CHAPTER 2
First Order Equations

IN THIS CHAPTER we study first order equations for which there are general methods of solution.

SECTION 2.1 deals with linear equations, the simplest kind of first order equations. In this section we introduce the method of variation of parameters. The idea underlying this method will be a unifying theme for our approach to solving many different kinds of differential equations throughout the book.

SECTION 2.2 deals with separable equations, the simplest nonlinear equations. In this section we introduce the idea of implicit and constant solutions of differential equations, and we point out some differences between the properties of linear and nonlinear equations.

SECTION 2.3 discusses existence and uniqueness of solutions of nonlinear equations. Although it may seem logical to place this section before Section 2.2, we presented Section 2.2 first so we could have illustrative examples in Section 2.3.

SECTION 2.4 deals with nonlinear equations that are not separable, but can be transformed into separable equations by a procedure similar to variation of parameters.

SECTION 2.5 covers exact differential equations, which are given this name because the method for solving them uses the idea of an exact differential from calculus.

SECTION 2.6 deals with equations that are not exact, but can made exact by multiplying them by a function known called *integrating factor*.

2.1 LINEAR FIRST ORDER EQUATIONS

A first order differential equation is said to be *linear* if it can be written as

$$y' + p(x)y = f(x). \tag{2.1.1}$$

A first order differential equation that can't be written like this is *nonlinear*. We say that (2.1.1) is *homogeneous* if $f \equiv 0$; otherwise it's *nonhomogeneous*. Since $y \equiv 0$ is obviously a solution of the homgeneous equation

$$y' + p(x)y = 0,$$

we call it the *trivial solution*. Any other solution is *nontrivial*.

Example 2.1.1 The first order equations

$$\begin{aligned} x^2y' + 3y &= x^2, \\ xy' - 8x^2y &= \sin x, \\ xy' + (\ln x)y &= 0, \\ y' &= x^2y - 2, \end{aligned}$$

are not in the form (2.1.1), but they are linear, since they can be rewritten as

$$\begin{aligned} y' + \frac{3}{x^2}y &= 1, \\ y' - 8xy &= \frac{\sin x}{x}, \\ y' + \frac{\ln x}{x}y &= 0, \\ y' - x^2y &= -2. \end{aligned}$$

Example 2.1.2 Here are some nonlinear first order equations:

$$\begin{aligned} xy' + 3y^2 &= 2x && \text{(because } y \text{ is squared)}, \\ yy' &= 3 && \text{(because of the product } yy'), \\ y' + xe^y &= 12 && \text{(because of } e^y). \end{aligned}$$

General Solution of a Linear First Order Equation

To motivate a definition that we'll need, consider the simple linear first order equation

$$y' = \frac{1}{x^2}. \tag{2.1.2}$$

From calculus we know that y satisfies this equation if and only if

$$y = -\frac{1}{x} + c, \tag{2.1.3}$$

where c is an arbitrary constant. We call c a *parameter* and say that (2.1.3) defines a *one–parameter family* of functions. For each real number c, the function defined by (2.1.3) is a solution of (2.1.2) on

$(-\infty, 0)$ and $(0, \infty)$; moreover, every solution of (2.1.2) on either of these intervals is of the form (2.1.3) for some choice of c. We say that (2.1.3) is *the general solution* of (2.1.2).

We'll see that a similar situation occurs in connection with any first order linear equation

$$y' + p(x)y = f(x); \tag{2.1.4}$$

that is, if p and f are continuous on some open interval (a, b) then there's a unique formula $y = y(x, c)$ analogous to (2.1.3) that involves x and a parameter c and has the these properties:

- For each fixed value of c, the resulting function of x is a solution of (2.1.4) on (a, b).
- If y is a solution of (2.1.4) on (a, b), then y can be obtained from the formula by choosing c appropriately.

We'll call $y = y(x, c)$ the *general solution* of (2.1.4).

When this has been established, it will follow that an equation of the form

$$P_0(x)y' + P_1(x)y = F(x) \tag{2.1.5}$$

has a general solution on any open interval (a, b) on which P_0, P_1, and F are all continuous and P_0 has no zeros, since in this case we can rewrite (2.1.5) in the form (2.1.4) with $p = P_1/P_0$ and $f = F/P_0$, which are both continuous on (a, b).

To avoid awkward wording in examples and exercises, we won't specify the interval (a, b) when we ask for the general solution of a specific linear first order equation. Let's agree that this always means that we want the general solution on every open interval on which p and f are continuous if the equation is of the form (2.1.4), or on which P_0, P_1, and F are continuous and P_0 has no zeros, if the equation is of the form (2.1.5). We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_0, P_1, and F are all continuous on an open interval (a, b), but P_0 *does* have a zero in (a, b), then (2.1.5) may fail to have a general solution on (a, b) in the sense just defined. Since this isn't a major point that needs to be developed in depth, we won't discuss it further; however, see Exercise 44 for an example.

Homogeneous Linear First Order Equations

We begin with the problem of finding the general solution of a homogeneous linear first order equation. The next example recalls a familiar result from calculus.

Example 2.1.3 Let a be a constant.

(a) Find the general solution of

$$y' - ay = 0. \tag{2.1.6}$$

(b) Solve the initial value problem

$$y' - ay = 0, \quad y(x_0) = y_0.$$

SOLUTION**(a)** You already know from calculus that if c is any constant, then $y = ce^{ax}$ satisfies (2.1.6). However, let's pretend you've forgotten this, and use this problem to illustrate a general method for solving a homogeneous linear first order equation.

We know that (2.1.6) has the trivial solution $y \equiv 0$. Now suppose y is a nontrivial solution of (2.1.6). Then, since a differentiable function must be continuous, there must be some open interval I on which y has no zeros. We rewrite (2.1.6) as

$$\frac{y'}{y} = a$$

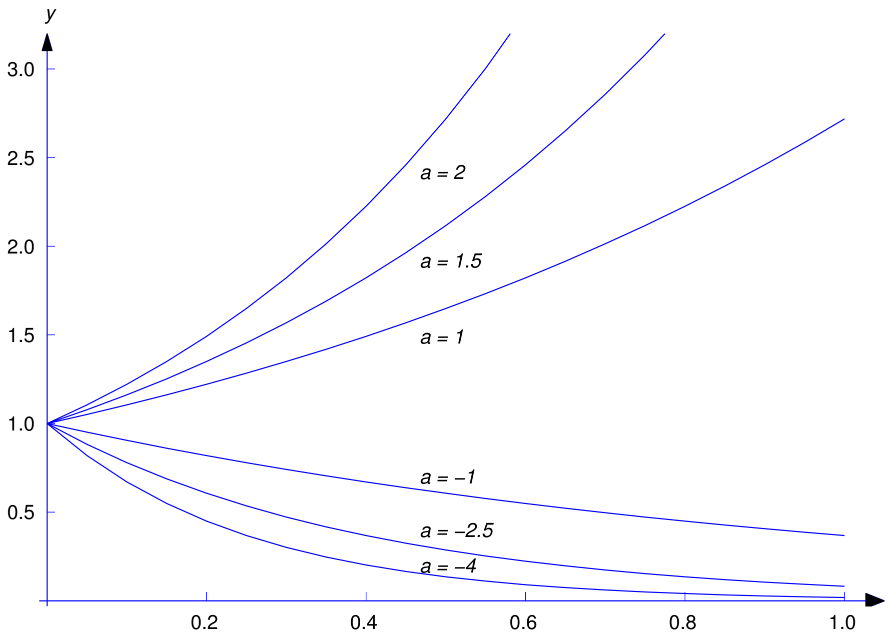

Figure 2.1.1 Solutions of $y' - ay = 0$, $y(0) = 1$

for x in I. Integrating this shows that

$$\ln|y| = ax + k, \quad \text{so} \quad |y| = e^k e^{ax},$$

where k is an arbitrary constant. Since e^{ax} can never equal zero, y has no zeros, so y is either always positive or always negative. Therefore we can rewrite y as

$$y = ce^{ax} \tag{2.1.7}$$

where

$$c = \begin{cases} e^k & \text{if } y > 0, \\ -e^k & \text{if } y < 0. \end{cases}$$

This shows that every nontrivial solution of (2.1.6) is of the form $y = ce^{ax}$ for some nonzero constant c. Since setting $c = 0$ yields the trivial solution, *all* solutions of (2.1.6) are of the form (2.1.7). Conversely, (2.1.7) is a solution of (2.1.6) for every choice of c, since differentiating (2.1.7) yields $y' = ace^{ax} = ay$.

SOLUTION(**b**) Imposing the initial condition $y(x_0) = y_0$ yields $y_0 = ce^{ax_0}$, so $c = y_0 e^{-ax_0}$ and

$$y = y_0 e^{-ax_0} e^{ax} = y_0 e^{a(x-x_0)}.$$

Figure 2.1.1 show the graphs of this function with $x_0 = 0$, $y_0 = 1$, and various values of a.

Example 2.1.4 **(a)** Find the general solution of

$$xy' + y = 0. \tag{2.1.8}$$

(b) Solve the initial value problem

$$xy' + y = 0, \quad y(1) = 3. \tag{2.1.9}$$

SOLUTION(**a**) We rewrite (2.1.8) as

$$y' + \frac{1}{x}y = 0, \tag{2.1.10}$$

where x is restricted to either $(-\infty, 0)$ or $(0, \infty)$. If y is a nontrivial solution of (2.1.10), there must be some open interval I on which y has no zeros. We can rewrite (2.1.10) as

$$\frac{y'}{y} = -\frac{1}{x}$$

for x in I. Integrating shows that

$$\ln|y| = -\ln|x| + k, \quad \text{so} \quad |y| = \frac{e^k}{|x|}.$$

Since a function that satisfies the last equation can't change sign on either $(-\infty, 0)$ or $(0, \infty)$, we can rewrite this result more simply as

$$y = \frac{c}{x} \tag{2.1.11}$$

where

$$c = \begin{cases} e^k & \text{if } y > 0, \\ -e^k & \text{if } y < 0. \end{cases}$$

We've now shown that every solution of (2.1.10) is given by (2.1.11) for some choice of c. (Even though we assumed that y was nontrivial to derive (2.1.11), we can get the trivial solution by setting $c = 0$ in (2.1.11).) Conversely, any function of the form (2.1.11) is a solution of (2.1.10), since differentiating (2.1.11) yields

$$y' = -\frac{c}{x^2},$$

and substituting this and (2.1.11) into (2.1.10) yields

$$\begin{aligned} y' + \frac{1}{x}y &= -\frac{c}{x^2} + \frac{1}{x}\frac{c}{x} \\ &= -\frac{c}{x^2} + \frac{c}{x^2} = 0. \end{aligned}$$

Figure 2.1.2 shows the graphs of some solutions corresponding to various values of c

SOLUTION(**b**) Imposing the initial condition $y(1) = 3$ in (2.1.11) yields $c = 3$. Therefore the solution of (2.1.9) is

$$y = \frac{3}{x}.$$

The interval of validity of this solution is $(0, \infty)$.

The results in Examples 2.1.3**(a)** and 2.1.4**(b)** are special cases of the next theorem.

Theorem 2.1.1 *If p is continuous on (a, b), then the general solution of the homogeneous equation*

$$y' + p(x)y = 0 \tag{2.1.12}$$

on (a, b) is

$$y = ce^{-P(x)},$$

where

$$P(x) = \int p(x)\,dx \tag{2.1.13}$$

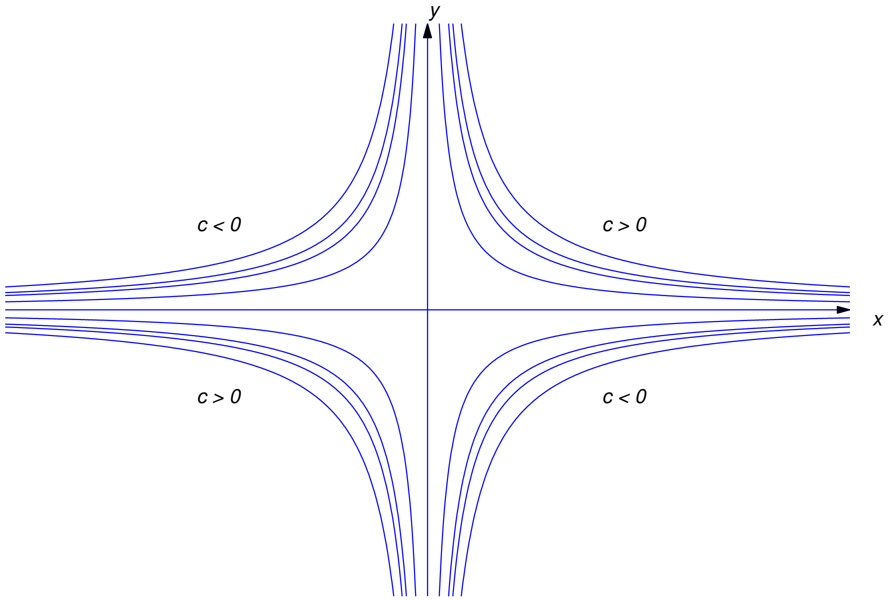

Figure 2.1.2 Solutions of $xy' + y = 0$ on $(0, \infty)$ and $(-\infty, 0)$

is any antiderivative of p on (a, b); that is,

$$P'(x) = p(x), \quad a < x < b. \tag{2.1.14}$$

Proof If $y = ce^{-P(x)}$, differentiating y and using (2.1.14) shows that

$$y' = -P'(x)ce^{-P(x)} = -p(x)ce^{-P(x)} = -p(x)y,$$

so $y' + p(x)y = 0$; that is, y is a solution of (2.1.12), for any choice of c.

