CHAPTER 5

Integrals

INTRODUCTION OF THE DEFINITE INTEGRAL

The geometric problems that motivated the development of the integral calculus (determination of lengths, areas, and volumes) arose in the ancient civilizations of Northern Africa. Where solutions were found, they related to concrete problems such as the measurement of a quantity of grain. Greek philosophers took a more abstract approach. In fact, Eudoxus (around 400 B.C.) and Archimedes (250 B.C.) formulated ideas of integration as we know it today.

Integral calculus developed independently, and without an obvious connection to differential calculus. The calculus became a "whole" in the last part of the seventeenth century when Isaac Barrow, Isaac Newton, and Gottfried Wilhelm Leibniz (with help from others) discovered that the integral of a function could be found by asking what was differentiated to obtain that function.

The following introduction of integration is the usual one. It displays the concept geometrically and then defines the integral in the nineteenth-century language of limits. This form of definition establishes the basis for a wide variety of applications.

Consider the area of the region bound by $y = f(x)$, the x-axis, and the joining vertical segments (ordinates) $x = a$ and $x = b$. (See Fig. 5-1.)

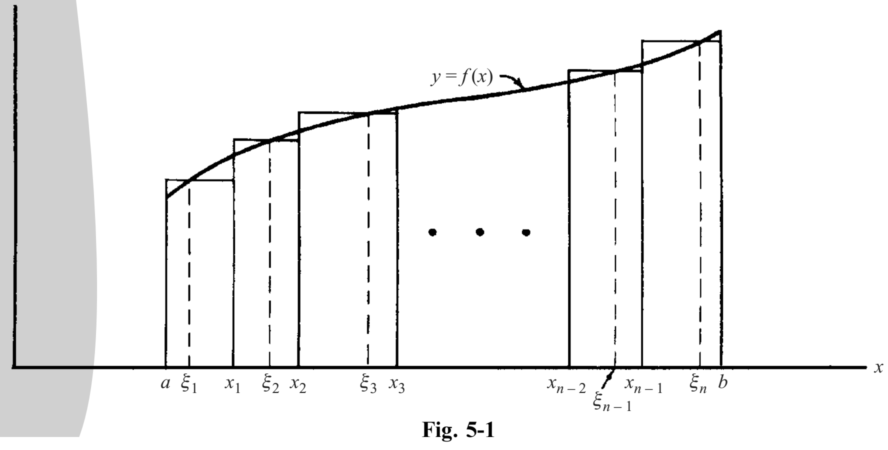


Fig. 5-1

Subdivide the interval $a \leqq x \leqq b$ into n sub-intervals by means of the points $x_1, x_2, \ldots, x_{n-1}$ chosen arbitrarily. In each of the new intervals $(a, x_1), (x_1, x_2), \ldots, (x_{n-1}, b)$ choose points $\xi_1, \xi_2, \ldots, \xi_n$ arbitrarily. Form the sum

$$f(\xi_1)(x_1 - a) + f(\xi_2)(x_2 - x_1) + f(\xi_3)(x_3 - x_2) + \cdots + f(\xi_n)(b - x_{n-1}) \tag{1}$$

By writing $x_0 = a$, $x_n = b$, and $x_k - x_{k-1} = \Delta x_k$, this can be written

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta x_k \tag{2}$$

Geometrically, this sum represents the total area of all rectangles in the above figure.

We now let the number of subdivisions n increase in such a way that each $\Delta x_k \to 0$. If as a result the sum (*1*) or (*2*) approaches a limit which does not depend on the mode of subdivision, we denote this limit by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k \tag{3}$$

This is called the *definite integral of $f(x)$ between a and b*. In this symbol $f(x)\,dx$ is called the *integrand*, and $[a, b]$ is called the *range of integration*. We call a and b the limits of integration, a being the lower limit of integration and b the upper limit.

The limit (*3*) exists whenever $f(x)$ is continuous (or piecewise continuous) in $a \leqq x \leqq b$ (see Problem 5.31). When this limit exists we say that f is *Riemann integrable* or simply *integrable* in $[a, b]$.

The definition of the definite integral as the limit of a sum was established by Cauchy around 1825. It was named for Riemann because he made extensive use of it in this 1850 exposition of integration.

Geometrically the value of this definite integral represents the area bounded by the curve $y = f(x)$, the x-axis and the ordinates at $x = a$ and $x = b$ only if $f(x) \geqq 0$. If $f(x)$ is sometimes positive and sometimes negative, the definite integral represents the algebraic sum of the areas above and below the x-axis, treating areas above the x-axis as positive and areas below the x-axis as negative.

MEASURE ZERO

A set of points on the x-axis is said to have *measure zero* if the sum of the lengths of intervals enclosing all the points can be made arbitrary small (less than any given positive number ϵ). We can show (see Problem 5.6) that any countable set of points on the real axis has measure zero. In particular, the set of rational numbers which is countable (see Problems 1.17 and 1.59, Chapter 1), has measure zero.

An important theorem in the theory of Riemann integration is the following:

Theorem. If $f(x)$ is bounded in $[a, b]$, then a necessary and sufficient condition for the existence of $\int_a^b f(x)\,dx$ is that the set of discontinuities of $f(x)$ have measure zero.

PROPERTIES OF DEFINITE INTEGRALS

If $f(x)$ and $g(x)$ are integrable in $[a, b]$ then

1. $\displaystyle\int_a^b \{f(x) \pm g(x)\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

2. $\displaystyle\int_a^b A f(x)\,dx = A\int_a^b f(x)\,dx$ where A is any constant

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ provided $f(x)$ is integrable in $[a, c]$ and $[c, b]$.

4. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

5. $\int_a^a f(x)\,dx = 0$

6. If in $a \leqq x \leqq b$, $m \leqq f(x) \leqq M$ where m and M are constants, then

$$m(b-a) \leqq \int_a^b f(x)\,dx \leqq M(b-a)$$

7. If in $a \leqq x \leqq b$, $f(x) \leqq g(x)$ then

$$\int_a^b f(x)\,dx \leqq \int_a^b g(x)\,dx$$

8. $\left|\int_a^b f(x)\,dx\right| \leqq \int_a^b |f(x)|\,dx$ if $a < b$

MEAN VALUE THEOREMS FOR INTEGRALS

As in differential calculus the mean value theorems listed below are existence theorems. The first one generalizes the idea of finding an arithmetic mean (i.e., an average value of a given set of values) to a continuous function over an interval. The second mean value theorem is an extension of the first one that defines a weighted average of a continuous function.

By analogy, consider determining the arithmetic mean (i.e., average value) of temperatures at noon for a given week. This question is resolved by recording the 7 temperatures, adding them, and dividing by 7. To generalize from the notion of arithmetic mean and ask for the average temperature for the week is much more complicated because the spectrum of temperatures is now continuous. However, it is reasonable to believe that there exists a time at which the *average* temperature takes place. The manner in which the integral can be employed to resolve the question is suggested by the following example.

Let f be continuous on the closed interval $a \leqq x \leqq b$. Assume the function is represented by the correspondence $y = f(x)$, with $f(x) > 0$. Insert points of equal subdivision, $a = x_0, x_1, \ldots, x_n = b$. Then all $\Delta x_k = x_k - x_{k-1}$ are equal and each can be designated by Δx. Observe that $b - a = n\Delta x$. Let ξ_k be the midpoint of the interval Δx_k and $f(\xi_k)$ the value of f there. Then the average of these functional values is

$$\frac{f(\xi_1) + \cdots + f(\xi_n)}{n} = \frac{[f(\xi_1) + \cdots + f(\xi_n)]\Delta x}{b-a} = \frac{1}{b-a}\sum_{k=1}^{n} f(\xi_\kappa)\Delta\xi_\kappa$$

This sum specifies the average value of the n functions at the midpoints of the intervals. However, we may abstract the last member of the string of equalities (dropping the special conditions) and define

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{k=1}^{n} f(\xi_\kappa)\Delta\xi_\kappa = \frac{1}{b-a}\int_a^b f(x)\,dx$$

as the average value of f on $[a, b]$.

Of course, the question of for what value $x = \xi$ the average is attained is not answered; and, in fact, in general, only existence not the value can be demonstrated. To see that there is a point $x = \xi$ such that $f(\xi)$ represents the average value of f on $[a, b]$, recall that a continuous function on a closed interval has maximum and minimum values, M and m, respectively. Thus (think of the integral as representing the area under the curve). (See Fig. 5-2.)

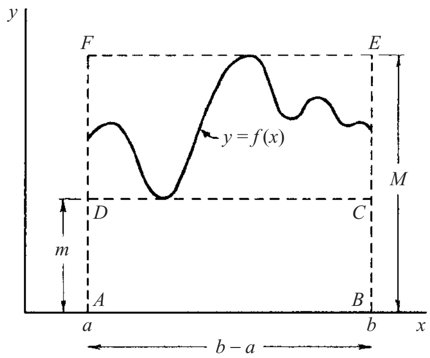


Fig. 5-2

$$m(b-a) \leqq \int_a^b f(x)\,dx \leqq M(b-a)$$

or

$$m \leqq \frac{1}{b-a}\int_a^b f(x)\,dx \leqq M$$

Since f is a continuous function on a closed interval, there exists a point $x = \xi$ in (a, b) intermediate to m and M such that

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

While this example is not a rigorous proof of the first mean value theorem, it motivates it and provides an interpretation. (See Chapter 3, Theorem 10.)

1. **First mean value theorem.** If $f(x)$ is continuous in $[a, b]$, there is a point ξ in (a, b) such that

$$\int_a^b f(x)\,dx = (b-a)f(\xi) \tag{4}$$

2. **Generalized first mean value theorem.** If $f(x)$ and $g(x)$ are continuous in $[a, b]$, and $g(x)$ does not change sign in the interval, then there is a point ξ in (a, b) such that

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx \tag{5}$$

This reduces to *(4)* if $g(x) = 1$.

CONNECTING INTEGRAL AND DIFFERENTIAL CALCULUS

In the late seventeenth century the key relationship between the derivative and the integral was established. The connection which is embodied in the fundamental theorem of calculus was responsible for the creation of a whole new branch of mathematics called analysis.

Definition: Any function F such that $F'(x) = f(x)$ is called an *antiderivative*, *primitive*, or *indefinite integral* of f.

The antiderivative of a function is not unique. This is clear from the observation that for any constant c

$$(F(x) + c)' = F'(x) = f(x)$$

The following theorem is an even stronger statement.

