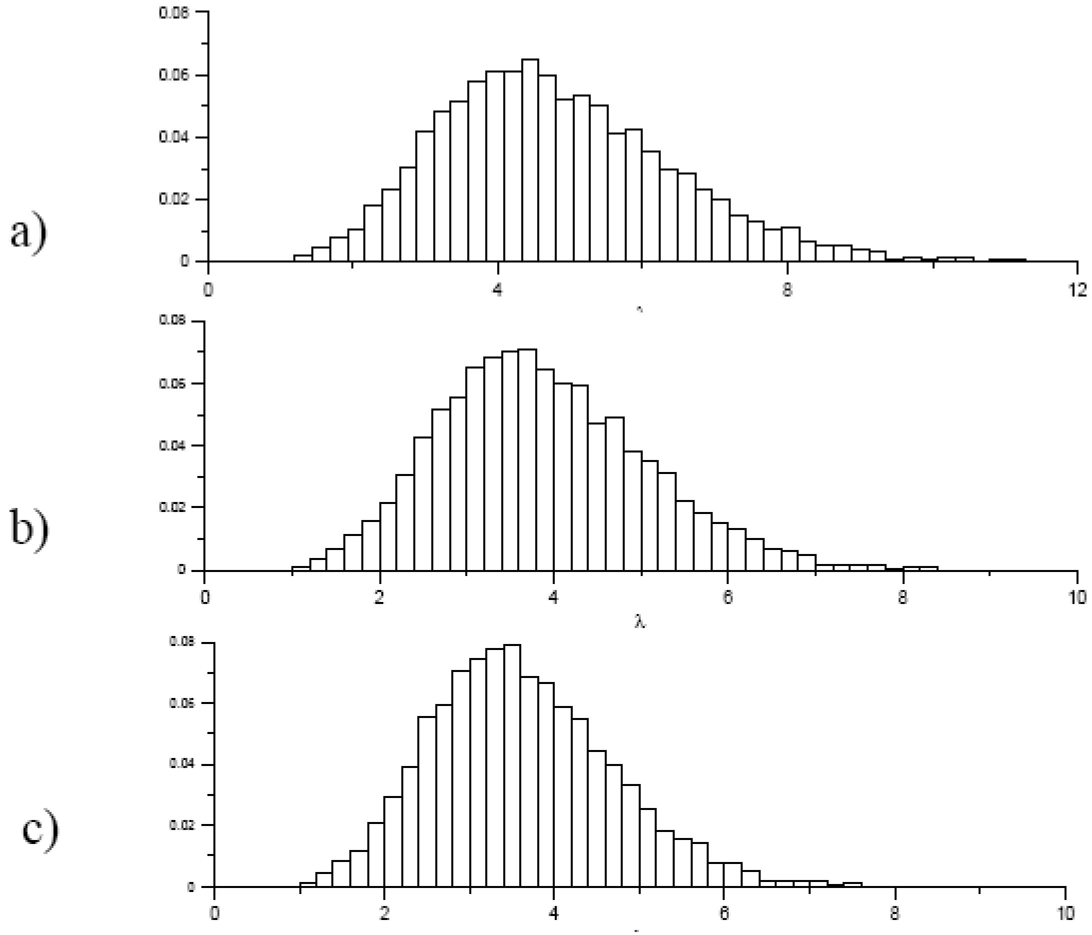
* Uni-parametrical load rise, up to allowable upper deck deflection u_0, $u_0 = H/400$, where H = 17.95 m is the colonnade height.

The single simulation step produces the limit load multiplier λ_i of this realization – a single value of the variable $\Lambda(\omega)$.

Three computational variants were provided, with respect to the wind load correlation coefficient r. The considered cases:

a) variables W_1 and W_2 uncorrelated – the coefficient $r = 0$
b) variables W_1 and W_2 correlated – the coefficient $r = 0.62$
c) variables W_1 and W_2 fully correlated – the coefficient $r = 1$

The SLS histograms are presented below



The results of the three variants of calculations are relative histograms of the limit state of the structure, statistical characteristics are collected in the table. It is worth pointing out that in the assumed structural and stochastic model each variant of calculation results in the probability of exceeding the allowable deflection lower than the accuracy of the method (the reciprocal of the number of realizations). On the basis of probabilistic limit state analysis it can be stated that the examined part of the structure is stiff enough to assure the proper structural service.

Variant of calculations	a)	b)	c)
Correlation coefficient of the wind load variables W_1 and W_2 : $r = \rho_{W_1W_2}$	0	0.62	1
Mean value	4.8133	3.8755	3.6128
Standard deviation	1.6106	1.1748	1.0540
Minimum value λ_{min}	1.1804	1.0427	1.0135
Maximum value λ_{max}	11.2338	8.3317	7.5443
Probability of failure	$<10^{-4}$	$<10^{-4}$	$<10^{-4}$

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REMARKS:

Sensitivity analysis of limit states of structures is proposed here, by means of a dedicated Monte Carlo algorithm. It leads to the 3rd level probabilistic methodology, i.e. the limit state histogram.

The procedure also allows to solve the problem limited to the reliability, or the probability of failure estimation.

Simulation-based limit state analysis usually means creating a population of structural states and choosing the failed cases, which determine the failure probability. The procedure proposed is modified and therefore developed. A group of dominant basic variables is chosen, in every simulation step. These variables increase uni-parametrically, to reach finally the structural limit state. Thus every simulated case is led to the limit state. The set of non-dimensional multipliers of dominant variables is the result of simulation. Its histogram serves as the estimator of the PDF of the structural limit state.

The proposed procedure makes it possible to perform the reliability assessment only. In this case techniques to reduce the number of simulations may be used.

ATTEMPTS TOWARDS A FULLY PROBABILISTIC DESIGN

The semi-probabilistic design procedures (for instance LRFD) make use of partial factors to depict random scatter of basic variables.

Values of loads and resistance coefficients are to be calibrated on the basis of statistical data.

The fully probabilistic design is the subject of a great number of present day's publications. Several international codes (e.g. ISO 2394: General principles of structural reliability, and EN 1990: Basis of structural design) present the overall design scheme exploring random analysis in greater extent. These documents serve as the code formats only (according to the JCSS nomenclature). They form the very basis of fully random design. No specific design codes exist up till now, referring to particular civil engineering branches (metal, concrete, timber structures, etc.) which really represent the fully probabilistic point of view. The transformation, described in numerous papers, seems to be a long-time process, requiring a huge effort of the code-writing committees and a population of professional designers, deeply educated the semi-probabilistic methods.