Now we'll show that any solution of (2.1.12) can be written as $y = ce^{-P(x)}$ for some constant c. The trivial solution can be written this way, with $c = 0$. Now suppose y is a nontrivial solution. Then there's an open subinterval I of (a, b) on which y has no zeros. We can rewrite (2.1.12) as

$$\frac{y'}{y} = -p(x) \tag{2.1.15}$$

for x in I. Integrating (2.1.15) and recalling (2.1.13) yields

$$\ln |y| = -P(x) + k,$$

where k is a constant. This implies that

$$|y| = e^k e^{-P(x)}.$$

Since P is defined for all x in (a, b) and an exponential can never equal zero, we can take $I = (a, b)$, so y has zeros on (a, b) (a, b), so we can rewrite the last equation as $y = ce^{-P(x)}$, where

$$c = \begin{cases} e^k & \text{if } y > 0 \text{ on } (a, b), \\ -e^k & \text{if } y < 0 \text{ on } (a, b). \end{cases}$$

REMARK: Rewriting a first order differential equation so that one side depends only on y and y' and the other depends only on x is called *separation of variables*. We did this in Examples 2.1.3 and 2.1.4, and in rewriting (2.1.12) as (2.1.15).We'llapply this method to nonlinear equations in Section 2.2.

Linear Nonhomogeneous First Order Equations

We'll now solve the nonhomogeneous equation

$$y' + p(x)y = f(x). \tag{2.1.16}$$

When considering this equation we call

$$y' + p(x)y = 0$$

the *complementary equation*.

We'll find solutions of (2.1.16) in the form $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation and u is to be determined. This method of using a solution of the complementary equation to obtain solutions of a nonhomogeneous equation is a special case of a method called *variation of parameters*, which you'll encounter several times in this book. (Obviously, u can't be constant, since if it were, the left side of (2.1.16) would be zero. Recognizing this, the early users of this method viewed u as a "parameter" that varies; hence, the name "variation of parameters.")

If

$$y = uy_1, \quad \text{then} \quad y' = u'y_1 + uy_1'.$$

Substituting these expressions for y and y' into (2.1.16) yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x),$$

which reduces to

$$u'y_1 = f(x), \tag{2.1.17}$$

since y_1 is a solution of the complementary equation; that is,

$$y_1' + p(x)y_1 = 0.$$

In the proof of Theorem 2.2.1 we saw that y_1 has no zeros on an interval where p is continuous. Therefore we can divide (2.1.17) through by y_1 to obtain

$$u' = f(x)/y_1(x).$$

We can integrate this (introducing a constant of integration), and multiply the result by y_1 to get the general solution of (2.1.16). Before turning to the formal proof of this claim, let's consider some examples.

Example 2.1.5 Find the general solution of

$$y' + 2y = x^3e^{-2x}. \tag{2.1.18}$$

By applying **(a)** of Example 2.1.3 with $a = -2$, we see that $y_1 = e^{-2x}$ is a solution of the complementary equation $y' + 2y = 0$. Therefore we seek solutions of (2.1.18) in the form $y = ue^{-2x}$, so that

$$y' = u'e^{-2x} - 2ue^{-2x} \quad \text{and} \quad y' + 2y = u'e^{-2x} - 2ue^{-2x} + 2ue^{-2x} = u'e^{-2x}. \tag{2.1.19}$$

Therefore y is a solution of (2.1.18) if and only if

$$u'e^{-2x} = x^3e^{-2x} \quad \text{or, equivalently,} \quad u' = x^3.$$

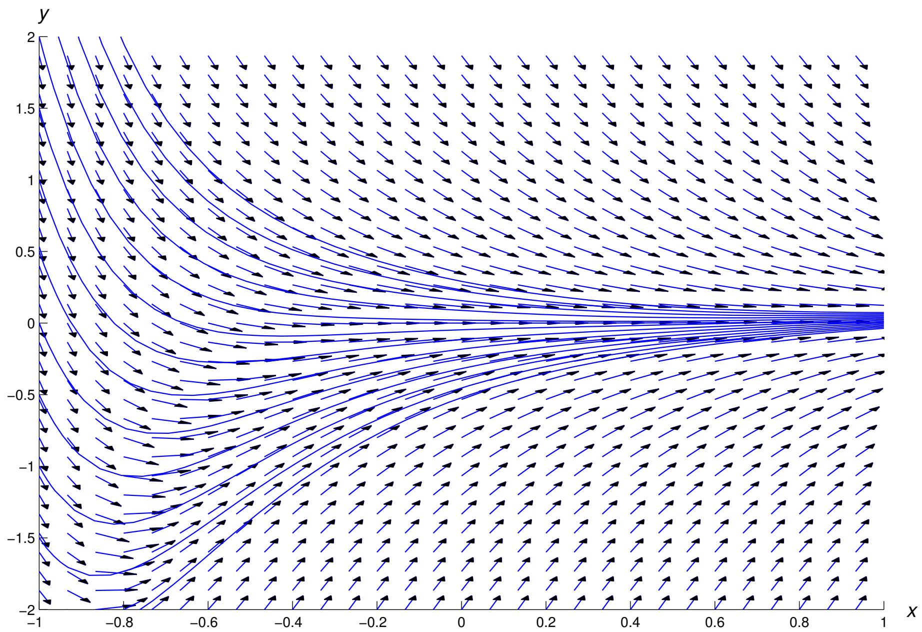

Figure 2.1.3 A direction field and integral curves for $y' + 2y = x^2e^{-2x}$

Therefore

$$u = \frac{x^4}{4} + c,$$

and

$$y = ue^{-2x} = e^{-2x}\left(\frac{x^4}{4} + c\right)$$

is the general solution of (2.1.18).

Figure 2.1.3 shows a direction field and some integral curves for (2.1.18).

Example 2.1.6

(a) Find the general solution

$$y' + (\cot x)y = x \csc x. \tag{2.1.20}$$

(b) Solve the initial value problem

$$y' + (\cot x)y = x \csc x, \quad y(\pi/2) = 1. \tag{2.1.21}$$

SOLUTION(**a**) Here $p(x) = \cot x$ and $f(x) = x \csc x$ are both continuous except at the points $x = r\pi$, where r is an integer. Therefore we seek solutions of (2.1.20) on the intervals $(r\pi, (r+1)\pi)$. We need a nontrival solution y_1 of the complementary equation; thus, y_1 must satisfy $y_1' + (\cot x)y_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = -\cot x = -\frac{\cos x}{\sin x}. \tag{2.1.22}$$

Integrating this yields

$$\ln |y_1| = -\ln |\sin x|,$$

where we take the constant of integration to be zero since we need only *one* function that satisfies (2.1.22). Clearly $y_1 = 1/\sin x$ is a suitable choice. Therefore we seek solutions of (2.1.20) in the form

$$y = \frac{u}{\sin x},$$

so that

$$y' = \frac{u'}{\sin x} - \frac{u\cos x}{\sin^2 x} \tag{2.1.23}$$

and

$$\begin{aligned} y' + (\cot x)y &= \frac{u'}{\sin x} - \frac{u\cos x}{\sin^2 x} + \frac{u\cot x}{\sin x} \\ &= \frac{u'}{\sin x} - \frac{u\cos x}{\sin^2 x} + \frac{u\cos x}{\sin^2 x} \\ &= \frac{u'}{\sin x}. \end{aligned} \tag{2.1.24}$$

Therefore y is a solution of (2.1.20) if and only if

$$u'/\sin x = x\csc x = x/\sin x \quad \text{or, equivalently,} \quad u' = x.$$

Integrating this yields

$$u = \frac{x^2}{2} + c, \quad \text{and} \quad y = \frac{u}{\sin x} = \frac{x^2}{2\sin x} + \frac{c}{\sin x}. \tag{2.1.25}$$

is the general solution of (2.1.20) on every interval $(r\pi, (r+1)\pi)$ (r =integer).

SOLUTION(**b**) Imposing the initial condition $y(\pi/2) = 1$ in (2.1.25) yields

$$1 = \frac{\pi^2}{8} + c \quad \text{or} \quad c = 1 - \frac{\pi^2}{8}.$$

Thus,

$$y = \frac{x^2}{2\sin x} + \frac{(1 - \pi^2/8)}{\sin x}$$

is a solution of (2.1.21). The interval of validity of this solution is $(0, \pi)$; Figure 2.1.4 shows its graph.

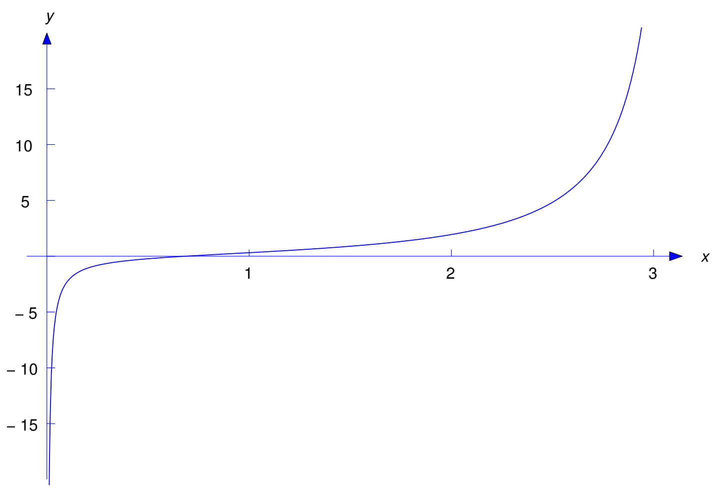

Figure 2.1.4 Solution of $y' + (\cot x)y = x\csc x$, $y(\pi/2) = 1$

REMARK: It wasn't necessary to do the computations (2.1.23) and (2.1.24) in Example 2.1.6, since we showed in the discussion preceding Example 2.1.5 that if $y = uy_1$ where $y_1' + p(x)y_1 = 0$, then $y' + p(x)y = u'y_1$. We did these computations so you would see this happen in this specific example. We recommend that you include these "unnecesary" computations in doing exercises, until you're confident that you really understand the method. After that, omit them.

We summarize the method of variation of parameters for solving

$$y' + p(x)y = f(x) \tag{2.1.26}$$

as follows:

(a) Find a function y_1 such that

$$\frac{y_1'}{y_1} = -p(x).$$

For convenience, take the constant of integration to be zero.

(b) Write

$$y = uy_1 \tag{2.1.27}$$

to remind yourself of what you're doing.

(c) Write $u'y_1 = f$ and solve for u'; thus, $u' = f/y_1$.

(d) Integrate u' to obtain u, with an arbitrary constant of integration.

(e) Substitute u into (2.1.27) to obtain y.

To solve an equation written as

$$P_0(x)y' + P_1(x)y = F(x),$$

we recommend that you divide through by $P_0(x)$ to obtain an equation of the form (2.1.26) and then follow this procedure.

Solutions in Integral Form

Sometimes the integrals that arise in solving a linear first order equation can't be evaluated in terms of elementary functions. In this case the solution must be left in terms of an integral.

Example 2.1.7

(a) Find the general solution of

$$y' - 2xy = 1.$$

(b) Solve the initial value problem

$$y' - 2xy = 1, \quad y(0) = y_0. \tag{2.1.28}$$

SOLUTION(**a**) To apply variation of parameters, we need a nontrivial solution y_1 of the complementary equation; thus, $y_1' - 2xy_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = 2x.$$

Integrating this and taking the constant of integration to be zero yields

$$\ln|y_1| = x^2, \quad \text{so} \quad |y_1| = e^{x^2}.$$

We choose $y_1 = e^{x^2}$ and seek solutions of (2.1.28) in the form $y = ue^{x^2}$, where

$$u'e^{x^2} = 1, \quad \text{so} \quad u' = e^{-x^2}.$$

Therefore

$$u = c + \int e^{-x^2}\,dx,$$

but we can't simplify the integral on the right because there's no elementary function with derivative equal to e^{-x^2}. Therefore the best available form for the general solution of (2.1.28) is

$$y = ue^{x^2} = e^{x^2}\left(c + \int e^{-x^2}\,dx\right). \tag{2.1.29}$$

SOLUTION(**b**) Since the initial condition in (2.1.28) is imposed at $x_0 = 0$, it is convenient to rewrite (2.1.29) as

$$y = e^{x^2}\left(c + \int_0^x e^{-t^2}\,dt\right), \quad \text{since} \quad \int_0^0 e^{-t^2}\,dt = 0.$$

Setting $x = 0$ and $y = y_0$ here shows that $c = y_0$. Therefore the solution of the initial value problem is

$$y = e^{x^2}\left(y_0 + \int_0^x e^{-t^2}\,dt\right). \tag{2.1.30}$$

For a given value of y_0 and each fixed x, the integral on the right can be evaluated by numerical methods. An alternate procedure is to apply the numerical integration procedures discussed in Chapter 3 directly to the initial value problem (2.1.28). Figure 2.1.5 shows graphs of of (2.1.30) for several values of y_0.