Theorem. Any two primitives (i.e., antiderivatives), F and G of f differ at most by a constant, i.e., $F(x) - G(x) = C$.

(See the problem set for the proof of this theorem.)

EXAMPLE. If $F'(x) = x^2$, then $F(x) = \int x^2 dx = \frac{x^3}{3} + c$ is an indefinite integral (antiderivative or primitive) of x^2.

The indefinite integral (which is a function) may be expressed as a definite integral by writing

$$\int f(x)\,dx = \int_c^x f(t)\,dt$$

The functional character is expressed through the upper limit of the definite integral which appears on the right-hand side of the equation.

This notation also emphasizes that the definite integral of a given function only depends on the limits of integration, and thus any symbol may be used as the variable of integration. For this reason, that variable is often called a *dummy* variable. The indefinite integral notation on the left depends on continuity of f on a domain that is not described. One can visualize the definite integral on the right by thinking of the dummy variable t as ranging over a subinterval $[c, x]$. (There is nothing unique about the letter t; any other convenient letter may represent the dummy variable.)

The previous terminology and explanation set the stage for the fundamental theorem. It is stated in two parts. The first states that the antiderivative of f is a new function, the integrand of which is the derivative of that function. Part two demonstrates how that primitive function (antiderivative) enables us to evaluate definite integrals.

THE FUNDAMENTAL THEOREM OF THE CALCULUS

Part 1 Let f be integrable on a closed interval $[a, b]$. Let c satisfy the condition $a \leqq c \leqq b$, and define a new function

$$F(x) = \int_c^x f(t)\,dt \quad \text{if} \quad a \leqq x \leqq b$$

Then the derivative $F'(x)$ exists at each point x in the open interval (a, b), where f is continuous and $F'(x) = f(x)$. (See Problem 5.10 for proof of this theorem.)

Part 2 As in Part 1, assume that f is integrable on the closed interval $[a, b]$ and continuous in the open interval (a, b). Let F be any antiderivative so that $F'(x) = f(x)$ for each x in (a, b). If $a < c < b$, then for any x in (a, b)

$$\int_c^x f(t)\,dt = F(x) - F(c)$$

If the open interval on which f is continuous includes a and b, then we may write

$$\int_a^b f(x)\,dx = F(b) - F(a). \qquad \text{(See Problem 5.11)}$$

This is the usual form in which the theorem is used.

EXAMPLE. To evaluate $\int_1^2 x^2\,dx$ we observe that $F'(x) = x^2$, $F(x) = \frac{x^3}{3} + c$ and $\int_1^2 x^2\,dx = \left(\frac{2^3}{3} + c\right) - \left(\frac{1^3}{3} + c\right) = \frac{7}{3}$. Since c subtracts out of this evaluation it is convenient to exclude it and simply write $\frac{2^3}{3} - \frac{1^3}{3}$.

GENERALIZATION OF THE LIMITS OF INTEGRATION

The upper and lower limits of integration may be variables. For example:

$$\int_{\sin x}^{\cos x} t\,dt = \left[\frac{t^2}{2}\right]_{\sin x}^{\cos x} = (\cos^2 x - \sin^2 x)/2$$

In general, if $F'(x) = f(x)$ then

$$\int_{u(x)}^{v(x)} f(t)\,dt = F[v(x)] = F[u(x)]$$

CHANGE OF VARIABLE OF INTEGRATION

If a determination of $\int f(x)\,dx$ is not immediately obvious in terms of elementary functions, useful results may be obtained by changing the variable from x to t according to the transformation $x = g(t)$. (This change of integrand that follows is suggested by the differential relation $dx = g'(t)\,dt$.) The fundamental theorem enabling us to do this is summarized in the statement

$$\int f(x)\,dx = \int f\{g(t)\}g'(t)\,dt \tag{6}$$

where after obtaining the indefinite integral on the right we replace t by its value in terms of x, i.e., $t = g^{-1}(x)$. This result is analogous to the chain rule for differentiation (see Page 69).

The corresponding theorem for definite integrals is

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f\{g(t)\}g'(t)\,dt \tag{7}$$

where $g(\alpha) = a$ and $g(\beta) = b$, i.e., $\alpha = g^{-1}(a)$, $\beta = g^{-1}(b)$. This result is certainly valid if $f(x)$ is continuous in $[a, b]$ and if $g(t)$ is continuous and has a continuous derivative in $\alpha \leqq t \leqq \beta$.

INTEGRALS OF ELEMENTARY FUNCTIONS

The following results can be demonstrated by differentiating both sides to produce an identity. In each case an arbitrary constant c (which has been omitted here) should be added.

1. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} \quad n \neq -1$
2. $\int \dfrac{du}{u} = \ln|u|$
3. $\int \sin u\,du = -\cos u$
4. $\int \cos u\,du = \sin u$
5. $\int \tan u\,du = \ln|\sec u| = -\ln|\cos u|$
6. $\int \cot u\,du = \ln|\sin u|$
7. $\int \sec u\,du = \ln|\sec u + \tan u| = \ln|\tan(u/2 + \pi/4)|$
8. $\int \csc u\,du = \ln|\csc u - \cot u| = \ln|\tan u/2|$
9. $\int \sec^2 u\,du = \tan u$
10. $\int \csc^2 u\,du = -\cot u$
11. $\int \sec u \tan u\,du = \sec u$
12. $\int \csc u \cot u\,du = -\csc u$
13. $\int a^u\,du = \dfrac{a^u}{\ln a} \quad a > 0, a \neq 1$
14. $\int e^u\,du = e^u$
15. $\int \sinh u\,du = \cosh u$
16. $\int \cosh u\,du = \sinh u$
17. $\int \tanh u\,du = \ln \cosh u$
18. $\int \coth u\,du = \ln|\sinh u|$
19. $\int \operatorname{sech} u\,du = \tan^{-1}(\sinh u)$
20. $\int \operatorname{csch} u\,du = -\coth^{-1}(\cosh u)$
21. $\int \operatorname{sech}^2 u\,du = \tanh u$
22. $\int \operatorname{csch}^2 u\,du = -\coth u$
23. $\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u$
24. $\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u$
25. $\int \dfrac{du}{\sqrt{s^2 - u^2}} = \sin^{-1}\dfrac{u}{a} \quad \text{or} \quad -\cos^{-1}\dfrac{u}{a}$
26. $\int \dfrac{du}{\sqrt{u^2 \pm a^2}} = \ln|u + \sqrt{u^2 \pm a^2}\,|$
27. $\int \dfrac{du}{u^2 + a^2} = \dfrac{1}{a}\tan^{-1}\dfrac{u}{a} \quad \text{or} \quad -\dfrac{1}{a}\cot^{-1}\dfrac{u}{a}$
28. $\int \dfrac{du}{u^2 - a^2} = \dfrac{1}{2a}\ln\left|\dfrac{u-a}{u+a}\right|$
29. $\int \dfrac{du}{u\sqrt{a^2 \pm u^2}} = \dfrac{1}{a}\ln\left|\dfrac{u}{a + \sqrt{a^2 \pm u^2}}\right|$
30. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\cos^{-1}\dfrac{a}{u} \quad \text{or} \quad \dfrac{1}{a}\sec^{-1}\dfrac{u}{a}$
31. $\int \sqrt{u^2 \pm a^2}\,du = \dfrac{u}{2}\sqrt{u^2 \pm a^2} \pm \dfrac{a^2}{2}\ln|u + \sqrt{u^2 \pm a^2}\,|$
32. $\int \sqrt{a^2 - u^2}\,du = \dfrac{u}{2}\sqrt{a^2 - u^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{u}{a}$
33. $\int e^{au}\sin bu\,du = \dfrac{e^{au}(a\sin bu - b\cos bu)}{a^2 + b^2}$
34. $\int e^{au}\cos bu\,du = \dfrac{e^{au}(a\cos bu + b\sin bu)}{a^2 + b^2}$

SPECIAL METHODS OF INTEGRATION

1. **Integration by parts.**
   Let u and v be differentiable functions. According to the product rule for differentials

$$d(uv) = u\,dv + v\,du$$

Upon taking the antiderivative of both sides of the equation, we obtain

$$uv = \int u\,dv + \int v\,du$$

This is the formula for integration by parts when written in the form

$$\int u\,dv = uv - \int v\,du \quad \text{or} \quad \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

where $u = f(x)$ and $v = g(x)$. The corresponding result for definite integrals over the interval $[a, b]$ is certainly valid if $f(x)$ and $g(x)$ are continuous and have continuous derivatives in $[a, b]$. See Problems 5.17 to 5.19.

2. **Partial fractions.** Any rational function $\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, with the degree of $P(x)$ less than that of $Q(x)$, can be written as the sum of rational functions having the form $\dfrac{A}{(ax+b)^r}, \dfrac{Ax+B}{(ax^2+bx+c)^r}$ where $r = 1, 2, 3, \ldots$ which can always be integrated in terms of elementary functions.

EXAMPLE 1. $\dfrac{3x-2}{(4x-3)(2x+5)^3} = \dfrac{A}{4x-3} + \dfrac{B}{(2x+5)^3} + \dfrac{C}{(2x+5)^2} + \dfrac{D}{2x+5}$

EXAMPLE 2. $\dfrac{5x^2-x+2}{(x^2+2x+4)^2(x-1)} = \dfrac{Ax+B}{(x^2+2x+4)^2} + \dfrac{Cx+D}{x^2+2x+4} + \dfrac{E}{x-1}$

The constants, A, B, C, etc., can be found by clearing of fractions and equating coefficients of like powers of x on both sides of the equation or by using special methods (see Problem 5.20).

3. **Rational functions of $\sin x$ and $\cos x$** can always be integrated in terms of elementary functions by the substitution $\tan x/2 = u$ (see Problem 5.21).
4. **Special devices** depending on the particular form of the integrand are often employed (see Problems 5.22 and 5.23).

IMPROPER INTEGRALS

If the range of integration $[a, b]$ is not finite or if $f(x)$ is not defined or not bounded at one or more points of $[a, b]$, then the integral of $f(x)$ over this range is called an *improper integral*. By use of appropriate limiting operations, we may define the integrals in such cases.

EXAMPLE 1. $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \lim_{M\to\infty} \int_0^M \frac{dx}{1+x^2} = \lim_{M\to\infty} \tan^{-1} x\Big|_0^M = \lim_{M\to\infty} \tan^{-1} M = \pi/2$

EXAMPLE 2. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0+} \int_\epsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0+} 2\sqrt{x}\Big|_\epsilon^1 = \lim_{\epsilon\to 0+} (2 - 2\sqrt{\epsilon}) = 2$

EXAMPLE 3. $\displaystyle\int_0^1 \frac{dx}{x} = \lim_{\epsilon\to 0+} \int_\epsilon^1 \frac{dx}{x} = \lim_{\epsilon\to 0+} \ln x\Big|_\epsilon^1 = \lim_{\epsilon\to 0+} (-\ln \epsilon)$

Since this limit does not exist we say that the integral diverges (i.e., does not converge).

For further examples, see Problems 5.29 and 5.74 through 5.76. For further discussion of improper integrals, see Chapter 12.

NUMERICAL METHODS FOR EVALUATING DEFINITE INTEGRALS

Numerical methods for evaluating definite integrals are available in case the integrals cannot be evaluated exactly. The following special numerical methods are based on subdividing the interval $[a, b]$ into n equal parts of length $\Delta x = (b-a)/n$. For simplicity we denote $f(a + k\Delta x) = f(x_k)$ by y_k, where $k = 0, 1, 2, \ldots, n$. The symbol $\approx$ means "approximately equal." In general, the approximation improves as n increases.

1. **Rectangular rule.**

$$\int_a^b f(x)\,dx \approx \Delta x\{y_0 + y_1 + y_2 + \cdots + y_{n-1}\} \quad \text{or} \quad \Delta x\{y_1 + y_2 + y_3 + \cdots + y_n\} \tag{8}$$

The geometric interpretation is evident from the figure on Page 90. When left endpoint function values $y_0, y_1, \ldots, y_{n-1}$ are used, the rule is called "the left-hand rule." Similarly, when right endpoint evaluations are employed, it is called "the right-hand rule."

2. **Trapezoidal rule.**

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\{y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\} \tag{9}$$

This is obtained by taking the mean of the approximations in (*8*). Geometrically this replaces the curve $y = f(x)$ by a set of approximating line segments.

3. **Simpson's rule.**

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\{y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\} \tag{10}$$

The above formula is obtained by approximating the graph of $y = g(x)$ by a set of parabolic arcs of the form $y = ax^2 + bx + c$. The correlation of two observations lead to 10. First,

$$\int_{-h}^{h} [ax^2 + bx + c]\,dx = \frac{h}{3}[2ah^2 + 6c]$$

The second observation is related to the fact that the vertical parabolas employed here are determined by three nonlinear points. In particular, consider $(-h, y_0)$, $(0, y_1)$, (h, y_2) then $y_0 = a(-h)^2 + b(-h) + c$, $y_1 = c$, $y_2 = ah^2 + bh + c$. Consequently, $y_0 + 4y_1 + y_2 = 2ah^2 + 6c$. Thus, this combination of ordinate values (corresponding to equally space domain values) yields the area bound by the parabola, vertical segments, and the x-axis. Now these ordinates may be interpreted as those of the function, f, whose integral is to be approximated. Then, as illustrated in Fig. 5-3:

$$\sum_{k=1}^{n} \frac{h}{3}[y_{k-1} + 4y_k + y_{k+1}] = \frac{\Delta x}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n]$$

The Simpson rule is likely to give a better approximation than the others for smooth curves.

APPLICATIONS

The use of the integral as a limit of a sum enables us to solve many physical or geometrical problems such as determination of areas, volumes, arc lengths, moments of intertia, centroids, etc.

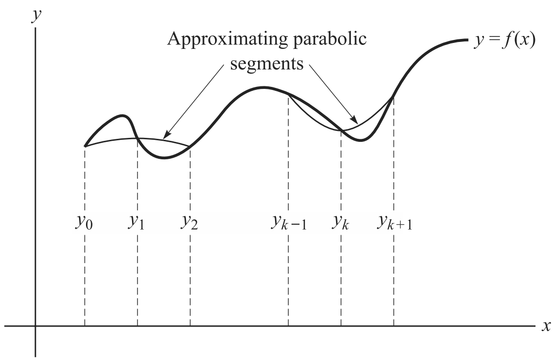


Fig. 5-3

ARC LENGTH

As you walk a twisting mountain trail, it is possible to determine the distance covered by using a pedometer. To create a geometric model of this event, it is necessary to describe the trail and a method of measuring distance along it. The trail might be referred to as a *path*, but in more exacting geometric terminology the word, *curve* is appropriate. That segment to be measured is an arc of the curve. The arc is subject to the following restrictions:

1. It does not intersect itself (i.e., it is a simple arc).
2. There is a tangent line at each point.
3. The tangent line varies continuously over the arc.

These conditions are satisfied with a parametric representation $x = f(t)$, $y = g(t)$, $z = h(t)$, $a \leqq t \leqq b$, where the functions f, g, and h have continuous derivatives that do not simultaneously vanish at any point. This arc is in Euclidean three space and will be discussed in Chapter 10. In this introduction to curves and their arc length, we let $z = 0$, thereby restricting the discussion to the plane.

A careful examination of your walk would reveal movement on a sequence of straight segments, each changed in direction from the previous one. This suggests that the length of the arc of a curve is obtained as the limit of a sequence of lengths of polygonal approximations. (The polygonal approximations are characterized by the number of divisions $n \to \infty$ and no subdivision is bound from zero. (See Fig. 5-4.)

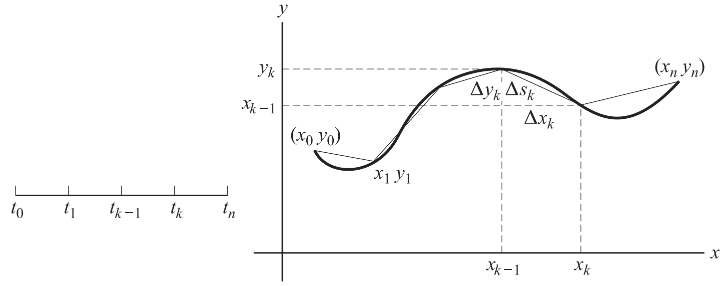


Fig. 5-4

Geometrically, the measurement of the kth segment of the arc, $0 \leqq t \leqq s$, is accomplished by employing the Pythagorean theorem, and thus, the measure is defined by

$$\lim_{n\to\infty}\sum_{k=1}^{n}\{(\Delta x_k)^2+(\Delta y_k)^2\}^{1/2}$$

or equivalently

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left\{1+\left(\frac{\Delta y_k}{\Delta x_k}\right)^2\right\}^{1/2}(\Delta x_k)$$

where $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$.

Thus, the length of the arc of a curve in rectangular Cartesian coordinates is

$$L=\int_a^b\{[f'(t)^2]+[g'(t)]^2\}^{1/2}\,dt=\int\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}^{1/2}dt$$

(This form may be generalized to any number of dimensions.)

Upon changing the variable of integration from t to x we obtain the planar form

$$L=\int_{f(a)}^{f(b)}\left\{1+\left[\frac{dy}{dx}\right]^2\right\}^{1/2}$$

(This form is only appropriate in the plane.)

The generic differential formula $ds^2 = dx^2 + dy^2$ is useful, in that various representations algebraically arise from it. For example,

$$\frac{ds}{dt}$$

expresses instantaneous speed.

AREA

Area was a motivating concept in introducing the integral. Since many applications of the integral are geometrically interpretable in the context of area, an extended formula is listed and illustrated below.

Let f and g be continuous functions whose graphs intersect at the graphical points corresponding to $x = a$ and $x = b$, $a < b$. If $g(x) \geqq f(x)$ on $[a, b]$, then the area bounded by $f(x)$ and $g(x)$ is

$$A=\int_a^b\{g(x)-f(x)\}\,dx$$

If the functions intersect in (a, b), then the integral yields an algebraic sum. For example, if $g(x) = \sin x$ and $f(x) = 0$ then:

$$\int_0^{2\pi}\sin x\,dx=\cos x\Big|_0^{2\pi}=0$$

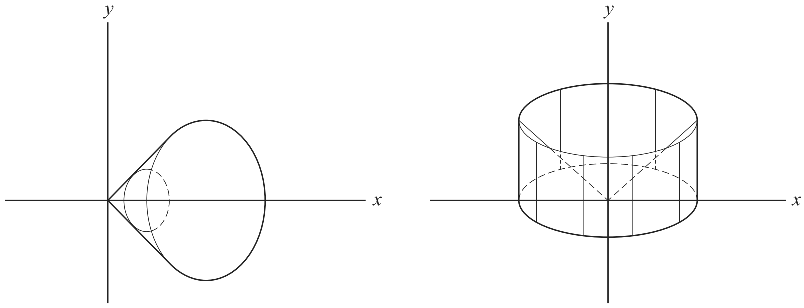
VOLUMES OF REVOLUTION

Disk Method

Assume that f is continuous on a closed interval $a \leqq x \leqq b$ and that $f(x) \geqq 0$. Then the solid realized through the revolution of a plane region R (bound by $f(x)$, the x-axis, and $x = a$ and $x = b$) about the x-axis has the volume

$$V=\pi\int_a^b[f(x)]^2\,dx$$

This method of generating a volume is called the *disk method* because the cross sections of revolution are circular disks. (See Fig. 5-5(a).)



(a) **Fig. 5-5** (b)

EXAMPLE. A solid cone is generated by revolving the graph of $y = kx$, $k > 0$ and $0 \leqq x \leqq b$, about the x-axis. Its volume is

$$V = \pi \int_0^b k^2x^2\,dx = \pi \frac{k^3x^3}{3}\bigg|_0^b = \pi \frac{k^3b^3}{3}$$

Shell Method

Suppose f is a continuous function on $[a, b]$, $a \geqq 0$, satisfying the condition $f(x) \geqq 0$. Let R be a plane region bound by $f(x)$, $x = a$, $x = b$, and the x-axis. The volume obtained by orbiting R about the y-axis is

$$V = \int_a^b 2\pi x f(x)\,dx$$

This method of generating a volume is called the *shell method* because of the cylindrical nature of the vertical lines of revolution. (See Fig. 5-5(b).)