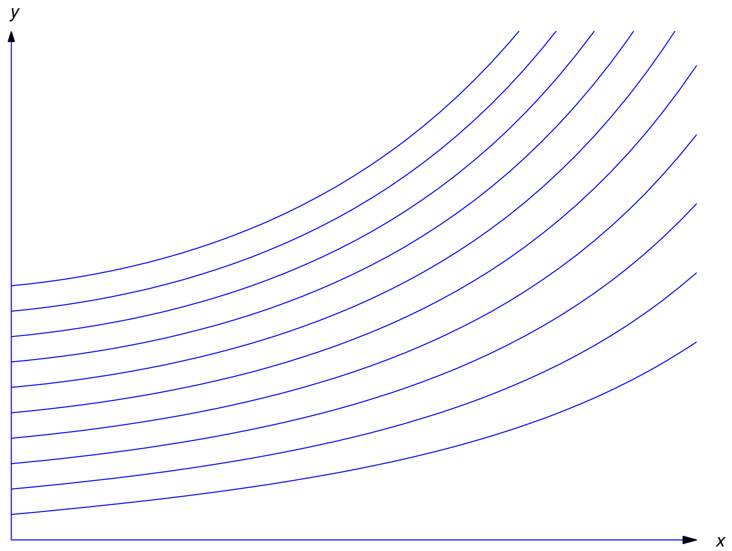

Figure 2.1.5 Solutions of $y' - 2xy = 1$, $y(0) = y_0$

An Existence and Uniqueness Theorem

The method of variation of parameters leads to this theorem.

Theorem 2.1.2 *Suppose p and f are continuous on an open interval (a,b), and let y_1 be any nontrivial solution of the complementary equation*

$$y' + p(x)y = 0$$

on (a,b). Then:

(a) *The general solution of the nonhomogeneous equation*

$$y' + p(x)y = f(x) \tag{2.1.31}$$

on (a,b) is

$$y = y_1(x)\left(c + \int f(x)/y_1(x)\,dx\right). \tag{2.1.32}$$

(b) *If x_0 is an arbitrary point in (a,b) and y_0 is an arbitrary real number, then the initial value problem*

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has the unique solution

$$y = y_1(x)\left(\frac{y_0}{y_1(x_0)} + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt\right)$$

on (a,b).

Proof **(a)** To show that (2.1.32) is the general solution of (2.1.31) on (a,b), we must prove that:

(i) If c is any constant, the function y in (2.1.32) is a solution of (2.1.31) on (a,b).

(ii) If y is a solution of (2.1.31) on (a,b) then y is of the form (2.1.32) for some constant c.

To prove **(i)**, we first observe that any function of the form (2.1.32) is defined on (a,b), since p and f are continuous on (a,b). Differentiating (2.1.32) yields

$$y' = y_1'(x)\left(c + \int f(x)/y_1(x)\,dx\right) + f(x).$$

Since $y_1' = -p(x)y_1$, this and (2.1.32) imply that

$$\begin{aligned} y' &= -p(x)y_1(x)\left(c + \int f(x)/y_1(x)\,dx\right) + f(x) \\ &= -p(x)y(x) + f(x), \end{aligned}$$

which implies that y is a solution of (2.1.31).

To prove **(ii)**, suppose y is a solution of (2.1.31) on (a,b). From the proof of Theorem 2.1.1, we know that y_1 has no zeros on (a,b), so the function $u = y/y_1$ is defined on (a,b). Moreover, since

$$y' = -py + f \quad \text{and} \quad y_1' = -py_1,$$

$$\begin{aligned} u' &= \frac{y_1 y' - y_1' y}{y_1^2} \\ &= \frac{y_1(-py + f) - (-py_1)y}{y_1^2} = \frac{f}{y_1}. \end{aligned}$$

Integrating $u' = f/y_1$ yields

$$u = \left(c + \int f(x)/y_1(x)\,dx\right),$$

which implies (2.1.32), since $y = uy_1$.

(b) We've proved **(a)**, where $\int f(x)/y_1(x)\,dx$ in (2.1.32) is an arbitrary antiderivative of f/y_1. Now it's convenient to choose the antiderivative that equals zero when $x = x_0$, and write the general solution of (2.1.31) as

$$y = y_1(x)\left(c + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt\right).$$

Since

$$y(x_0) = y_1(x_0)\left(c + \int_{x_0}^{x_0} \frac{f(t)}{y_1(t)}\,dt\right) = cy_1(x_0),$$

we see that $y(x_0) = y_0$ if and only if $c = y_0/y_1(x_0)$.

2.1 Exercises

In Exercises 1–5 find the general solution.

1. $y' + ay = 0$ (a=constant)

2. $y' + 3x^2y = 0$

3. $xy' + (\ln x)y = 0$

4. $xy' + 3y = 0$

5. $x^2y' + y = 0$

In Exercises 6–11 solve the initial value problem.

6. $y' + \left(\frac{1+x}{x}\right)y = 0, \quad y(1) = 1$

7. $xy' + \left(1 + \frac{1}{\ln x}\right)y = 0, \quad y(e) = 1$

8. $xy' + (1 + x\cot x)y = 0, \quad y\left(\frac{\pi}{2}\right) = 2$

9. $y' - \left(\frac{2x}{1+x^2}\right)y = 0, \quad y(0) = 2$

10. $y' + \frac{k}{x}y = 0, \quad y(1) = 3 \quad$ (k= constant)

11. $y' + (\tan kx)y = 0, \quad y(0) = 2 \quad$ (k = constant)

In Exercises 12 –15 find the general solution. Also, plot a direction field and some integral curves on the rectangular region $\{-2 \le x \le 2,\ -2 \le y \le 2\}$.

12. C/G $y' + 3y = 1$

13. C/G $y' + \left(\frac{1}{x} - 1\right)y = -\frac{2}{x}$

14. C/G $y' + 2xy = xe^{-x^2}$

15. C/G $y' + \frac{2x}{1+x^2}y = \frac{e^{-x}}{1+x^2}$

In Exercises 16 –24 find the general solution.

16. $y' + \frac{1}{x}y = \frac{7}{x^2} + 3$

17. $y' + \frac{4}{x-1}y = \frac{1}{(x-1)^5} + \frac{\sin x}{(x-1)^4}$

18. $xy' + (1+2x^2)y = x^3e^{-x^2}$

19. $xy' + 2y = \frac{2}{x^2} + 1$

20. $y' + (\tan x)y = \cos x$

21. $(1+x)y' + 2y = \frac{\sin x}{1+x}$

22. $(x-2)(x-1)y' - (4x-3)y = (x-2)^3$

23. $y' + (2\sin x\cos x)y = e^{-\sin^2 x}$

24. $x^2y' + 3xy = e^x$

In Exercises 25–29 solve the initial value problem and sketch the graph of the solution.

25. C/G $y' + 7y = e^{3x}, \quad y(0) = 0$

26. C/G $(1+x^2)y' + 4xy = \frac{2}{1+x^2}, \quad y(0) = 1$

27. C/G $xy' + 3y = \frac{2}{x(1+x^2)}, \quad y(-1) = 0$

28. C/G $y' + (\cot x)y = \cos x, \quad y\left(\frac{\pi}{2}\right) = 1$

29. C/G $y' + \frac{1}{x}y = \frac{2}{x^2} + 1, \quad y(-1) = 0$

In Exercises 30–37 solve the initial value problem.

30. $(x-1)y' + 3y = \frac{1}{(x-1)^3} + \frac{\sin x}{(x-1)^2}, \quad y(0) = 1$

31. $xy' + 2y = 8x^2, \quad y(1) = 3$

32. $xy' - 2y = -x^2, \quad y(1) = 1$

33. $y' + 2xy = x, \quad y(0) = 3$

34. $(x-1)y' + 3y = \frac{1+(x-1)\sec^2 x}{(x-1)^3}, \quad y(0) = -1$

35. $(x+2)y' + 4y = \frac{1+2x^2}{x(x+2)^3}, \quad y(-1) = 2$

36. $(x^2-1)y' - 2xy = x(x^2-1), \quad y(0) = 4$

37. $(x^2-5)y' - 2xy = -2x(x^2-5), \quad y(2) = 7$

In Exercises 38–42 solve the initial value problem and leave the answer in a form involving a definite integral. (You can solve these problems numerically by methods discussed in Chapter 3.)

38. $y' + 2xy = x^2, \quad y(0) = 3$

39. $y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2$

40. $y' + y = \dfrac{e^{-x}\tan x}{x}, \quad y(1) = 0$

41. $y' + \dfrac{2x}{1+x^2}y = \dfrac{e^x}{(1+x^2)^2}, \quad y(0) = 1$

42. $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2$

43. Experiments indicate that glucose is absorbed by the body at a rate proportional to the amount of glucose present in the bloodstream. Let λ denote the (positive) constant of proportionality. Now suppose glucose is injected into a patient's bloodstream at a constant rate of r units per unit of time. Let $G = G(t)$ be the number of units in the patient's bloodstream at time $t > 0$. Then

$$G' = -\lambda G + r,$$

where the first term on the right is due to the absorption of the glucose by the patient's body and the second term is due to the injection. Determine G for $t > 0$, given that $G(0) = G_0$. Also, find $\lim_{t\to\infty} G(t)$.

44. **(a)** L Plot a direction field and some integral curves for

$$xy' - 2y = -1 \tag{A}$$

on the rectangular region $\{-1 \le x \le 1, -.5 \le y \le 1.5\}$. What do all the integral curves have in common?

(b) Show that the general solution of (A) on $(-\infty, 0)$ and $(0, \infty)$ is

$$y = \frac{1}{2} + cx^2.$$

(c) Show that y is a solution of (A) on $(-\infty, \infty)$ if and only if

$$y = \begin{cases} \dfrac{1}{2} + c_1x^2, & x \ge 0, \\ \dfrac{1}{2} + c_2x^2, & x < 0, \end{cases}$$

where c_1 and c_2 are arbitrary constants.

(d) Conclude from **(c)** that all solutions of (A) on $(-\infty, \infty)$ are solutions of the initial value problem

$$xy' - 2y = -1, \quad y(0) = \frac{1}{2}.$$

(e) Use **(b)** to show that if $x_0 \ne 0$ and y_0 is arbitrary, then the initial value problem

$$xy' - 2y = -1, \quad y(x_0) = y_0$$

has infinitely many solutions on $(-\infty, \infty)$. Explain why this does'nt contradict Theorem 2.1.1**(b)**.

45. Suppose f is continuous on an open interval (a, b) and α is a constant.

(a) Derive a formula for the solution of the initial value problem

$$y' + \alpha y = f(x), \quad y(x_0) = y_0, \tag{A}$$

where x_0 is in (a, b) and y_0 is an arbitrary real number.

(b) Suppose $(a,b)=(a,\infty)$, $\alpha>0$ and $\lim_{x\to\infty} f(x)=L$. Show that if y is the solution of (A), then $\lim_{x\to\infty} y(x)=L/\alpha$.

46. Assume that all functions in this exercise are defined on a common interval (a,b).

(a) Prove: If y_1 and y_2 are solutions of

$$y'+p(x)y=f_1(x)$$

and

$$y'+p(x)y=f_2(x)$$

respectively, and c_1 and c_2 are constants, then $y=c_1y_1+c_2y_2$ is a solution of

$$y'+p(x)y=c_1f_1(x)+c_2f_2(x).$$

(This is the*principle of superposition.*)

(b) Use **(a)** to show that if y_1 and y_2 are solutions of the nonhomogeneous equation

$$y'+p(x)y=f(x), \tag{A}$$

then y_1-y_2 is a solution of the homogeneous equation

$$y'+p(x)y=0. \tag{B}$$

(c) Use **(a)** to show that if y_1 is a solution of (A) and y_2 is a solution of (B), then y_1+y_2 is a solution of (A).

47. Some nonlinear equations can be transformed into linear equations by changing the dependent variable. Show that if

$$g'(y)y'+p(x)g(y)=f(x)$$

where y is a function of x and g is a function of y, then the new dependent variable $z=g(y)$ satisfies the linear equation

$$z'+p(x)z=f(x).$$

48. Solve by the method discussed in Exercise 47.

(a) $(\sec^2 y)y'-3\tan y=-1$ **(b)** $e^{y^2}\left(2yy'+\dfrac{2}{x}\right)=\dfrac{1}{x^2}$

(c) $\dfrac{xy'}{y}+2\ln y=4x^2$ **(d)** $\dfrac{y'}{(1+y)^2}-\dfrac{1}{x(1+y)}=-\dfrac{3}{x^2}$

49. We've shown that if p and f are continuous on (a,b) then every solution of

$$y'+p(x)y=f(x) \tag{A}$$

on (a,b) can be written as $y=uy_1$, where y_1 is a nontrivial solution of the complementary equation for (A) and $u'=f/y_1$. Now suppose $f, f', \dots, f^{(m)}$ and $p, p', \dots, p^{(m-1)}$ are continuous on (a,b), where m is a positive integer, and define

$$\begin{aligned} f_0 &= f, \\ f_j &= f'_{j-1}+pf_{j-1}, \quad 1\le j\le m. \end{aligned}$$

Show that

$$u^{(j+1)}=\frac{f_j}{y_1}, \quad 0\le j\le m.$$

2.2 SEPARABLE EQUATIONS

A first order differential equation is *separable* if it can be written as

$$h(y)y' = g(x), \tag{2.2.1}$$

where the left side is a product of y' and a function of y and the right side is a function of x. Rewriting a separable differential equation in this form is called *separation of variables.* In Section 2.1 we used separation of variables to solve homogeneous linear equations. In this section we'll apply this method to nonlinear equations.