EXAMPLE. If the region bounded by $y = kx$, $0 \leqq x \leqq b$ and $x = b$ (with the same conditions as in the previous example) is orbited about the y-axis the volume obtained is

$$V = 2\pi \int_0^b x(kx)\,dx = 2\pi k \frac{x^3}{3}\bigg|_0^b = 2\pi k \frac{b^3}{3}$$

By comparing this example with that in the section on the disk method, it is clear that for the same plane region the disk method and the shell method produce different solids and hence different volumes.

Moment of Inertia

Moment of inertia is an important physical concept that can be studied through its idealized geometric form. This form is abstracted in the following way from the physical notions of kinetic energy, $K = \frac{1}{2}mv^2$, and angular velocity, $v = \omega r$. (m represents mass and v signifies linear velocity). Upon substituting for v

$$K = \tfrac{1}{2}m\omega^2r^2 = \tfrac{1}{2}(mr^2)\omega^2$$

When this form is compared to the original representation of kinetic energy, it is reasonable to identify mr^2 as rotational mass. It is this quantity, $I = mr^2$ that we call the *moment of inertia*.

Then in a purely geometric sense, we denote a plane region R described through continuous functions f and g on $[a, b]$, where $a > 0$ and $f(x)$ and $g(x)$ intersect at a and b only. For simplicity, assume $g(x) \geqq f(x) > 0$. Then

$$I = \int_a^b x^2[g(x) - f(x)]\,dx$$

By idealizing the plane region, R, as a volume with uniform density *one*, the expression $[f(x) - g(x)]\,dx$ stands in for mass and r^2 has the coordinate representation x^2. (See Problem 5.25(*b*) for more details.)

Solved Problems

DEFINITION OF A DEFINITE INTEGRAL

5.1. If $f(x)$ is continuous in $[a, b]$ prove that

$$\lim_{n\to\infty} \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) = \int_a^b f(x)\,dx$$

Since $f(x)$ is continuous, the limit exists independent of the mode of subdivision (see Problem 5.31). Choose the subdivision of $[a, b]$ into n equal parts of equal length $\Delta x = (b-a)/n$ (see Fig. 5-1, Page 90). Let $\xi_k = a + k(b-a)/n$, $k = 1, 2, \ldots, n$. Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k = \lim_{n\to\infty} \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) = \int_a^b f(x)\,dx$$

5.2. Express $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$ as a definite integral.

Let $a = 0$, $b = 1$ in Problem 1. Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx$$

5.3. (*a*) Express $\int_0^1 x^2\,dx$ as a limit of a sum, and use the result to evaluate the given definite integral.

(*b*) Interpret the result geometrically.

(*a*) If $f(x) = x^2$, then $f(k/n) = (k/n)^2 = k^2/n^2$. Thus by Problem 5.2,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{k^2}{n^2} = \int_0^1 x^2\,dx$$

This can be written, using Problem 1.29 of Chapter 1,

$$\begin{aligned}\int_0^1 x^2\,dx &= \lim_{n\to\infty} \frac{1}{n}\left(\frac{1^2}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{n^2}{n^2}\right) = \lim_{n\to\infty} \frac{1^2 + 2^2 + \cdots + n^2}{n^3} \\ &= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} \\ &= \lim_{n\to\infty} \frac{(1+1/n)(2+1/n)}{6} = \frac{1}{3}\end{aligned}$$

which is the required limit.

Note: By using the fundamental theorem of the calculus, we observe that $\int_0^1 x^2\,dx = (x^3/3)|_0^1 = 1^3/3 - 0^3/3 = 1/3$.

(*b*) The area bounded by the curve $y = x^2$, the x-axis and the line $x = 1$ is equal to $\frac{1}{3}$.

5.4. Evaluate $\lim_{n\to\infty}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right\}$.

The required limit can be written

$$\lim_{n\to\infty}\frac{1}{n}\left\{\frac{1}{1+1/n}+\frac{1}{1+2/n}+\cdots+\frac{1}{1+n/n}\right\}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+k/n}$$
$$=\int_0^1\frac{dx}{1+x}=\ln(1+x)\Big|_0^1=\ln 2$$

using Problem 5.2 and the fundamental theorem of the calculus.

5.5. Prove that $\lim_{n\to\infty}\frac{1}{n}\left\{\sin\frac{t}{n}+\sin\frac{2t}{n}+\cdots+\sin\frac{(n-1)t}{n}\right\}=\frac{1-\cos t}{t}$.

Let $a=0, b=t, f(x)=\sin x$ in Problem 1. Then

$$\lim_{n\to\infty}\frac{t}{n}\sum_{k=1}^{n}\sin\frac{kt}{n}=\int_0^t\sin x\,dx=1-\cos t$$

and so

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}\sin\frac{kt}{n}=\frac{1-\cos t}{t}$$

using the fact that $\lim_{n\to\infty}\frac{\sin t}{n}=0$.

MEASURE ZERO

5.6. Prove that a countable point set has measure zero.

Let the point set be denoted by $x_1, x_2, x_3, x_4, \ldots$ and suppose that intervals of lengths less than $\epsilon/2, \epsilon/4, \epsilon/8, \epsilon/16, \ldots$ respectively enclose the points, where ϵ is any positive number. Then the sum of the lengths of the intervals is less than $\epsilon/2+\epsilon/4+\epsilon/8+\cdots=\epsilon$ (let $a=\epsilon/2$ and $r=\frac{1}{2}$ in Problem 2.25(*a*) of Chapter 2), showing that the set has measure zero.

PROPERTIES OF DEFINITE INTEGRALS

5.7. Prove that $\left|\int_a^b f(x)\,dx\right| \leqq \int_a^b |f(x)|\,dx$ if $a<b$.

By absolute value property 2, Page 3,

$$\left|\sum_{k=1}^{n}f(\xi_k)\Delta x_k\right| \leqq \sum_{k=1}^{n}|f(\xi_k)\Delta x_k| = \sum_{k=1}^{n}|f(\xi_k)|\Delta x_k$$

Taking the limit as $n\to\infty$ and each $\Delta x_k\to 0$, we have the required result.

5.8. Prove that $\lim_{n\to\infty}\int_0^{2\pi}\frac{\sin nx}{x^2+n^2}\,dx=0$.

$$\left|\int_0^{2\pi}\frac{\sin nx}{x^2+n^2}\,dx\right| \leqq \int_0^{2\pi}\left|\frac{\sin nx}{x^2+n^2}\right|dx \leqq \int_0^{2\pi}\frac{dx}{n^2}=\frac{2\pi}{n^2}$$

Then $\lim_{n\to\infty}\left|\int_0^{2\pi}\frac{\sin nx}{x^2+n^2}\,dx\right|=0$, and so the required result follows.

MEAN VALUE THEOREMS FOR INTEGRALS

5.9. Given the right triangle pictured in Fig. 5-6: (*a*) Find the average value of h. (*b*) At what point does this average value occur? (*c*) Determine the average value of $f(x) = \sin^{-1} x, 0 \leqq x \leqq \frac{1}{2}$. (Use integration by parts.) (*d*) Determine the average value of $f(x) = \cos^2 x, 0 \leqq x \leqq \dfrac{\pi}{2}$.

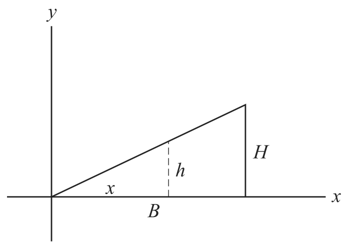


Fig. 5-6

(*a*) $h(x) = \dfrac{H}{B}x$. According to the mean value theorem for integrals, the average value of the function h on the interval $[0, B]$ is

$$A = \frac{1}{B}\int_0^B \frac{H}{B}x\,dx = \frac{H}{2}$$

(*b*) The point, ξ, at which the average value of h occurs may be obtained by equating $f(\xi)$ with that average value, i.e., $\dfrac{H}{B}\xi = \dfrac{H}{2}$. Thus, $\xi = \dfrac{B}{2}$.

FUNDAMENTAL THEOREM OF THE CALCULUS

5.10. If $F(x) = \displaystyle\int_a^x f(t)\,dt$ where $f(x)$ is continuous in $[a, b]$, prove that $F'(x) = f(x)$.

$$\begin{aligned}\frac{F(x+h) - F(x)}{h} &= \frac{1}{h}\left\{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right\} = \frac{1}{h}\int_x^{x+h} f(t)\,dt \\ &= f(\xi) \qquad \xi \text{ between } x \text{ and } x+h\end{aligned}$$

by the first mean value theorem for integrals (Page 93).

Then if x is any point interior to $[a, b]$,

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} f(\xi) = f(x)$$

since f is continuous.

If $x = a$ or $x = b$, we use right- or left-hand limits, respectively, and the result holds in these cases as well.

5.11. Prove the fundamental theorem of the calculus, Part 2 (Pages 94 and 95).

By Problem 5.10, if $F(x)$ is *any* function whose derivative is $f(x)$, we can write

$$F(x) = \int_a^x f(t)\,dt + c$$

where c is any constant (see last line of Problem 22, Chapter 4).

Since $F(a) = c$, it follows that $F(b) = \displaystyle\int_a^b f(t)\,dt + F(a)$ or $\displaystyle\int_a^b f(t)\,dt = F(b) - F(a)$.

5.12. If $f(x)$ is continuous in $[a, b]$, prove that $F(x) = \displaystyle\int_a^x f(t)\,dt$ is continuous in $[a, b]$.

If x is any point interior to $[a, b]$, then as in Problem 5.10,

$$\lim_{h\to 0} F(x+h) - F(x) = \lim_{h\to 0} hf(\xi) = 0$$

and $F(x)$ is continuous.

If $x = a$ and $x = b$, we use right- and left-hand limits, respectively, to show that $F(x)$ is continuous at $x = a$ and $x = b$.

Another method:

By Problem 5.10 and Problem 4.3, Chapter 4, it follows that $F'(x)$ exists and so $F(x)$ must be continuous.

CHANGE OF VARIABLES AND SPECIAL METHODS OF INTEGRATION

5.13. Prove the result (*7*), Page 95, for changing the variable of integration.

Let $F(x) = \int_a^x f(x)\,dx$ and $G(t) = \int_\alpha^t f\{g(t)\}\,g'(t)\,dt$, where $x = g(t)$.

Then $dF = f(x)\,dx$, $dG = f\{g(t)\}\,g'(t)\,dt$.

Since $dx = g'(t)\,dt$, it follows that $f(x)\,dx = f\{g(t)\}\,g'(t)\,dt$ so that $dF(x) = dG(t)$, from which $F(x) = G(t) + c$.