To see how to solve (2.2.1), let's first assume that y is a solution. Let $G(x)$ and $H(y)$ be antiderivatives of $g(x)$ and $h(y)$; that is,

$$H'(y) = h(y) \quad \text{and} \quad G'(x) = g(x). \tag{2.2.2}$$

Then, from the chain rule,

$$\frac{d}{dx}H(y(x)) = H'(y(x))y'(x) = h(y)y'(x).$$

Therefore (2.2.1) is equivalent to

$$\frac{d}{dx}H(y(x)) = \frac{d}{dx}G(x).$$

Integrating both sides of this equation and combining the constants of integration yields

$$H(y(x)) = G(x) + c. \tag{2.2.3}$$

Although we derived this equation on the assumption that y is a solution of (2.2.1), we can now view it differently: Any differentiable function y that satisfies (2.2.3) for some constant c is a solution of (2.2.1). To see this, we differentiate both sides of (2.2.3), using the chain rule on the left, to obtain

$$H'(y(x))y'(x) = G'(x),$$

which is equivalent to

$$h(y(x))y'(x) = g(x)$$

because of (2.2.2).

In conclusion, to solve (2.2.1) it suffices to find functions $G = G(x)$ and $H = H(y)$ that satisfy (2.2.2). Then any differentiable function $y = y(x)$ that satisfies (2.2.3) is a solution of (2.2.1).

Example 2.2.1 Solve the equation

$$y' = x(1 + y^2).$$

Solution Separating variables yields

$$\frac{y'}{1 + y^2} = x.$$

Integrating yields

$$\tan^{-1} y = \frac{x^2}{2} + c$$

Therefore

$$y = \tan\left(\frac{x^2}{2} + c\right).$$

Example 2.2.2

(a) Solve the equation

$$y' = -\frac{x}{y}. \tag{2.2.4}$$

(b) Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = 1. \tag{2.2.5}$$

(c) Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = -2. \tag{2.2.6}$$

SOLUTION(**a**) Separating variables in (2.2.4) yields

$$yy' = -x.$$

Integrating yields

$$\frac{y^2}{2} = -\frac{x^2}{2} + c, \quad \text{or, equivalently,} \quad x^2 + y^2 = 2c.$$

The last equation shows that c must be positive if y is to be a solution of (2.2.4) on an open interval. Therefore we let $2c = a^2$ (with $a > 0$) and rewrite the last equation as

$$x^2 + y^2 = a^2. \tag{2.2.7}$$

This equation has two differentiable solutions for y in terms of x:

$$y = \sqrt{a^2 - x^2}, \quad -a < x < a, \tag{2.2.8}$$

and

$$y = -\sqrt{a^2 - x^2}, \quad -a < x < a. \tag{2.2.9}$$

The solution curves defined by (2.2.8) are semicircles above the x-axis and those defined by (2.2.9) are semicircles below the x-axis (Figure 2.2.1).

SOLUTION(**b**) The solution of (2.2.5) is positive when $x = 1$; hence, it is of the form (2.2.8). Substituting $x = 1$ and $y = 1$ into (2.2.7) to satisfy the initial condition yields $a^2 = 2$; hence, the solution of (2.2.5) is

$$y = \sqrt{2 - x^2}, \quad -\sqrt{2} < x < \sqrt{2}.$$

SOLUTION(**c**) The solution of (2.2.6) is negative when $x = 1$ and is therefore of the form (2.2.9). Substituting $x = 1$ and $y = -2$ into (2.2.7) to satisfy the initial condition yields $a^2 = 5$. Hence, the solution of (2.2.6) is

$$y = -\sqrt{5 - x^2}, \quad -\sqrt{5} < x < \sqrt{5}.$$

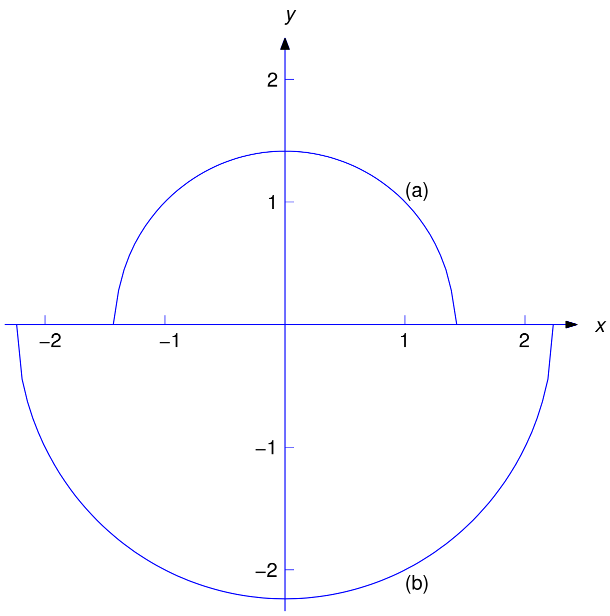

Figure 2.2.1 **(a)** $y = \sqrt{2 - x^2}$, $-\sqrt{2} < x < \sqrt{2}$; **(b)** $y = -\sqrt{5 - x^2}$, $-\sqrt{5} < x < \sqrt{5}$

Implicit Solutions of Separable Equations

In Examples 2.2.1 and 2.2.2 we were able to solve the equation $H(y) = G(x) + c$ to obtain explicit formulas for solutions of the given separable differential equations. As we'll see in the next example, this isn't always possible. In this situation we must broaden our definition of a solution of a separable equation. The next theorem provides the basis for this modification. We omit the proof, which requires a result from advanced calculus called as the *implicit function theorem*.

Theorem 2.2.1 *Suppose $g = g(x)$ is continous on (a, b) and $h = h(y)$ are continuous on (c, d). Let G be an antiderivative of g on (a, b) and let H be an antiderivative of h on (c, d). Let x_0 be an arbitrary point in (a, b), let y_0 be a point in (c, d) such that $h(y_0) \neq 0$, and define*

$$c = H(y_0) - G(x_0). \tag{2.2.10}$$

Then there's a function $y = y(x)$ defined on some open interval (a_1, b_1), where $a \le a_1 < x_0 < b_1 \le b$, such that $y(x_0) = y_0$ and

$$H(y) = G(x) + c \tag{2.2.11}$$

for $a_1 < x < b_1$. Therefore y is a solution of the initial value problem

$$h(y)y' = g(x), \quad y(x_0) = x_0. \tag{2.2.12}$$

It's convenient to say that (2.2.11) with c arbitrary is an *implicit solution* of $h(y)y' = g(x)$. Curves defined by (2.2.11) are integral curves of $h(y)y' = g(x)$. If c satisfies (2.2.10), we'll say that (2.2.11) is an *implicit solution of the initial value problem* (2.2.12). However, keep these points in mind:

- For some choices of c there may not be any differentiable functions y that satisfy (2.2.11).

- The function y in (2.2.11) (not (2.2.11) itself) is a solution of $h(y)y' = g(x)$.

Example 2.2.3

(a) Find implicit solutions of

$$y' = \frac{2x+1}{5y^4+1}. \tag{2.2.13}$$

(b) Find an implicit solution of

$$y' = \frac{2x+1}{5y^4+1}, \quad y(2) = 1. \tag{2.2.14}$$

SOLUTION(**a**) Separating variables yields

$$(5y^4+1)y' = 2x+1.$$

Integrating yields the implicit solution

$$y^5 + y = x^2 + x + c. \tag{2.2.15}$$

of (2.2.13).

SOLUTION(**b**) Imposing the initial condition $y(2) = 1$ in (2.2.15) yields $1 + 1 = 4 + 2 + c$, so $c = -4$. Therefore

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem (2.2.14). Although more than one differentiable function $y = y(x)$ satisfies 2.2.13) near $x = 1$, it can be shown that there's only one such function that satisfies the initial condition $y(1) = 2$.

Figure 2.2.2 shows a direction field and some integral curves for (2.2.13).

Constant Solutions of Separable Equations

An equation of the form

$$y' = g(x)p(y)$$

is separable, since it can be rewritten as

$$\frac{1}{p(y)}y' = g(x).$$

However, the division by $p(y)$ is not legitimate if $p(y) = 0$ for some values of y. The next two examples show how to deal with this problem.

Example 2.2.4 Find all solutions of

$$y' = 2xy^2. \tag{2.2.16}$$

Solution Here we must divide by $p(y) = y^2$ to separate variables. This isn't legitimate if y is a solution of (2.2.16) that equals zero for some value of x. One such solution can be found by inspection: $y \equiv 0$. Now suppose y is a solution of (2.2.16) that isn't identically zero. Since y is continuous there must be an interval on which y is never zero. Since division by y^2 is legitimate for x in this interval, we can separate variables in (2.2.16) to obtain

$$\frac{y'}{y^2} = 2x.$$

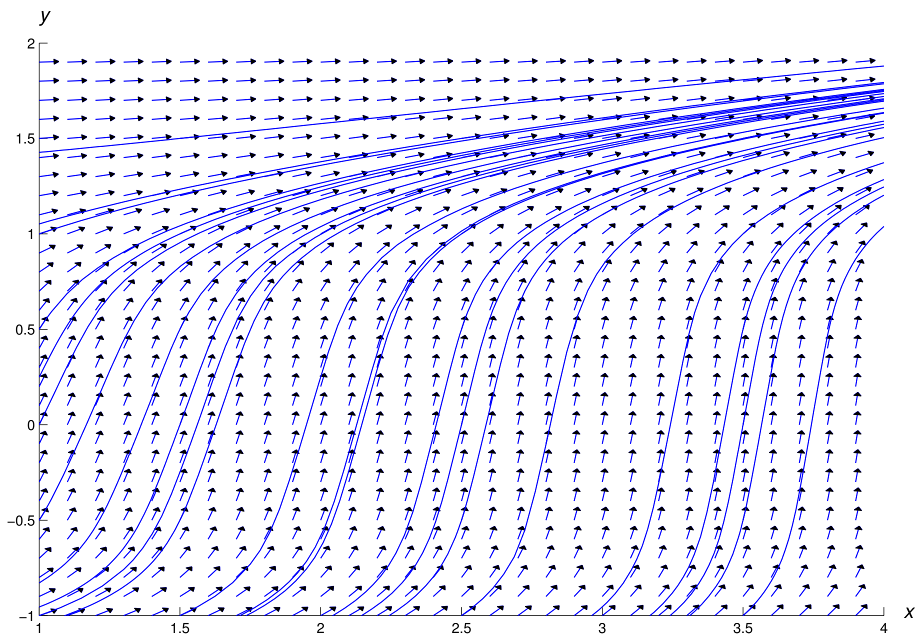

Figure 2.2.2 A direction field and integral curves for $y' = \dfrac{2x+1}{5y^4+1}$

Integrating this yields

$$-\frac{1}{y} = x^2 + c,$$

which is equivalent to

$$y = -\frac{1}{x^2+c}. \tag{2.2.17}$$

We've now shown that if y is a solution of (2.2.16) that is not identically zero, then y must be of the form (2.2.17). By substituting (2.2.17) into (2.2.16), you can verify that (2.2.17) is a solution of (2.2.16). Thus, solutions of (2.2.16) are $y \equiv 0$ and the functions of the form (2.2.17). Note that the solution $y \equiv 0$ isn't of the form (2.2.17) for any value of c.