Now when $x = a$, $t = \alpha$ or $F(a) = G(\alpha) + c$. But $F(a) = G(\alpha) = 0$, so that $c = 0$. Hence $F(x) = G(t)$. Since $x = b$ when $t = \beta$, we have

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f\{g(t)\}\,g'(t)\,dt$$

as required.

5.14. Evaluate:

(*a*) $\displaystyle\int (x+2)\sin(x^2+4x-6)\,dx$ (*c*) $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt{(x+2)(3-x)}}$ (*e*) $\displaystyle\int_0^{1/\sqrt{2}} \frac{x\sin^{-1}x^2}{\sqrt{1-x^4}}\,dx$

(*b*) $\displaystyle\int \frac{\cot(\ln x)}{x}\,dx$ (*d*) $\displaystyle\int 2^{-x}\tanh 2^{1-x}\,dx$ (*f*) $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+x+1}}$

(*a*) **Method 1:** Let $x^2 + 4x - 6 = u$. Then $(2x+4)\,dx = du$, $(x+2)\,dx = \frac{1}{2}\,du$ and the integral becomes

$$\frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos(x^2+4x-6) + c$$

Method 2:

$$\int (x+2)\,\sin(x^2+4x-6)\,dx = \frac{1}{2}\int \sin(x^2+4x-6)d(x^2+4x-6) = -\frac{1}{2}\cos(x^2+4x-6) + c$$

(*b*) Let $\ln x = u$. Then $(dx)/x = du$ and the integral becomes

$$\int \cot u\,du = \ln|\sin u| + c = \ln|\sin(\ln x)| + c$$

(*c*) **Method 1:** $\displaystyle\int \frac{dx}{\sqrt{(x+2)(3-x)}} = \int \frac{dx}{\sqrt{6+x-x^2}} = \int \frac{dx}{\sqrt{6-(x^2-x)}} = \int \frac{dx}{\sqrt{25/4-(x-\frac{1}{2})^2}}$

Letting $x - \frac{1}{2} = u$, this becomes

$$\int \frac{du}{\sqrt{25/4-u^2}} = \sin^{-1}\frac{u}{5/2} + c = \sin^{-1}\left(\frac{2x-1}{5}\right) + c$$

Then

$$\int_{-1}^{1} \frac{dx}{\sqrt{(x+2)(3-x)}} = \sin^{-1}\left(\frac{2x-1}{5}\right)\Bigg|_{-1}^{1} = \sin^{-1}\left(\frac{1}{5}\right) - \sin^{-1}\left(-\frac{3}{5}\right)$$
$$= \sin^{-1}.2 + \sin^{-1}.6$$

Method 2: Let $x - \frac{1}{2} = u$ as in Method 1. Now when $x = -1$, $u = -\frac{3}{2}$; and when $x = 1$, $u = \frac{1}{2}$. Thus by Formula 25, Page 96.

$$\int_{-1}^{1} \frac{dx}{\sqrt{(x+2)(3-x)}} = \int_{-1}^{1} \frac{dx}{\sqrt{25/4 - (x-\frac{1}{2})^2}} = \int_{-3/2}^{1/2} \frac{du}{\sqrt{25/4 - u^2}} = \sin^{-1} \frac{u}{5/2}\bigg|_{-3/2}^{1/2}$$
$$= \sin^{1} .2 + \sin^{-1} .6$$

(*d*) Let $2^{1-x} = u$. Then $-2^{1-x}(\ln 2)dx = du$ and $2^{-x}dx = -\dfrac{du}{2\ln 2}$, so that the integral becomes

$$-\frac{1}{2\ln 2}\int \tanh u\, du = -\frac{1}{2\ln 2}\ln\cosh 2^{1-x} + c$$

(*e*) Let $\sin^{-1} x^2 = u$. Then $du = \dfrac{1}{\sqrt{1-(x^2)^2}}2x\,dx = \dfrac{2x\,dx}{\sqrt{1-x^4}}$ and the integral becomes

$$\frac{1}{2}\int u\,du = \frac{1}{4}u^2 + c = \frac{1}{4}(\sin^{-1} x^2)^2 + c$$

Thus $\displaystyle\int_0^{1/\sqrt{2}} \frac{x \sin^{-1} x^2}{\sqrt{1-x^4}}\,dx = \frac{1}{4}(\sin^{-1} x^2)^2\bigg|_0^{1/\sqrt{2}} = \frac{1}{4}\left(\sin^{-1}\frac{1}{2}\right)^2 = \frac{\pi^2}{144}$.

(*f*)
$$\int \frac{x\,dx}{\sqrt{x^2+x+1}} = \frac{1}{2}\int \frac{2x+1-1}{\sqrt{x^2+x+1}}\,dx = \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx - \frac{1}{2}\frac{dx}{\sqrt{x^2+x+1}}$$
$$= \frac{1}{2}\int (x^2+x+1)^{-1/2} d(x^2+x+1) - \frac{1}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2 + \frac{3}{4}}}$$
$$= \sqrt{x^2+x+1} - \tfrac{1}{2}\ln|x + \tfrac{1}{2} + \sqrt{(x+\tfrac{1}{2})^2 + \tfrac{3}{4}}| + c$$

5.15. Show that $\displaystyle\int_1^2 \frac{dx}{(x^2-2x+4)^{3/2}} = \frac{1}{6}$.

Write the integral as $\displaystyle\int_1^2 \frac{dx}{[(x-1)^2+3]^{3/2}}$. Let $x - 1 = \sqrt{3}\tan u$, $dx = \sqrt{3}\sec^2 u\,du$. When $x = 1$, $u = \tan^{-1} 0 = 0$; when $x = 2$, $u = \tan^{-1} 1/\sqrt{3} = \pi/6$. Then the integral becomes

$$\int_0^{\pi/6} \frac{\sqrt{3}\sec^2 u\,du}{[3 + 3\tan^2 u]^{3/2}} = \int_0^{\pi/6} \frac{\sqrt{3}\sec^2 u\,du}{[3\sec^2 u]^{3/2}} = \frac{1}{3}\int_0^{\pi/6} \cos u\,du = \frac{1}{3}\sin u\bigg|_0^{\pi/6} = \frac{1}{6}$$

5.16. Determine $\displaystyle\int_e^{e^2} \frac{dx}{x(\ln x)^3}$.

Let $\ln x = y$, $(dx)/x = dy$. When $x = e$, $y = 1$; when $x = e^2$, $y = 2$. Then the integral becomes

$$\int_1^2 \frac{dy}{y^3} = \frac{y^{-2}}{-2}\bigg|_1^2 = \frac{3}{8}$$

5.17. Find $\displaystyle\int x^n \ln x\,dx$ if (*a*) $n \neq -1$, (*b*) $n = -1$.

(*a*) Use integration by parts, letting $u = \ln x$, $dv = x^n\,dx$, so that $du = (dx)/x$, $v = x^{n+1}/(n+1)$. Then

$$\int x^n \ln x\,dx = \int u\,dv = uv - \int v\,du = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^{n+1}}{n+1}\cdot\frac{dx}{x}$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + c$$

(*b*) $\displaystyle\int x^{-1}\ln x\,dx = \int \ln x\,d(\ln x) = \frac{1}{2}(\ln x)^2 + c.$

5.18. Find $\displaystyle\int 3^{\sqrt{2x+1}}\,dx$.

Let $\sqrt{2x+1} = y$, $2x+1 = y^2$. Then $dx = y\,dy$ and the integral becomes $\displaystyle\int 3^y \cdot y\,dy$.

Integrate by parts, letting $u = y$, $dv = 3^y\,dy$; then $du = dy$, $v = 3^y/(\ln 3)$, and we have

$$\int 3^y\cdot y\,dy = \int u\,dv = uv - \int v\,du = \frac{y\cdot 3^y}{\ln 3} - \int \frac{3^y}{\ln 3}\,dy = \frac{y\cdot 3^y}{\ln 3} - \frac{3^y}{(\ln 3)^2} + c$$

5.19. Find $\displaystyle\int_0^1 x\ln(x+3)\,dx$.

Let $u = \ln(x+3)$, $dv = x\,dx$. Then $du = \dfrac{dx}{x+3}$, $v = \dfrac{x^2}{2}$. Hence on integrating by parts,

$$\int x\ln(x+3)\,dx = \frac{x^2}{2}\ln(x+3) - \frac{1}{2}\int\frac{x^2\,dx}{x+3} = \frac{x^2}{2}\ln(x+3) - \frac{1}{2}\int\left(x - 3 + \frac{9}{x+3}\right)dx$$
$$= \frac{x^2}{2}\ln(x+3) - \frac{1}{2}\left\{\frac{x^2}{2} - 3x + 9\ln(x+3)\right\} + c$$

Then
$$\int_0^1 x\ln(x+3)\,dx = \frac{5}{4} - 4\ln 4 + \frac{9}{2}\ln 3$$

5.20. Determine $\displaystyle\int \frac{6-x}{(x-3)(2x+5)}\,dx$.

Use the method of *partial fractions*. Let $\dfrac{6-x}{(x-3)(2x+5)} = \dfrac{A}{x-3} + \dfrac{B}{2x+5}$.

Method 1: To determine the constants A and B, multiply both sides by $(x-3)(2x+5)$ to obtain

$$6 - x = A(2x+5) + B(x-3) \quad \text{or} \quad 6 - x = 5A - 3B + (2A + B)x \qquad (1)$$

Since this is an identity, $5A - 3B = 6$, $2A + B = -1$ and $A = 3/11$, $B = -17/11$. Then

$$\int\frac{6-x}{(x-3)(2x+5)}\,dx = \int\frac{3/11}{x-3}\,dx + \int\frac{-17/11}{2x+5}\,dx = \frac{3}{11}\ln|x-3| - \frac{17}{22}\ln|2x+5| + c$$

Method 2: Substitute suitable values for x in the identity (*1*). For example, letting $x = 3$ and $x = -5/2$ in (*1*), we find at once $A = 3/11$, $B = -17/11$.

5.21. Evaluate $\displaystyle\int\frac{dx}{5 + 3\cos x}$ by using the substitution $\tan x/2 = u$.

From Fig. 5-7 we see that

$$\sin x/2 = \frac{u}{\sqrt{1+u^2}}, \qquad \cos x/2 = \frac{1}{\sqrt{1+u^2}}$$

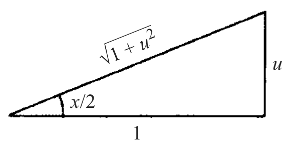


Fig. 5-7

Then $\cos x = \cos^2 x/2 - \sin^2 x/2 = \dfrac{1-u^2}{1+u^2}$.