Figure 2.2.3 shows a direction field and some integral curves for (2.2.16)

Example 2.2.5 Find all solutions of

$$y' = \frac{1}{2}x(1-y^2). \tag{2.2.18}$$

Solution Here we must divide by $p(y) = 1 - y^2$ to separate variables. This isn't legitimate if y is a solution of (2.2.18) that equals ± 1 for some value of x. Two such solutions can be found by inspection: $y \equiv 1$ and $y \equiv -1$. Now suppose y is a solution of (2.2.18) such that $1 - y^2$ isn't identically zero. Since $1 - y^2$ is continuous there must be an interval on which $1 - y^2$ is never zero. Since division by $1 - y^2$ is legitimate for x in this interval, we can separate variables in (2.2.18) to obtain

$$\frac{2y'}{y^2-1} = -x.$$

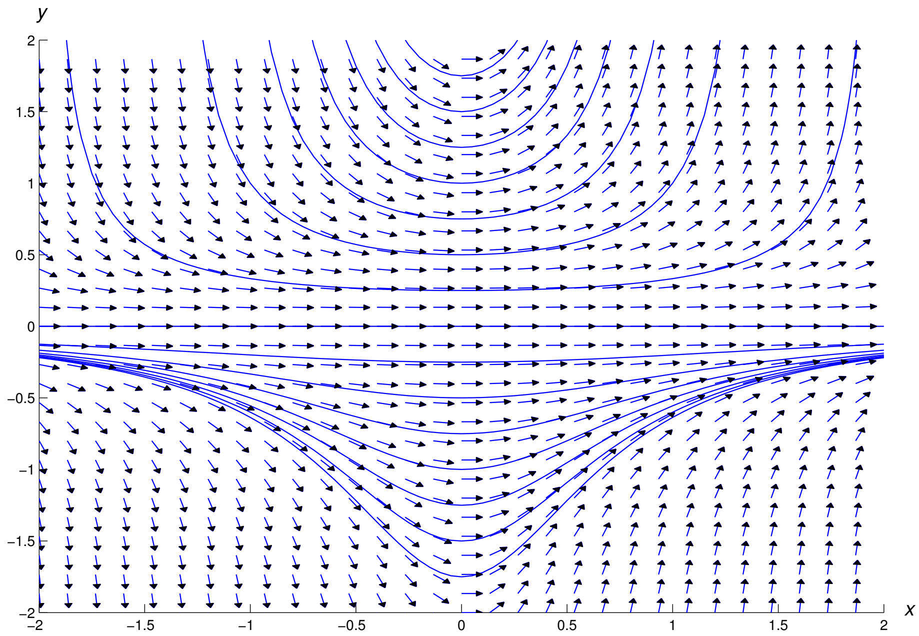

Figure 2.2.3 A direction field and integral curves for $y' = 2xy^2$

A partial fraction expansion on the left yields

$$\left[\frac{1}{y-1} - \frac{1}{y+1}\right] y' = -x,$$

and integrating yields

$$\ln\left|\frac{y-1}{y+1}\right| = -\frac{x^2}{2} + k;$$

hence,

$$\left|\frac{y-1}{y+1}\right| = e^k e^{-x^2/2}.$$

Since $y(x) \neq \pm 1$ for x on the interval under discussion, the quantity $(y-1)/(y+1)$ can't change sign in this interval. Therefore we can rewrite the last equation as

$$\frac{y-1}{y+1} = ce^{-x^2/2},$$

where $c = \pm e^k$, depending upon the sign of $(y-1)/(y+1)$ on the interval. Solving for y yields

$$y = \frac{1 + ce^{-x^2/2}}{1 - ce^{-x^2/2}}. \tag{2.2.19}$$

We've now shown that if y is a solution of (2.2.18) that is not identically equal to ± 1, then y must be as in (2.2.19). By substituting (2.2.19) into (2.2.18) you can verify that (2.2.19) is a solution of (2.2.18). Thus, the solutions of (2.2.18) are $y \equiv 1$, $y \equiv -1$ and the functions of the form (2.2.19). Note that the constant solution $y \equiv 1$ can be obtained from this formula by taking $c = 0$; however, the other constant solution, $y \equiv -1$, can't be obtained in this way.

Figure 2.2.4 shows a direction field and some integrals for (2.2.18).

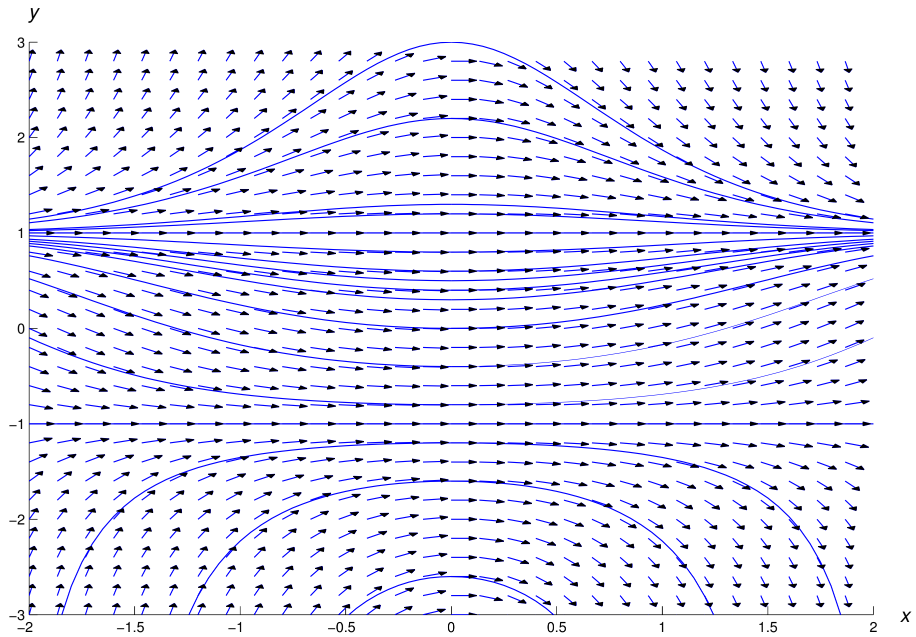

Figure 2.2.4 A direction field and integral curves for $y' = \dfrac{x(1-y^2)}{2}$

Differences Between Linear and Nonlinear Equations

Theorem 2.1.2 states that if p and f are continuous on (a,b) then every solution of

$$y' + p(x)y = f(x)$$

on (a,b) can be obtained by choosing a value for the constant c in the general solution, and if x_0 is any point in (a,b) and y_0 is arbitrary, then the initial value problem

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has a solution on (a,b).

The not true for nonlinear equations. First, we saw in Examples 2.2.4 and 2.2.5 that a nonlinear equation may have solutions that can't be obtained by choosing a specific value of a constant appearing in a one-parameter family of solutions. Second, it is in general impossible to determine the interval of validity of a solution to an initial value problem for a nonlinear equation by simply examining the equation, since the interval of validity may depend on the initial condition. For instance, in Example 2.2.2 we saw that the solution of

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(x_0) = y_0$$

is valid on $(-a,a)$, where $a = \sqrt{x_0^2 + y_0^2}$.

Example 2.2.6 Solve the initial value problem

$$y' = 2xy^2, \quad y(0) = y_0$$

and determine the interval of validity of the solution.

Solution First suppose $y_0 \neq 0$. From Example 2.2.4, we know that y must be of the form

$$y = -\frac{1}{x^2 + c}. \tag{2.2.20}$$

Imposing the initial condition shows that $c = -1/y_0$. Substituting this into (2.2.20) and rearranging terms yields the solution

$$y = \frac{y_0}{1 - y_0 x^2}.$$

This is also the solution if $y_0 = 0$. If $y_0 < 0$, the denominator isn't zero for any value of x, so the the solution is valid on $(-\infty, \infty)$. If $y_0 > 0$, the solution is valid only on $(-1/\sqrt{y_0}, 1/\sqrt{y_0})$.

2.2 Exercises

In Exercises 1–6 find all solutions.

1. $y' = \dfrac{3x^2 + 2x + 1}{y - 2}$

2. $(\sin x)(\sin y) + (\cos y)y' = 0$

3. $xy' + y^2 + y = 0$

4. $y' \ln|y| + x^2 y = 0$

5. $(3y^3 + 3y\cos y + 1)y' + \dfrac{(2x+1)y}{1 + x^2} = 0$

6. $x^2 y y' = (y^2 - 1)^{3/2}$

In Exercises 7–10 find all solutions. Also, plot a direction field and some integral curves on the indicated rectangular region.

7. C/G $y' = x^2(1 + y^2)$; $\{-1 \le x \le 1, \ -1 \le y \le 1\}$

8. C/G $y'(1 + x^2) + xy = 0$; $\{-2 \le x \le 2, \ -1 \le y \le 1\}$

9. C/G $y' = (x - 1)(y - 1)(y - 2)$; $\{-2 \le x \le 2, \ -3 \le y \le 3\}$

10. C/G $(y - 1)^2 y' = 2x + 3$; $\{-2 \le x \le 2, \ -2 \le y \le 5\}$

In Exercises 11 and 12 solve the initial value problem.

11. $y' = \dfrac{x^2 + 3x + 2}{y - 2}, \quad y(1) = 4$

12. $y' + x(y^2 + y) = 0, \quad y(2) = 1$

In Exercises 13-16 solve the initial value problem and graph the solution.

13. C/G $(3y^2 + 4y)y' + 2x + \cos x = 0, \quad y(0) = 1$

14. C/G $y' + \dfrac{(y+1)(y-1)(y-2)}{x+1} = 0, \quad y(1) = 0$

15. C/G $y' + 2x(y+1) = 0, \quad y(0) = 2$

16. C/G $y' = 2xy(1+y^2), \quad y(0) = 1$

In Exercises 17–23 solve the initial value problem and find the interval of validity of the solution.

17. $y'(x^2+2) + 4x(y^2+2y+1) = 0, \quad y(1) = -1$

18. $y' = -2x(y^2-3y+2), \quad y(0) = 3$

19. $y' = \dfrac{2x}{1+2y}, \quad y(2) = 0$

20. $y' = 2y - y^2, \quad y(0) = 1$

21. $x + yy' = 0, \quad y(3) = -4$

22. $y' + x^2(y+1)(y-2)^2 = 0, \quad y(4) = 2$

23. $(x+1)(x-2)y' + y = 0, \quad y(1) = -3$

24. Solve $y' = \dfrac{(1+y^2)}{(1+x^2)}$ explicitly. HINT: *Use the identity* $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

25. Solve $y'\sqrt{1-x^2} + \sqrt{1-y^2} = 0$ explicitly. HINT: *Use the identity* $\sin(A-B) = \sin A \cos B - \cos A \sin B$.

26. Solve $y' = \dfrac{\cos x}{\sin y}, \quad y(\pi) = \dfrac{\pi}{2}$ explicitly. HINT: *Use the identity* $\cos(x + \pi/2) = -\sin x$ *and the periodicity of the cosine.*

27. Solve the initial value problem

$$y' = ay - by^2, \quad y(0) = y_0.$$

Discuss the behavior of the solution if **(a)** $y_0 \ge 0$; **(b)** $y_0 < 0$.

28. The population $P = P(t)$ of a species satisfies the logistic equation

$$P' = aP(1 - \alpha P)$$

and $P(0) = P_0 > 0$. Find P for $t > 0$, and find $\lim_{t\to\infty} P(t)$.

29. An epidemic spreads through a population at a rate proportional to the product of the number of people already infected and the number of people susceptible, but not yet infected. Therefore, if S denotes the total population of susceptible people and $I = I(t)$ denotes the number of infected people at time t, then

$$I' = rI(S - I),$$

where r is a positive constant. Assuming that $I(0) = I_0$, find $I(t)$ for $t > 0$, and show that $\lim_{t\to\infty} I(t) = S$.

30. L The result of Exercise 29 is discouraging: if any susceptible member of the group is initially infected, then in the long run all susceptible members are infected! On a more hopeful note, suppose the disease spreads according to the model of Exercise 29, but there's a medication that cures the infected population at a rate proportional to the number of infected individuals. Now the equation for the number of infected individuals becomes

$$I' = rI(S - I) - qI \tag{A}$$

where q is a positive constant.

(a) Choose r and S positive. By plotting direction fields and solutions of (A) on suitable rectangular grids

$$R = \{0 \le t \le T,\ 0 \le I \le d\}$$

in the (t, I)-plane, verify that if I is any solution of (A) such that $I(0) > 0$, then $\lim_{t\to\infty} I(t) = S - q/r$ if $q < rS$ and $\lim_{t\to\infty} I(t) = 0$ if $q \ge rS$.

(b) To verify the experimental results of **(a)**, use separation of variables to solve (A) with initial condition $I(0) = I_0 > 0$, and find $\lim_{t\to\infty} I(t)$. HINT: *There are three cases to consider:* **(i)** $q < rS$; **(ii)** $q > rS$; **(iii)** $q = rS$.

31. L Consider the differential equation

$$y' = ay - by^2 - q, \tag{A}$$

where a, b are positive constants, and q is an arbitrary constant. Suppose y denotes a solution of this equation that satisfies the initial condition $y(0) = y_0$.

(a) Choose a and b positive and $q < a^2/4b$. By plotting direction fields and solutions of (A) on suitable rectangular grids

$$R = \{0 \le t \le T,\ c \le y \le d\} \tag{B}$$

in the (t, y)-plane, discover that there are numbers y_1 and y_2 with $y_1 < y_2$ such that if $y_0 > y_1$ then $\lim_{t\to\infty} y(t) = y_2$, and if $y_0 < y_1$ then $y(t) = -\infty$ for some finite value of t. (What happens if $y_0 = y_1$?)

(b) Choose a and b positive and $q = a^2/4b$. By plotting direction fields and solutions of (A) on suitable rectangular grids of the form (B), discover that there's a number y_1 such that if $y_0 \ge y_1$ then $\lim_{t\to\infty} y(t) = y_1$, while if $y_0 < y_1$ then $y(t) = -\infty$ for some finite value of t.

(c) Choose positive a, b and $q > a^2/4b$. By plotting direction fields and solutions of (A) on suitable rectangular grids of the form (B), discover that no matter what y_0 is, $y(t) = -\infty$ for some finite value of t.

(d) Verify your results experiments analytically. Start by separating variables in (A) to obtain

$$\frac{y'}{ay - by^2 - q} = 1.$$

To decide what to do next you'll have to use the quadratic formula. This should lead you to see why there are three cases. Take it from there!

Because of its role in the transition between these three cases, $q_0 = a^2/4b$ is called a *bifurcation value* of q. In general, if q is a parameter in any differential equation, q_0 is said to be a bifurcation value of q if the nature of the solutions of the equation with $q < q_0$ is qualitatively different from the nature of the solutions with $q > q_0$.

32. L By plotting direction fields and solutions of

$$y' = qy - y^3,$$

convince yourself that $q_0 = 0$ is a bifurcation value of q for this equation. Explain what makes you draw this conclusion.