Also $du = \dfrac{1}{2}\sec^2 x/2\,dx$ or $dx = 2\cos^2 x/2\,du = \dfrac{2\,du}{1+u^2}$.

Thus the integral becomes $\displaystyle\int \frac{du}{u^2+4} = \frac{1}{2}\tan^{-1} u/2 + c = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan x/2\right) + c.$

5.22. Evaluate $\displaystyle\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$.

Let $x = \pi - y$. Then

$$\begin{aligned} I &= \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi - y)\sin y}{1+\cos^2 y}\,dy = \pi\int_0^{\pi} \frac{\sin y}{1+\cos^2 y}\,dy - \int_0^{\pi} \frac{y\sin y}{1+\cos^2 y}\,dy \\ &= -\pi\int_0^{\pi} \frac{d(\cos y)}{1+\cos^2 y} - I = -\pi\tan^{-1}(\cos y)|_0^{\pi} - I = \pi^2/2 - I \end{aligned}$$

i.e., $I = \pi^2/2 - I$ or $I = \pi^2/4$.

5.23. Prove that $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \frac{\pi}{4}$.

Letting $x = \pi/2 - y$, we have

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos y}}{\sqrt{\cos y}+\sqrt{\sin y}}\,dy = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

Then

$$\begin{aligned} I + I &= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \\ &= \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \end{aligned}$$

from which $2I = \pi/2$ and $I = \pi/4$.

The same method can be used to prove that for all real values of m,

$$\int_0^{\pi/2} \frac{\sin^m x}{\sin^m x + \cos^m x}\,dx = \frac{\pi}{4}$$

(see Problem 5.89).

Note: This problem and Problem 5.22 show that some definite integrals can be evaluated without first finding the corresponding indefinite integrals.

NUMERICAL METHODS FOR EVALUATING DEFINITE INTEGRALS

5.24. Evaluate $\displaystyle\int_0^1 \frac{dx}{1+x^2}$ approximately, using (*a*) the trapezoidal rule, (*b*) Simpson's rule, where the interval [0, 1] is divided into $n = 4$ equal parts.

Let $f(x) = 1/(1+x^2)$. Using the notation on Page 98, we find $\Delta x = (b-a)/n = (1-0)/4 = 0.25$. Then keeping 4 decimal places, we have: $y_0 = f(0) = 1.0000$, $y_1 = f(0.25) = 0.9412$, $y_2 = f(0.50) = 0.8000$, $y_3 = f(0.75) = 0.6400$, $y_4 = f(1) = 0.50000$.

(*a*) The trapezoidal rule gives

$$\frac{\Delta x}{2}\{y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\} = \frac{0.25}{2}\{1.0000 + 2(0.9412) + 2(0.8000) + 2(0.6400) + 0.500\}$$
$$= 0.7828.$$

(*b*) Simpson's rule gives

$$\frac{\Delta x}{3}\{y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\} = \frac{0.25}{3}\{1.0000 + 4(0.9412) + 2(0.8000) + 4(0.6400) + 0.5000\}$$
$$= 0.7854.$$

The true value is $\pi/4 \approx 0.7854$.

APPLICATIONS (AREA, ARC LENGTH, VOLUME, MOMENT OF INTERTIA)

5.25. Find the (*a*) area and (*b*) moment of inertia about the y-axis of the region in the xy plane bounded by $y = 4 - x^2$ and the x-axis.

(*a*) Subdivide the region into rectangles as in the figure on Page 90. A typical rectangle is shown in the adjoining Fig. 5-8. Then

$$\text{Required area} = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)\,\Delta x_k$$
$$= \lim_{n\to\infty}\sum_{k=1}^{n}(4 - \xi_k^2)\,\Delta x_k$$
$$= \int_{-2}^{2}(4 - x^2)\,dx = \frac{32}{3}$$

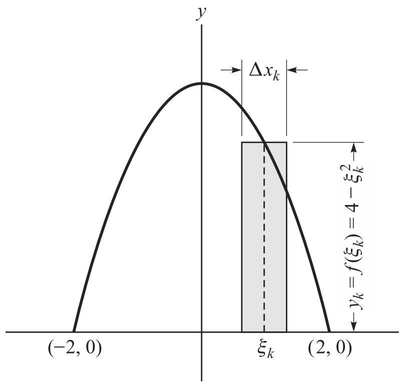


Fig. 5-8

(*b*) Assuming unit density, the moment of inertia about the y-axis of the typical rectangle shown above is $\xi_k^2 f(\xi_k)\,\Delta x_k$. Then

$$\text{Required moment of inertia} = \lim_{n\to\infty}\sum_{k=1}^{n}\xi_k^2 f(\xi_k)\,\Delta x_k = \lim_{n\to\infty}\sum_{k=1}^{n}\xi_k^2(4 - \xi_k^2)\,\Delta x_k$$
$$= \int_{-2}^{2} x^2(4 - x^2)\,dx = \frac{128}{15}$$

5.26. Find the length of arc of the parabola $y = x^2$ from $x = 0$ to $x = 1$.

$$\text{Required arc length} = \int_0^1 \sqrt{1 + (dy/dx)^2}\,dx = \int_0^1 \sqrt{1 + (2x)^2}\,dx$$
$$= \int_0^1 \sqrt{1 + 4x^2}\,dx = \frac{1}{2}\int_0^2 \sqrt{1 + u^2}\,du$$
$$= \tfrac{1}{2}\{\tfrac{1}{2}u\sqrt{1 + u^2} + \tfrac{1}{2}\ln(u + \sqrt{1 + u^2})\}|_0^2 = \tfrac{1}{2}\sqrt{5} + \tfrac{1}{4}\ln(2 + \sqrt{5})$$

5.27. (*a*) (Disk Method) Find the volume generated by revolving the region of Problem 5.25 about the x-axis.

$$\text{Required volume} = \lim_{n\to\infty}\sum_{k=1}^{n}\pi y_k^2\Delta x_k = \pi\int_{-2}^{2}(4 - x^2)^2\,dx = 512\pi/15.$$

(*b*) (Disk Method) Find the volume of the frustrum of a paraboloid obtained by revolving $f(x) = \sqrt{kx}$, $0 < a \leqq x \leqq b$ about the x-axis.

$$V = \pi \int_a^b kx\,dx = \frac{\pi k}{2}(b^2 - a^2).$$

(*c*) (Shell Method) Find the volume obtained by orbiting the region of part (*b*) about the *y*-axis. Compare this volume with that obtained in part (*b*).

$$V = 2\pi \int_0^b x(kx)\,dx = 2\pi k b^3/3$$

The solids generated by the two regions are different, as are the volumes.

MISCELLANEOUS PROBLEMS

5.28 If $f(x)$ and $g(x)$ are continuous in $[a, b]$, prove *Schwarz's inequality for integrals*:

$$\left(\int_a^b f(x)\,g(x)\,dx\right)^2 \leqq \int_a^b \{f(x)\}^2\,dx \int_a^b \{g(x)\}^2\,dx$$

We have

$$\int_a^b \{f(x) + \lambda g(x)\}^2\,dx = \int_a^b \{f(x)\}^2\,dx + 2\lambda \int_a^b f(x)\,g(x)\,dx + \lambda^2 \int_a^b \{g(x)\}^2\,dx \geqq 0$$

for all real values of λ. Hence by Problem 1.13 of Chapter 1, using (*1*) with

$$A^2 = \int_a^b g(x)\}^2\,dx, \qquad B^2 = \int_a^b \{f(x)\}^2\,dx, \qquad C = \int_a^b f(x)\,g(x)\,dx$$

we find $C^2 \leqq A^2B^2$, which gives the required result.

5.29. Prove that $\lim\limits_{M\to\infty} \int_0^M \frac{dx}{x^4 + 4} = \frac{\pi}{8}$.

We have $x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2 + 2)^2 - (2x)^2 = (x^2 + 2 + 2x)(x^2 + 2 - 2x)$.

According to the method of partial fractions, assume

$$\frac{1}{x^4 + 4} = \frac{Ax + B}{x^2 + 2x + 2} + \frac{Cx + D}{x^2 - 2x + 2}$$

Then $1 = (A + C)x^3 + (B - 2A + 2C + D)x^2 + (2A - 2B + 2C + 2D)x + 2B + 2D$

so that $A + C = 0$, $B - 2A + 2C + D = 0$, $2A - 2B + 2C + 2D = 0$, $2B + 2D = 1$

Solving simultaneously, $A = \frac{1}{8}$, $B = \frac{1}{4}$, $C = -\frac{1}{8}$, $D = \frac{1}{4}$. Thus

$$\begin{aligned}
\int \frac{dx}{x^4 + 4} &= \frac{1}{8}\int \frac{x + 2}{x^2 + 2x + 2}\,dx - \frac{1}{8}\int \frac{x - 2}{x^2 - 2x + 2}\,dx \\
&= \frac{1}{8}\int \frac{x + 1}{(x + 1)^2 + 1}\,dx + \frac{1}{8}\int \frac{dx}{(x + 1)^2 + 1} - \frac{1}{8}\int \frac{x - 1}{(x - 1)^2 + 1}\,dx + \frac{1}{8}\int \frac{dx}{(x - 1)^2 + 1} \\
&= \frac{1}{16}\ln(x^2 + 2x + 2) + \frac{1}{8}\tan^{-1}(x + 1) - \frac{1}{16}\ln(x^2 - 2x + 2) + \frac{1}{8}\tan^{-1}(x - 1) + C
\end{aligned}$$

Then

$$\lim_{M\to\infty} \int_0^M \frac{dx}{x^4 + 4} = \lim_{M\to\infty} \left\{\frac{1}{16}\ln\left(\frac{M^2 + 2M + 2}{M^2 - 2M + 2}\right) + \frac{1}{8}\tan^{-1}(M + 1) + \frac{1}{8}\tan^{-1}(M - 1)\right\} = \frac{\pi}{8}$$

We denote this limit by $\int_0^\infty \frac{dx}{x^4 + 4}$, called an *improper integral of the first kind*. Such integrals are considered further in Chapter 12. See also Problem 5.74.