33. Suppose a disease spreads according to the model of Exercise 29, but there's a medication that cures the infected population at a constant rate of q individuals per unit time, where $q > 0$. Then the equation for the number of infected individuals becomes

$$I' = rI(S - I) - q.$$

Assuming that $I(0) = I_0 > 0$, use the results of Exercise 31 to describe what happens as $t \to \infty$.

34. Assuming that $p \not\equiv 0$, state conditions under which the linear equation

$$y' + p(x)y = f(x)$$

is separable. If the equation satisfies these conditions, solve it by separation of variables and by the method developed in Section 2.1.

Solve the equations in Exercises 35–38 using variation of parameters followed by separation of variables.

35. $y' + y = \dfrac{2xe^{-x}}{1 + ye^x}$

36. $xy' - 2y = \dfrac{x^6}{y + x^2}$

37. $y' - y = \dfrac{(x+1)e^{4x}}{(y + e^x)^2}$

38. $y' - 2y = \dfrac{xe^{2x}}{1 - ye^{-2x}}$

39. Use variation of parameters to show that the solutions of the following equations are of the form $y = uy_1$, where u satisfies a separable equation $u' = g(x)p(u)$. Find y_1 and g for each equation.

(a) $xy' + y = h(x)p(xy)$

(b) $xy' - y = h(x)p\left(\dfrac{y}{x}\right)$

(c) $y' + y = h(x)p(e^x y)$

(d) $xy' + ry = h(x)p(x^r y)$

(e) $y' + \dfrac{v'(x)}{v(x)}y = h(x)p\left(v(x)y\right)$

2.3 EXISTENCE AND UNIQUENESS OF SOLUTIONS OF NONLINEAR EQUATIONS

Although there are methods for solving some nonlinear equations, it's impossible to find useful formulas for the solutions of most. Whether we're looking for exact solutions or numerical approximations, it's useful to know conditions that imply the existence and uniqueness of solutions of initial value problems for nonlinear equations. In this section we state such a condition and illustrate it with examples.

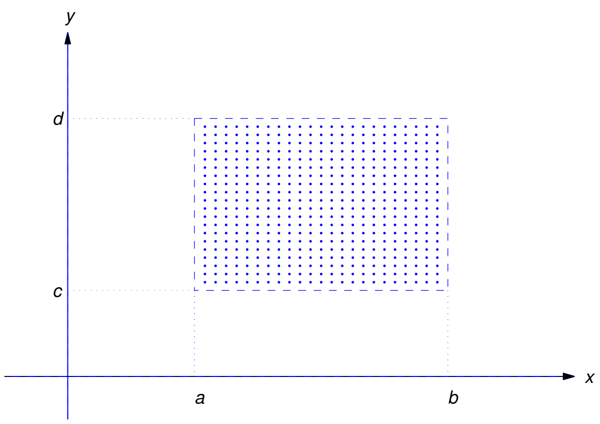

Figure 2.3.1 An open rectangle

Some terminology: an *open rectangle R* is a set of points (x, y) such that

$$a < x < b \quad \text{and} \quad c < y < d$$

(Figure 2.3.1). We'll denote this set by $R : \{a < x < b, c < y < d\}$. "Open" means that the boundary rectangle (indicated by the dashed lines in Figure 2.3.1) isn't included in R .

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for first order nonlinear differential equations. We omit the proof, which is beyond the scope of this book.

Theorem 2.3.1

(a) *If f is continuous on an open rectangle*

$$R : \{a < x < b, c < y < d\}$$

that contains (x_0, y_0) then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2.3.1}$$

has at least one solution on some open subinterval of (a, b) that contains x_0.

(b) *If both f and f_y are continuous on R then (2.3.1) has a unique solution on some open subinterval of (a, b) that contains x_0.*

It's important to understand exactly what Theorem 2.3.1 says.

- **(a)** is an *existence theorem*. It guarantees that a solution exists on some open interval that contains x_0, but provides no information on how to find the solution, or to determine the open interval on which it exists. Moreover, **(a)** provides no information on the number of solutions that (2.3.1) may have. It leaves open the possibility that (2.3.1) may have two or more solutions that differ for values of x arbitrarily close to x_0. We will see in Example 2.3.6 that this can happen.

- **(b)** is a *uniqueness theorem*. It guarantees that (2.3.1) has a unique solution on some open interval (a,b) that contains x_0. However, if $(a, b) \neq (-\infty, \infty)$, (2.3.1) may have more than one solution on a larger interval that contains (a, b). For example, it may happen that $b < \infty$ and all solutions have the same values on (a, b), but two solutions y_1 and y_2 are defined on some interval (a, b_1) with $b_1 > b$, and have different values for $b < x < b_1$; thus, the graphs of the y_1 and y_2 "branch off" in different directions at $x = b$. (See Example 2.3.7 and Figure 2.3.3). In this case, continuity implies that $y_1(b) = y_2(b)$ (call their common value $\overline{y}$), and y_1 and y_2 are both solutions of the initial value problem

$$y' = f(x, y), \quad y(b) = \overline{y} \tag{2.3.2}$$

  that differ on every open interval that contains b. Therefore f or f_y must have a discontinuity at some point in each open rectangle that contains $(b, \overline{y})$, since if this were not so, (2.3.2) would have a unique solution on some open interval that contains b. We leave it to you to give a similar analysis of the case where $a > -\infty$.

Example 2.3.1 Consider the initial value problem

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0. \tag{2.3.3}$$

Since

$$f(x, y) = \frac{x^2 - y^2}{1 + x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{2y(1 + 2x^2)}{(1 + x^2 + y^2)^2}$$

are continuous for all (x, y), Theorem 2.3.1 implies that if (x_0, y_0) is arbitrary, then (2.3.3) has a unique solution on some open interval that contains x_0.

Example 2.3.2 Consider the initial value problem

$$y' = \frac{x^2 - y^2}{x^2 + y^2}, \quad y(x_0) = y_0. \tag{2.3.4}$$

Here

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{4x^2 y}{(x^2 + y^2)^2}$$

are continuous everywhere except at $(0, 0)$. If $(x_0, y_0) \neq (0, 0)$, there's an open rectangle R that contains (x_0, y_0) that does not contain $(0, 0)$. Since f and f_y are continuous on R, Theorem 2.3.1 implies that if $(x_0, y_0) \neq (0, 0)$ then (2.3.4) has a unique solution on some open interval that contains x_0.

Example 2.3.3 Consider the initial value problem

$$y' = \frac{x + y}{x - y}, \quad y(x_0) = y_0. \tag{2.3.5}$$

Here

$$f(x, y) = \frac{x + y}{x - y} \quad \text{and} \quad f_y(x, y) = \frac{2x}{(x - y)^2}$$

are continuous everywhere except on the line $y = x$. If $y_0 \neq x_0$, there's an open rectangle R that contains (x_0, y_0) that does not intersect the line $y = x$. Since f and f_y are continuous on R, Theorem 2.3.1 implies that if $y_0 \neq x_0$, (2.3.5) has a unique solution on some open interval that contains x_0.

Example 2.3.4 In Example 2.2.4 we saw that the solutions of

$$y' = 2xy^2 \tag{2.3.6}$$

are

$$y \equiv 0 \quad \text{and} \quad y = -\frac{1}{x^2 + c},$$

where c is an arbitrary constant. In particular, this implies that no solution of (2.3.6) other than $y \equiv 0$ can equal zero for any value of x. Show that Theorem 2.3.1**(b)** implies this.

Solution We'll obtain a contradiction by assuming that (2.3.6) has a solution y_1 that equals zero for some value of x, but isn't identically zero. If y_1 has this property, there's a point x_0 such that $y_1(x_0) = 0$, but $y_1(x) \neq 0$ for some value of x in every open interval that contains x_0. This means that the initial value problem

$$y' = 2xy^2, \quad y(x_0) = 0 \tag{2.3.7}$$

has two solutions $y \equiv 0$ and $y = y_1$ that differ for some value of x on every open interval that contains x_0. This contradicts Theorem 2.3.1(b), since in (2.3.6) the functions

$$f(x, y) = 2xy^2 \quad \text{and} \quad f_y(x, y) = 4xy.$$

are both continuous for all (x, y), which implies that (2.3.7) has a unique solution on some open interval that contains x_0.

Example 2.3.5 Consider the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(x_0) = y_0. \tag{2.3.8}$$

(a) For what points (x_0, y_0) does Theorem 2.3.1**(a)** imply that (2.3.8) has a solution?

(b) For what points (x_0, y_0) does Theorem 2.3.1**(b)** imply that (2.3.8) has a unique solution on some open interval that contains x_0?

SOLUTION(**a**) Since

$$f(x, y) = \frac{10}{3}xy^{2/5}$$

is continuous for all (x, y), Theorem 2.3.1 implies that (2.3.8) has a solution for every (x_0, y_0).

SOLUTION(**b**) Here

$$f_y(x, y) = \frac{4}{3}xy^{-3/5}$$

is continuous for all (x, y) with $y \neq 0$. Therefore, if $y_0 \neq 0$ there's an open rectangle on which both f and f_y are continuous, and Theorem 2.3.1 implies that (2.3.8) has a unique solution on some open interval that contains x_0.

If $y = 0$ then $f_y(x, y)$ is undefined, and therefore discontinuous; hence, Theorem 2.3.1 does not apply to (2.3.8) if $y_0 = 0$.

Example 2.3.6 Example 2.3.5 leaves open the possibility that the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 0 \tag{2.3.9}$$

has more than one solution on every open interval that contains $x_0 = 0$. Show that this is true.

Solution By inspection, $y \equiv 0$ is a solution of the differential equation

$$y' = \frac{10}{3}xy^{2/5}. \tag{2.3.10}$$

Since $y \equiv 0$ satisfies the initial condition $y(0) = 0$, it's a solution of (2.3.9).

Now suppose y is a solution of (2.3.10) that isn't identically zero. Separating variables in (2.3.10) yields

$$y^{-2/5}y' = \frac{10}{3}x$$

on any open interval where y has no zeros. Integrating this and rewriting the arbitrary constant as $5c/3$ yields

$$\frac{5}{3}y^{3/5} = \frac{5}{3}(x^2 + c).$$

Therefore

$$y = (x^2 + c)^{5/3}. \tag{2.3.11}$$

Since we divided by y to separate variables in (2.3.10), our derivation of (2.3.11) is legitimate only on open intervals where y has no zeros. However, (2.3.11) actually defines y for all x, and differentiating (2.3.11) shows that

$$y' = \frac{10}{3}x(x^2 + c)^{2/3} = \frac{10}{3}xy^{2/5}, \ -\infty < x < \infty.$$

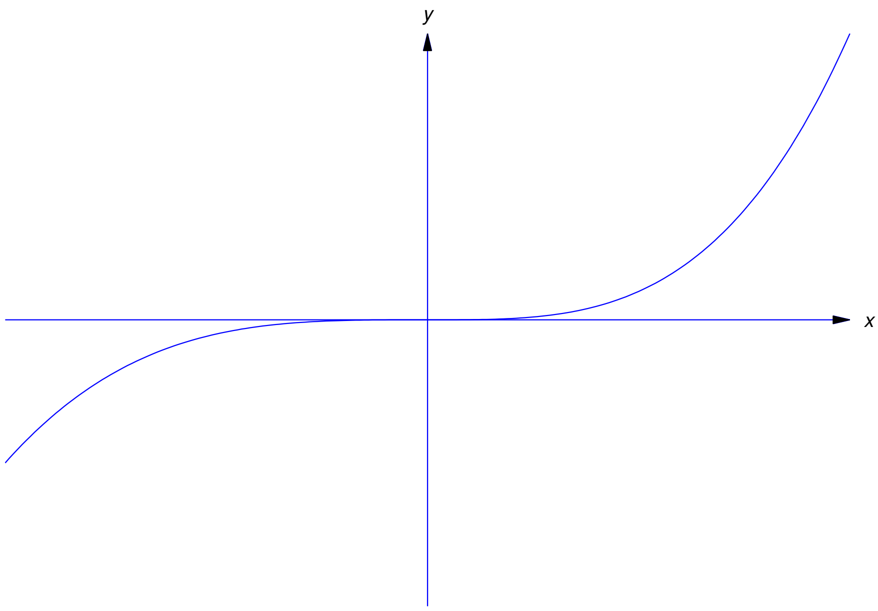

Figure 2.3.2 Two solutions ($y = 0$ and $y = x^{1/2}$) of (2.3.9) that differ on every interval containing $x_0 = 0$

Therefore (2.3.11) satisfies (2.3.10) on $(-\infty, \infty)$ even if $c \le 0$, so that $y(\sqrt{|c|}) = y(-\sqrt{|c|}) = 0$. In particular, taking $c = 0$ in (2.3.11) yields

$$y = x^{10/3}$$

as a second solution of (2.3.9). Both solutions are defined on $(-\infty, \infty)$, and they differ on every open interval that contains $x_0 = 0$ (see Figure 2.3.2.) In fact, there are *four* distinct solutions of (2.3.9) defined on $(-\infty, \infty)$ that differ from each other on every open interval that contains $x_0 = 0$. Can you identify the other two?

Example 2.3.7 From Example 2.3.5, the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = -1 \tag{2.3.12}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find a solution and determine the largest open interval (a, b) on which it's unique.