5.30. Evaluate $\lim_{x \to 0} \dfrac{\int_0^x \sin t^3 \, dt}{x^4}$.

The conditions of L'Hospital's rule are satisfied, so that the required limit is

$$\lim_{x\to 0} \frac{\dfrac{d}{dx}\displaystyle\int_0^x \sin t^3\,dt}{\dfrac{d}{dx}(x^4)} = \lim_{x\to 0}\frac{\sin x^3}{4x^3} = \lim_{x\to 0}\frac{\dfrac{d}{dx}(\sin x^3)}{\dfrac{d}{dx}(4x^3)} = \lim_{x\to 0}\frac{3x^2\cos x^3}{12x^2} = \frac{1}{4}$$

5.31. Prove that if $f(x)$ is continuous in $[a, b]$ then $\int_a^b f(x)\,dx$ exists.

Let $\sigma = \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k$, using the notation of Page 91. Since $f(x)$ is continuous we can find numbers M_k and m_k representing the l.u.b. and g.l.b. of $f(x)$ in the interval $[x_{k-1}, x_k]$, i.e., such that $m_k \leqq f(x) \leqq M_k$. We then have

$$m(b-a) \leqq s = \sum_{k=1}^{n} m_k \Delta x_k \leqq \sigma \leqq \sum_{k=1}^{n} M_k \Delta x_k = S \leqq M(b-a) \qquad (1)$$

where m and M are the g.l.b. and l.u.b. of $f(x)$ in $[a, b]$. The sums s and S are sometimes called the *lower* and *upper sums*, respectively.

Now choose a second mode of subdivision of $[a, b]$ and consider the corresponding lower and upper sums denoted by s' and S' respectively. We have must

$$s' \leqq S \quad \text{and} \quad S' \geqq s \qquad (2)$$

To prove this we choose a third mode of subdivision obtained by using the division points of both the first and second modes of subdivision and consider the corresponding lower and upper sums, denoted by t and T, respectively. By Problem 5.84, we have

$$s \leqq t \leqq T \leqq S' \quad \text{and} \quad s' \leqq t \leqq T \leqq S \qquad (3)$$

which proves (*2*).

From (*2*) it is also clear that as the number of subdivisions is increased, the upper sums are monotonic decreasing and the lower sums are monotonic increasing. Since according to (*1*) these sums are also bounded, it follows that they have limiting values which we shall call $\bar{s}$ and $\underline{S}$ respectively. By Problem 5.85, $\bar{s} \leqq \underline{S}$. In order to prove that the integral exists, we must show that $\bar{s} = \underline{S}$.

Since $f(x)$ is continuous in the closed interval $[a, b]$, it is uniformly continuous. Then given any $\epsilon > 0$, we can take each Δx_k so small that $M_k - m_k < \epsilon/(b-a)$. It follows that

$$S - s = \sum_{k=1}^{n}(M_k - m_k)\Delta x_k < \frac{\epsilon}{b-a}\sum_{k=1}^{n}\Delta x_k = \epsilon \qquad (4)$$

Now $S - s = (S - \underline{S}) + (\underline{S} - \bar{s}) + (\bar{s} - s)$ and it follows that each term in parentheses is positive and so is less than ϵ by (*4*). In particular, since $\underline{S} - \bar{s}$ is a definite number it must be zero, i.e., $\underline{S} = \bar{s}$. Thus, the limits of the upper and lower sums are equal and the proof is complete.

Supplementary Problems

DEFINITION OF A DEFINITE INTEGRAL

5.32. (*a*) Express $\int_0^1 x^3\,dx$ as a limit of a sum. (*b*) Use the result of (*a*) to evaluate the given definite integral. (*c*) Interpret the result geometrically.
Ans. (*b*) $\frac{1}{4}$

5.33. Using the definition, evaluate (*a*) $\int_0^2 (3x+1)\,dx$, (*b*) $\int_3^6 (x^2 - 4x)\,dx$.
Ans. (*a*) 8, (*b*) 9

5.34. Prove that $\lim_{n\to\infty}\left\{\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right\}=\frac{\pi}{4}$.

5.35. Prove that $\lim_{n\to\infty}\left\{\frac{1^p+2^p+3^p+\cdots+n^p}{n^{p+1}}=\frac{1}{p+1}\right\}$ if $p>-1$.

5.36. Using the definition, prove that $\int_a^b e^x\,dx = e^b - e^a$.

5.37. Work Problem 5.5 directly, using Problem 1.94 of Chapter 1.

5.38. Prove that $\lim_{n\to\infty}\left\{\frac{1}{\sqrt{n^2+1^2}}+\frac{1}{\sqrt{n^2+2^2}}+\cdots+\frac{1}{\sqrt{n^2+n^2}}\right\}=\ln(1+\sqrt{2})$.

5.39. Prove that $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2x^2}=\frac{\tan^{-1}x}{x}$ if $x\neq 0$.

PROPERTIES OF DEFINITE INTEGRALS

5.40. Prove (*a*) Property 2, (*b*) Property 3 on Pages 91 and 92.

5.41. If $f(x)$ is integrable in (a, c) and (c, b), prove that $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

5.42. If $f(x)$ and $g(x)$ are integrable in $[a, b]$ and $f(x) \leqq g(x)$, prove that $\int_a^b f(x)\,dx \leqq \int_a^b g(x)\,dx$.

5.43. Prove that $1-\cos x \geqq x^2/\pi$ for $0 \leqq x \leqq \pi/2$.

5.44. Prove that $\left|\int_0^1 \frac{\cos nx}{x+1}\,dx\right| \leqq \ln 2$ for all n.

5.45. Prove that $\left|\int_1^{\sqrt{3}} \frac{e^{-x}\sin x}{x^2+1}\,dx\right| \leqq \frac{\pi}{12e}$.

MEAN VALUE THEOREMS FOR INTEGRALS

5.46. Prove the result (5), Page 92. [Hint: If $m \leqq f(x) \leqq M$, then $mg(x) \leqq f(x)g(x) \leqq Mg(x)$. Now integrate and divide by $\int_a^b g(x)\,dx$. Then apply Theorem 9 in Chapter 3.

5.47. Prove that there exist values ξ_1 and ξ_2 in $0 \leqq x \leqq 1$ such that

$$\int_0^1 \frac{\sin \pi x}{x^2+1}\,dx = \frac{2}{\pi(\xi_1^2+1)} = \frac{\pi}{4}\sin \pi\xi_2$$

Hint: Apply the first mean value theorem.

5.48. (*a*) Prove that there is a value ξ in $0 \leqq x \leqq \pi$ such that $\int_0^\pi e^{-x}\cos x\,dx = \sin\xi$. (*b*) Suppose a wedge in the shape of a right triangle is idealized by the region bound by the x-axis, $f(x) = x$, and $x = L$. Let the weight distribution for the wedge be defined by $W(x) = x^2 + 1$. Use the generalized mean value theorem to show that the point at which the weighted value occurs is $\frac{3L}{4}\frac{L^2+2}{L^2+3}$.

CHANGE OF VARIABLES AND SPECIAL METHODS OF INTEGRATION

5.49. Evaluate: (*a*) $\int x^2 e^{\sin x^3} \cos x^3\,dx$, (*b*) $\int_0^1 \frac{\tan^{-1} t}{1+t^2}\,dt$, (*c*) $\int_1^3 \frac{dx}{\sqrt{4x-x^2}}$, (*d*) $\int \frac{\operatorname{csch}^2\sqrt{u}}{\sqrt{u}}\,du$, (*e*) $\int_{-2}^{2} \frac{dx}{16-x^2}$.

Ans. (*a*) $\frac{1}{3}e^{\sin x^3}+c$, (*b*) $\pi^2/32$, (*c*) $\pi/3$, (*d*) $-2\coth\sqrt{u}+c$, (*e*) $\frac{1}{4}\ln 3$.

5.50. Show that (*a*) $\int_0^1 \frac{dx}{(3+2x-x^2)^{3/2}} = \frac{\sqrt{3}}{12}$, (*b*) $\int \frac{dx}{x^2\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x}+c$.

5.51. Prove that (*a*) $\int \sqrt{u^2 \pm a^2}\,du = \frac{1}{2}u\sqrt{u^2\pm a^2} \pm \frac{1}{2}a^2 \ln|u+\sqrt{u^2\pm a^2}|$

(*b*) $\int \sqrt{a^2-u^2}\,du = \frac{1}{2}u\sqrt{a^2-u^2}+\frac{1}{2}a^2\sin^{-1} u/a + c, \quad a>0.$

5.52. Find $\int \frac{x\,dx}{\sqrt{x^2+2x+5}}$. *Ans.* $\sqrt{x^2+2x+5} - \ln|x+1+\sqrt{x^2+2x+5}|+c$.

5.53. Establish the validity of the method of integration by parts.

5.54. Evaluate (*a*) $\int_0^{\pi} x\cos 3x\,dx$, (*b*) $\int x^3 e^{-2x}\,dx$. *Ans.* (*a*) $-2/9$, (*b*) $-\frac{1}{3}e^{-2x}(4x^3+6x^2+6x+3)+c$

5.55. Show that (*a*) $\int_0^1 x^2\tan^{-1} x\,dx = \frac{1}{12}\pi - \frac{1}{6}+\frac{1}{6}\ln 2$

(*b*) $\int_{-2}^{2} \sqrt{x^2+x+1}\,dx = \frac{5\sqrt{7}}{4}+\frac{3\sqrt{3}}{4}+\frac{3}{8}\ln\left(\frac{5+2\sqrt{7}}{2\sqrt{3}-3}\right).$

5.56. (*a*) If $u=f(x)$ and $v=g(x)$ have continuous nth derivatives, prove that

$$\int uv^{(n)}\,dx = uv^{(n-1)} - u'v^{(n-2)} + u''v^{(n-3)} - \cdots - (-1)^n \int u^{(n)} v\,dx$$

called *generalized integration by parts*. (*b*) What simplifications occur if $u^{(n)}=0$? Discuss. (*c*) Use (*a*) to evaluate $\int_0^{\pi} x^4\sin x\,dx$. *Ans.* (*c*) $\pi^4-12\pi^2+48$

5.57. Show that $\int_0^1 \frac{x\,dx}{(x+1)^2(x^2+1)} = \frac{\pi-2}{8}$.

[Hint: Use partial fractions, i.e., assume $\frac{x}{(x+1)^2(x^2+1)} = \frac{A}{(x+1)^2}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$ and find A, B, C, D.]

5.58. Prove that $\int_0^{\pi} \frac{dx}{\alpha-\cos x} = \frac{\pi}{\sqrt{\alpha^2-1}}, \quad \alpha>1.$

NUMERICAL METHODS FOR EVALUATING DEFINITE INTEGRALS

5.59. Evaluate $\int_0^1 \frac{dx}{1+x}$ approximately, using (*a*) the trapezoidal rule, (*b*) Simpson's rule, taking $n=4$. Compare with the exact value, $\ln 2 = 0.6931$.

5.60. Using (*a*) the trapezoidal rule, (*b*) Simpson's rule evaluate $\int_0^{\pi/2} \sin^2 x\,dx$ by obtaining the values of $\sin^2 x$ at $x = 0^\circ, 10^\circ, \ldots, 90^\circ$ and compare with the exact value $\pi/4$.

5.61. Prove the (*a*) rectangular rule, (*b*) trapezoidal rule, i.e., (*16*) and (*17*) of Page 98.

5.62. Prove Simpson's rule.

5.63. Evaluate to 3 decimal places using numerical integration: (*a*) $\int_1^2 \frac{dx}{1+x^2}$, (*b*) $\int_0^1 \cosh x^2\, dx$.
Ans. (*a*) 0.322, (*b*) 1.105.

APPLICATIONS

5.64. Find the (*a*) area and (*b*) moment of inertia about the *y*-axis of the region in the *xy* plane bounded by $y = \sin x$, $0 \leqq x \leqq \pi$ and the *x*-axis, assuming unit density.
Ans. (*a*) 2, (*b*) $\pi^2 - 4$

5.65. Find the moment of inertia about the *x*-axis of the region bounded by $y = x^2$ and $y = x$, if the density is proportional to the distance from the *x*-axis.
Ans. $\frac{1}{8}M$, where M = mass of the region.

5.66. (*a*) Show that the arc length of the *catenary* $y = \cosh x$ from $x = 0$ to $x = \ln 2$ is $\frac{3}{4}$. (*b*) Show that the length of arc of $y = x^{3/2}$, $2 \leqq x \leqq 5$ is $\frac{343}{27} - 2\sqrt{2}\, 11^{3/2}$.

5.67. Show that the length of one arc of the *cycloid* $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $(0 \leqq \theta \leqq 2\pi)$ is $8a$.

5.68. Prove that the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$ is πab.

5.69. (*a*) (Disk Method) Find the volume of the region obtained by revolving the curve $y = \sin x$, $0 \leqq x \leqq \pi$, about the *x*-axis. *Ans.* (*a*) $\pi^2/2$
(*b*) (Disk Method) Show that the volume of the frustrum of a paraboloid obtained by revolving $f(x) = \sqrt{kx}$, $0 < a \leqq x \leqq b$, about the *x*-axis is $\pi \int_a^b kx\, dx = \frac{\pi k}{2}(b^2 - a^2)$. (*c*) Determine the volume obtained by rotating the region bound by $f(x) = 3$, $g(x) = 5 - x^2$ on $-\sqrt{2} \leqq x \leqq \sqrt{2}$. (*d*) (Shell Method) A spherical bead of radius *a* has a circular cylindrical hole of radius *b*, $b < a$, through the center. Find the volume of the remaining solid by the shell method. (*e*) (Shell Method) Find the volume of a solid whose outer boundary is a torus (i.e., the solid is generated by orbiting a circle $(x-a)^2 + y^2 = b^2$ about the *y*-axis $(a > b)$.

5.70. Prove that the centroid of the region bounded by $y = \sqrt{a^2 - x^2}$, $-a \leqq x \leqq a$ and the *x*-axis is located at $(0, 4a/3\pi)$.

5.71. (*a*) If $\rho = f(\phi)$ is the equation of a curve in polar coordinates, show that the area bounded by this curve and the lines $\phi = \phi_1$ and $\phi = \phi_2$ is $\frac{1}{2}\int_{\phi_1}^{\phi_2} \rho^2 d\phi$. (*b*) Find the area bounded by one loop of the *lemniscate* $\rho^2 = a^2 \cos 2\phi$.
Ans. (*b*) a^2

5.72. (*a*) Prove that the arc length of the curve in Problem 5.71(*a*) is $\int_{\phi_1}^{\phi_2} \sqrt{\rho^2 + (d\rho/d\phi)^2}\, d\phi$. (*b*) Find the length of arc of the *cardioid* $\rho = a(1 - \cos\phi)$.
Ans. (*b*) $8a$

MISCELLANEOUS PROBLEMS

5.73. Establish the mean value theorem for derivatives from the first mean value theorem for integrals. [Hint: Let $f(x) = F'(x)$ in (*4*), Page 93.]

5.74. Prove that (*a*) $\lim_{\epsilon \to 0+} \int_0^{4-\epsilon} \frac{dx}{\sqrt{4-x}} = 4$, (*b*) $\lim_{\epsilon \to 0+} \int_\epsilon^3 \frac{dx}{\sqrt[3]{x}} = 6$, (*c*) $\lim_{\epsilon \to 0+} \int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$ and give a geometric interpretation of the results.

[These limits, denoted usually by $\int_0^4 \frac{dx}{\sqrt{4-x}}$, $\int_0^3 \frac{dx}{\sqrt[3]{x}}$ and $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ respectively, are called *improper integrals of the second kind* (see Problem 5.29) since the integrands are not bounded in the range of integration. For further discussion of improper integrals, see Chapter 12.]

5.75. Prove that (*a*) $\lim_{M \to \infty} \int_0^M x^5 e^{-x}\, dx = 4! = 24$, (*b*) $\lim_{\epsilon \to 0+} \int_1^{2-\epsilon} \frac{dx}{\sqrt{x(2-x)}} = \frac{\pi}{2}$.

5.76. Evaluate (*a*) $\displaystyle\int_0^\infty \frac{dx}{1+x^3}$, (*b*) $\displaystyle\int_0^{\pi/2} \frac{\sin 2x}{(\sin x)^{4/3}}\,dx$, (*c*) $\displaystyle\int_0^\infty \frac{dx}{x+\sqrt{x^2+1}}$.

Ans. (*a*) $\dfrac{2\pi}{3\sqrt{3}}$ (*b*) 3 (*c*) does not exist

5.77. Evaluate $\displaystyle\lim_{x\to\pi/2} \frac{ex^2/\pi - e\pi/4 + \int_x^{\pi/2} e^{\sin t}\,dt}{1+\cos 2x}$. *Ans.* $e/2\pi$

5.78. Prove: (*a*) $\displaystyle\frac{d}{dx}\int_{x^2}^{x^3}(t^2+t+1)\,dt = 3x^3+x^5-2x^3+3x^2-2x$, (*b* $\displaystyle\frac{d}{dx}\int_x^{x^2}\cos t^2\,dt = 2x\cos x^4 - \cos x^2$.

5.79. Prove that (*a*) $\displaystyle\int_0^\pi \sqrt{1+\sin x}\,dx = 4$, (*b*) $\displaystyle\int_0^{\pi/2}\frac{dx}{\sin x+\cos x} = \sqrt{2}\ln(\sqrt{2}+1)$.

5.80. Explain the fallacy: $\displaystyle I = \int_{-1}^{1}\frac{dx}{1+x^2} = -\int_{-1}^{1}\frac{dy}{1+y^2} = -I$, using the transformation $x = 1/y$. Hence $I = 0$. But $I = \tan^{-1}(1) - \tan^{-1}(-1) = \pi/4 - (-\pi/4) = \pi/2$. Thus $\pi/2 = 0$.

5.81. Prove that $\displaystyle\int_0^{1/2}\frac{\cos\pi x}{\sqrt{1+x^2}}\,dx \leqq \frac{1}{4}\tan^{-1}\frac{1}{2}$.

5.82. Evaluate $\displaystyle\lim_{n\to\infty}\left\{\frac{\sqrt{n+1}+\sqrt{n+2}+\cdots+\sqrt{2n-1}}{n^{3/2}}\right\}$. *Ans.* $\frac{2}{3}(2\sqrt{2}-1)$

5.83. Prove that $f(x) = \begin{cases} 1 \text{ if } x \text{ is irrational} \\ 0 \text{ if } x \text{ is rational}\end{cases}$ is not Riemann integrable in [0, 1].

[Hint: In (*2*), Page 91, let ξ_k, $k = 1, 2, 3, \ldots, n$ be first rational and then irrational points of subdivision and examine the lower and upper sums of Problem 5.31.]

5.84. Prove the result (*3*) of Problem 5.31. [Hint: First consider the effect of only one additional point of subdivision.]

5.85. In Problem 5.31, prove that $\bar{s} \leqq \underline{S}$. [Hint: Assume the contrary and obtain a contradiction.]

5.86. If $f(x)$ is sectionally continuous in [*a*, *b*], prove that $\displaystyle\int_a^b f(x)\,dx$ exists. [Hint: Enclose each point of discontinuity in an interval, noting that the sum of the lengths of such intervals can be made arbitrarily small. Then consider the difference between the upper and lower sums.

5.87. If $f(x) = \begin{cases} 2x & 0 < x < 1 \\ 3 & x = 1 \\ 6x-1 & 1 < x < 2\end{cases}$, find $\displaystyle\int_0^2 f(x)\,dx$. Interpret the result graphically. *Ans.* 9

5.88. Evaluate $\displaystyle\int_0^3 \{x - [x] + \tfrac{1}{2}\}\,dx$ where $[x]$ denotes the greatest integer less than or equal to x. Interpret the result graphically. *Ans.* 3

5.89. (*a*) Prove that $\displaystyle\int_0^{\pi/2}\frac{\sin^m x}{\sin^m x+\cos^m x}\,dx = \frac{\pi}{4}$ for all real values of m.

(*b*) Prove that $\displaystyle\int_0^{2\pi}\frac{dx}{1+\tan^4 x} = \pi$.

5.90. Prove that $\displaystyle\int_0^{\pi/2}\frac{\sin x}{x}\,dx$ exists.

5.91. Show that $\displaystyle\int_0^{0.5}\frac{\tan^{-1} x}{x}\,dx = 0.4872$ approximately.

5.92. Show that $\displaystyle\int_0^\pi \frac{x\,dx}{1+\cos^2 x} = \frac{\pi^2}{2\sqrt{2}}$.