Solution Let y be any solution of (2.3.12). Because of the initial condition $y(0) = -1$ and the continuity of y, there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form (2.3.11). Setting $x = 0$ and $y = -1$ in (2.3.11) yields $c = -1$, so

$$y = (x^2 - 1)^{5/3} \tag{2.3.13}$$

for x in I. Therefore every solution of (2.3.12) differs from zero and is given by (2.3.13) on $(-1, 1)$; that is, (2.3.13) is the unique solution of (2.3.12) on $(-1, 1)$. This is the largest open interval on which

(2.3.12) has a unique solution. To see this, note that (2.3.13) is a solution of (2.3.12) on $(-\infty, \infty)$. From Exercise 2.2.15, there are infinitely many other solutions of (2.3.12) that differ from (2.3.13) on every open interval larger than $(-1, 1)$. One such solution is

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 \le x \le 1, \\ 0, & |x| > 1. \end{cases}$$

(Figure 2.3.3).

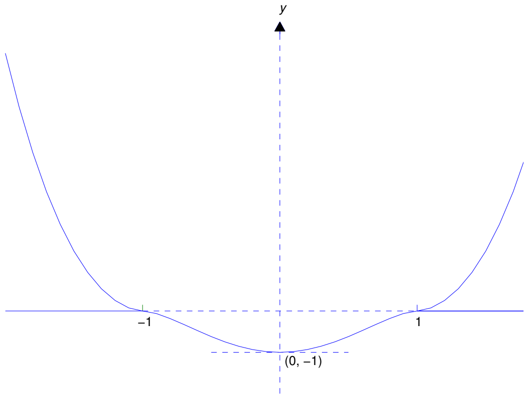

Figure 2.3.3 Two solutions of (2.3.12) on $(-\infty, \infty)$ that coincide on $(-1, 1)$, but on no larger open interval

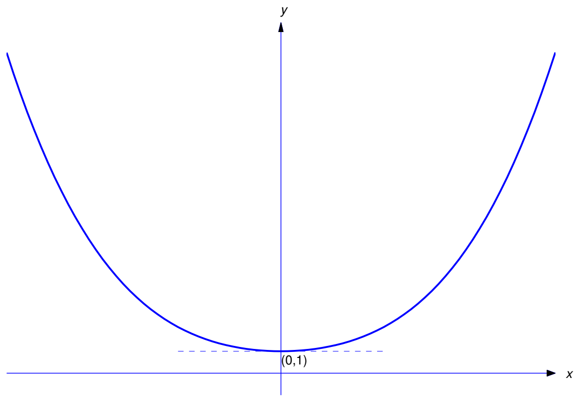

Figure 2.3.4 The unique solution of (2.3.14)

Example 2.3.8 From Example 2.3.5, the initial value problem

$$y' = \frac{10}{3} x y^{2/5}, \quad y(0) = 1 \tag{2.3.14}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it's unique.

Solution Let y be any solution of (2.3.14). Because of the initial condition $y(0) = 1$ and the continuity of y, there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form (2.3.11). Setting $x = 0$ and $y = 1$ in (2.3.11) yields $c = 1$, so

$$y = (x^2 + 1)^{5/3} \tag{2.3.15}$$

for x in I. Therefore every solution of (2.3.14) differs from zero and is given by (2.3.15) on $(-\infty, \infty)$; that is, (2.3.15) is the unique solution of (2.3.14) on $(-\infty, \infty)$. Figure 2.3.4 shows the graph of this solution.

2.3 Exercises

In Exercises 1-13 find all (x_0, y_0) for which Theorem 2.3.1 implies that the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ has **(a)** a solution **(b)** a unique solution on some open interval that contains x_0.

1. $y' = \dfrac{x^2 + y^2}{\sin x}$

2. $y' = \dfrac{e^x + y}{x^2 + y^2}$

3. $y' = \tan xy$

4. $y' = \dfrac{x^2 + y^2}{\ln xy}$

5. $y' = (x^2 + y^2)y^{1/3}$

6. $y' = 2xy$

7. $y' = \ln(1 + x^2 + y^2)$

8. $y' = \dfrac{2x + 3y}{x - 4y}$

9. $y' = (x^2 + y^2)^{1/2}$

10. $y' = x(y^2 - 1)^{2/3}$

11. $y' = (x^2 + y^2)^2$

12. $y' = (x + y)^{1/2}$

13. $y' = \dfrac{\tan y}{x - 1}$

14. Apply Theorem 2.3.1 to the initial value problem

$$y' + p(x)y = q(x), \quad y(x_0) = y_0$$

for a linear equation, and compare the conclusions that can be drawn from it to those that follow from Theorem 2.1.2.

15. **(a)** Verify that the function

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 < x < 1, \\ 0, & |x| \geq 1, \end{cases}$$

is a solution of the initial value problem

$$y' = \frac{10}{3} x y^{2/5}, \quad y(0) = -1$$

on $(-\infty, \infty)$. HINT: *You'll need the definition*

$$y'(\overline{x}) = \lim_{x \to \overline{x}} \frac{y(x) - y(\overline{x})}{x - \overline{x}}$$

to verify that y *satisfies the differential equation at* $\overline{x} = \pm 1$.

(b) Verify that if $\epsilon_i = 0$ or 1 for $i = 1, 2$ and $a, b > 1$, then the function

$$y = \begin{cases} \epsilon_1 (x^2 - a^2)^{5/3}, & -\infty < x < -a, \\ 0, & -a \leq x \leq -1, \\ (x^2 - 1)^{5/3}, & -1 < x < 1, \\ 0, & 1 \leq x \leq b, \\ \epsilon_2 (x^2 - b^2)^{5/3}, & b < x < \infty, \end{cases}$$

is a solution of the initial value problem of **(a)** on $(-\infty, \infty)$.

16. Use the ideas developed in Exercise 15 to find infinitely many solutions of the initial value problem

$$y' = y^{2/5}, \quad y(0) = 1$$

on $(-\infty, \infty)$.

17. Consider the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(x_0) = y_0. \tag{A}$$

(a) For what points (x_0, y_0) does Theorem 2.3.1 imply that (A) has a solution?

(b) For what points (x_0, y_0) does Theorem 2.3.1 imply that (A) has a unique solution on some open interval that contains x_0?

18. Find nine solutions of the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(0) = 1$$

that are all defined on $(-\infty, \infty)$ and differ from each other for values of x in every open interval that contains $x_0 = 0$.

19. From Theorem 2.3.1, the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(0) = 9$$

has a unique solution on an open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it's unique.

20. **(a)** From Theorem 2.3.1, the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(3) = -7 \tag{A}$$

has a unique solution on some open interval that contains $x_0 = 3$. Determine the largest such open interval, and find the solution on this interval.

(b) Find infinitely many solutions of (A), all defined on $(-\infty, \infty)$.

21. Prove:

(a) If

$$f(x, y_0) = 0, \quad a < x < b, \tag{A}$$

and x_0 is in (a, b), then $y \equiv y_0$ is a solution of

$$y' = f(x, y), \quad y(x_0) = y_0$$

on (a, b).

(b) If f and f_y are continuous on an open rectangle that contains (x_0, y_0) and (A) holds, no solution of $y' = f(x, y)$ other than $y \equiv y_0$ can equal y_0 at any point in (a, b).

2.4 TRANSFORMATION OF NONLINEAR EQUATIONS INTO SEPARABLE EQUATIONS

In Section 2.1 we found that the solutions of a linear nonhomogeneous equation

$$y' + p(x)y = f(x)$$

are of the form $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation

$$y' + p(x)y = 0 \tag{2.4.1}$$

and u is a solution of

$$u'y_1(x) = f(x).$$

Note that this last equation is separable, since it can be rewritten as

$$u' = \frac{f(x)}{y_1(x)}.$$

In this section we'll consider nonlinear differential equations that are not separable to begin with, but can be solved in a similar fashion by writing their solutions in the form $y = uy_1$, where y_1 is a suitably chosen known function and u satisfies a separable equation. We'llsay in this case that we *transformed* the given equation into a separable equation.

Bernoulli Equations

A *Bernoulli equation* is an equation of the form

$$y' + p(x)y = f(x)y^r, \tag{2.4.2}$$

where r can be any real number other than 0 or 1. (Note that (2.4.2) is linear if and only if $r = 0$ or $r = 1$.) We can transform (2.4.2) into a separable equation by variation of parameters: if y_1 is a nontrivial solution of (2.4.1), substituting $y = uy_1$ into (2.4.2) yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x)(uy_1)^r,$$

which is equivalent to the separable equation

$$u'y_1(x) = f(x)\,(y_1(x))^r\,u^r \quad \text{or} \quad \frac{u'}{u^r} = f(x)\,(y_1(x))^{r-1},$$

since $y_1' + p(x)y_1 = 0$.

Example 2.4.1 Solve the Bernoulli equation

$$y' - y = xy^2. \tag{2.4.3}$$

Solution Since $y_1 = e^x$ is a solution of $y' - y = 0$, we look for solutions of (2.4.3) in the form $y = ue^x$, where

$$u'e^x = xu^2e^{2x} \quad \text{or, equivalently,} \quad u' = xu^2e^x.$$

Separating variables yields

$$\frac{u'}{u^2} = xe^x,$$

and integrating yields

$$-\frac{1}{u} = (x-1)e^x + c.$$

Hence,

$$u = -\frac{1}{(x-1)e^x + c}$$

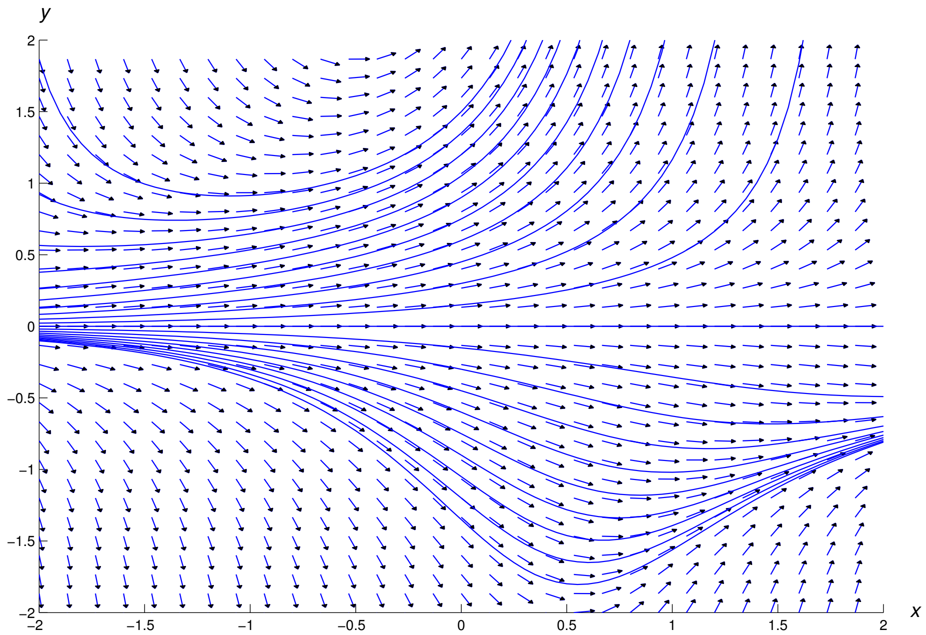

Figure 2.4.1 A direction field and integral curves for $y' - y = xy^2$

and

$$y = -\frac{1}{x - 1 + ce^{-x}}.$$

Figure 2.4.1 shows direction field and some integral curves of (2.4.3).

Other Nonlinear Equations That Can be Transformed Into Separable Equations

We've seen that the nonlinear Bernoulli equation can be transformed into a separable equation by the substitution $y = uy_1$ if y_1 is suitably chosen. Now let's discover a sufficient condition for a nonlinear first order differential equation

$$y' = f(x, y) \tag{2.4.4}$$

to be transformable into a separable equation in the same way. Substituting $y = uy_1$ into (2.4.4) yields

$$u'y_1(x) + uy_1'(x) = f(x, uy_1(x)),$$

which is equivalent to

$$u'y_1(x) = f(x, uy_1(x)) - uy_1'(x). \tag{2.4.5}$$

If

$$f(x, uy_1(x)) = q(u)y_1'(x)$$

for some function q, then (2.4.5) becomes

$$u'y_1(x) = (q(u) - u)y_1'(x), \tag{2.4.6}$$

which is separable. After checking for constant solutions $u \equiv u_0$ such that $q(u_0) = u_0$, we can separate variables to obtain

$$\frac{u'}{q(u) - u} = \frac{y_1'(x)}{y_1(x)}.$$

Homogeneous Nonlinear Equations

In the text we'll consider only the most widely studied class of equations for which the method of the preceding paragraph works. Other types of equations appear in Exercises 44–51.

The differential equation (2.4.4) is said to be *homogeneous* if x and y occur in f in such a way that $f(x, y)$ depends only on the ratio y/x; that is, (2.4.4) can be written as

$$y' = q(y/x), \tag{2.4.7}$$

where $q = q(u)$ is a function of a single variable. For example,

$$y' = \frac{y + xe^{-y/x}}{x} = \frac{y}{x} + e^{-y/x}$$

and

$$y' = \frac{y^2 + xy - x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$$

are of the form (2.4.7), with

$$q(u) = u + e^{-u} \quad \text{and} \quad q(u) = u^2 + u - 1,$$

respectively. The general method discussed above can be applied to (2.4.7) with $y_1 = x$ (and therefore $y_1' = 1$). Thus, substituting $y = ux$ in (2.4.7) yields

$$u'x + u = q(u),$$

and separation of variables (after checking for constant solutions $u \equiv u_0$ such that $q(u_0) = u_0$) yields

$$\frac{u'}{q(u) - u} = \frac{1}{x}.$$

Before turning to examples, we point out something that you may've have already noticed: the definition of *homogeneous equation* given here isn't the same as the definition given in Section 2.1, where we said that a linear equation of the form

$$y' + p(x)y = 0$$

is homogeneous. We make no apology for this inconsistency, since we didn't create it historically, *homogeneous* has been used in these two inconsistent ways. The one having to do with linear equations is the most important. This is the only section of the book where the meaning defined here will apply.

Since y/x is in general undefined if $x = 0$, we'll consider solutions of nonhomogeneous equations only on open intervals that do not contain the point $x = 0$.

Example 2.4.2 Solve

$$y' = \frac{y + xe^{-y/x}}{x}. \tag{2.4.8}$$

Solution Substituting $y = ux$ into (2.4.8) yields

$$u'x + u = \frac{ux + xe^{-ux/x}}{x} = u + e^{-u}.$$

Simplifying and separating variables yields

$$e^u u' = \frac{1}{x}.$$

Integrating yields $e^u = \ln|x| + c$. Therefore $u = \ln(\ln|x| + c)$ and $y = ux = x\ln(\ln|x| + c)$.
Figure 2.4.2 shows a direction field and integral curves for (2.4.8).

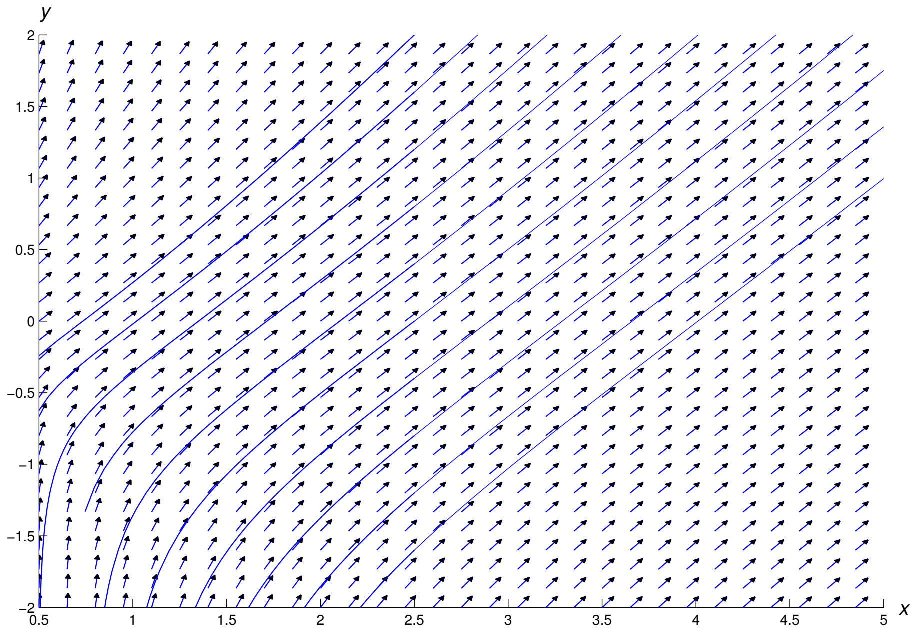

Figure 2.4.2 A direction field and some integral curves for $y' = \dfrac{y + xe^{-y/x}}{x}$

Example 2.4.3

(a) Solve

$$x^2y' = y^2 + xy - x^2. \tag{2.4.9}$$

(b) Solve the initial value problem

$$x^2y' = y^2 + xy - x^2, \quad y(1) = 2. \tag{2.4.10}$$

SOLUTION(**a**) We first find solutions of (2.4.9) on open intervals that don't contain $x = 0$. We can rewrite (2.4.9) as

$$y' = \frac{y^2 + xy - x^2}{x^2}$$

for x in any such interval. Substituting $y = ux$ yields

$$u'x + u = \frac{(ux)^2 + x(ux) - x^2}{x^2} = u^2 + u - 1,$$

so

$$u'x = u^2 - 1. \tag{2.4.11}$$

By inspection this equation has the constant solutions $u \equiv 1$ and $u \equiv -1$. Therefore $y = x$ and $y = -x$ are solutions of (2.4.9). If u is a solution of (2.4.11) that doesn't assume the values ± 1 on some interval, separating variables yields

$$\frac{u'}{u^2 - 1} = \frac{1}{x},$$

or, after a partial fraction expansion,

$$\frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right]u' = \frac{1}{x}.$$

Multiplying by 2 and integrating yields

$$\ln\left|\frac{u-1}{u+1}\right| = 2\ln|x| + k,$$

or

$$\left|\frac{u-1}{u+1}\right| = e^k x^2,$$

which holds if

$$\frac{u-1}{u+1} = cx^2 \tag{2.4.12}$$

where c is an arbitrary constant. Solving for u yields

$$u = \frac{1 + cx^2}{1 - cx^2}.$$

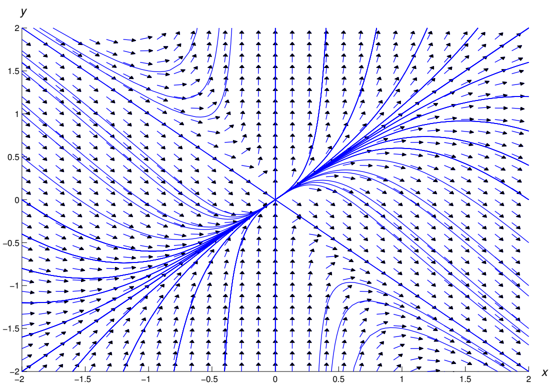

Figure 2.4.3 A direction field and integral curves for $x^2y' = y^2 + xy - x^2$

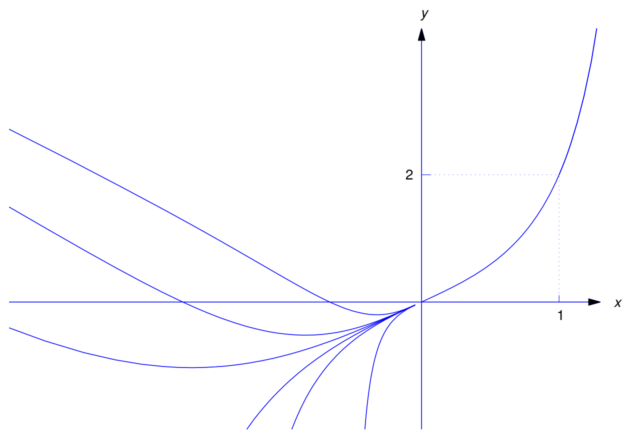

Figure 2.4.4 Solutions of $x^2y' = y^2 + xy - x^2$, $y(1) = 2$

Therefore

$$y = ux = \frac{x(1 + cx^2)}{1 - cx^2} \tag{2.4.13}$$

is a solution of (2.4.10) for any choice of the constant c. Setting $c = 0$ in (2.4.13) yields the solution $y = x$. However, the solution $y = -x$ can't be obtained from (2.4.13). Thus, the solutions of (2.4.9) on intervals that don't contain $x = 0$ are $y = -x$ and functions of the form (2.4.13).

The situation is more complicated if $x = 0$ is the open interval. First, note that $y = -x$ satisfies (2.4.9) on $(-\infty, \infty)$. If c_1 and c_2 are arbitrary constants, the function

$$y = \begin{cases} \dfrac{x(1+c_1x^2)}{1-c_1x^2}, & a < x < 0, \\ \dfrac{x(1+c_2x^2)}{1-c_2x^2}, & 0 \le x < b, \end{cases} \tag{2.4.14}$$

is a solution of (2.4.9) on (a, b), where

$$a = \begin{cases} -\dfrac{1}{\sqrt{c_1}} & \text{if } c_1 > 0, \\ -\infty & \text{if } c_1 \le 0, \end{cases} \quad \text{and} \quad b = \begin{cases} \dfrac{1}{\sqrt{c_2}} & \text{if } c_2 > 0, \\ \infty & \text{if } c_2 \le 0. \end{cases}$$

We leave it to you to verify this. To do so, note that if y is any function of the form (2.4.13) then $y(0) = 0$ and $y'(0) = 1$.

Figure 2.4.3 shows a direction field and some integral curves for (2.4.9).

SOLUTION(**b**) We could obtain c by imposing the initial condition $y(1) = 2$ in (2.4.13), and then solving for c. However, it's easier to use (2.4.12). Since $u = y/x$, the initial condition $y(1) = 2$ implies that $u(1) = 2$. Substituting this into (2.4.12) yields $c = 1/3$. Hence, the solution of (2.4.10) is

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}.$$

The interval of validity of this solution is $(-\sqrt{3}, \sqrt{3})$. However, the largest interval on which (2.4.10) has a unique solution is $(0, \sqrt{3})$. To see this, note from (2.4.14) that any function of the form

$$y = \begin{cases} \dfrac{x(1+cx^2)}{1-cx^2}, & a < x \le 0, \\ \dfrac{x(1+x^2/3)}{1-x^2/3}, & 0 \le x < \sqrt{3}, \end{cases} \tag{2.4.15}$$

is a solution of (2.4.10) on $(a, \sqrt{3})$, where $a = -1/\sqrt{c}$ if $c > 0$ or $a = -\infty$ if $c \le 0$. (Why doesn't this contradict Theorem 2.3.1?)

Figure 2.4.4 shows several solutions of the initial value problem (2.4.10). Note that these solutions coincide on $(0, \sqrt{3})$.

In the last two examples we were able to solve the given equations explicitly. However, this isn't always possible, as you'll see in the exercises.

2.4 Exercises

In Exercises 1–4 solve the given Bernoulli equation.

1. $y' + y = y^2$

2. $7xy' - 2y = -\dfrac{x^2}{y^6}$

3. $x^2y' + 2y = 2e^{1/x}y^{1/2}$

4. $(1 + x^2)y' + 2xy = \dfrac{1}{(1 + x^2)y}$

In Exercises 5 and 6 find all solutions. Also, plot a direction field and some integral curves on the indicated rectangular region.

5. C/G $y' - xy = x^3y^3; \quad \{-3 \le x \le 3, 2 \le y \ge 2\}$

6. C/G $y' - \dfrac{1+x}{3x}y = y^4; \quad \{-2 \le x \le 2, -2 \le y \le 2\}$

In Exercises 7–11 solve the initial value problem.

7. $y' - 2y = xy^3, \quad y(0) = 2\sqrt{2}$

8. $y' - xy = xy^{3/2}, \quad y(1) = 4$

9. $xy' + y = x^4y^4, \quad y(1) = 1/2$

10. $y' - 2y = 2y^{1/2}, \quad y(0) = 1$

11. $y' - 4y = \dfrac{48x}{y^2}, \quad y(0) = 1$

In Exercises 12 and 13 solve the initial value problem and graph the solution.

12. C/G $x^2y' + 2xy = y^3, \quad y(1) = 1/\sqrt{2}$

13. C/G $y' - y = xy^{1/2}, \quad y(0) = 4$

14. You may have noticed that the logistic equation

$$P' = aP(1 - \alpha P)$$

from Verhulst's model for population growth can be written in Bernoulli form as

$$P' - aP = -a\alpha P^2.$$

This isn't particularly interesting, since the logistic equation is separable, and therefore solvable by the method studied in Section 2.2. So let's consider a more complicated model, where a is a positive constant and α is a positive continuous function of t on $[0, \infty)$. The equation for this model is

$$P' - aP = -a\alpha(t)P^2,$$

a non-separable Bernoulli equation.

(a) Assuming that $P(0) = P_0 > 0$, find P for $t > 0$. HINT: *Express your result in terms of the integral* $\int_0^t \alpha(\tau)e^{a\tau}\,d\tau$.

(b) Verify that your result reduces to the known results for the Malthusian model where $\alpha = 0$, and the Verhulst model where α is a nonzero constant.

(c) Assuming that

$$\lim_{t\to\infty} e^{-at}\int_0^t \alpha(\tau)e^{a\tau}\,d\tau = L$$

exists (finite or infinite), find $\lim_{t\to\infty} P(t)$.

In Exercises 15–18 solve the equation explicitly.

15. $y' = \dfrac{y+x}{x}$

16. $y' = \dfrac{y^2 + 2xy}{x^2}$

17. $xy^3y' = y^4 + x^4$

18. $y' = \dfrac{y}{x} + \sec\dfrac{y}{x}$

In Exercises 19-21 solve the equation explicitly. Also, plot a direction field and some integral curves on the indicated rectangular region.

19. C/G $x^2y' = xy + x^2 + y^2; \quad \{-8 \le x \le 8, -8 \le y \le 8\}$

20. C/G $xyy' = x^2 + 2y^2; \quad \{-4 \le x \le 4, -4 \le y \le 4\}$

21. C/G $y' = \dfrac{2y^2 + x^2e^{-(y/x)^2}}{2xy}; \quad \{-8 \le x \le 8, -8 \le y \le 8\}$

In Exercises 22–27 solve the initial value problem.

22. $y' = \dfrac{xy + y^2}{x^2}, \quad y(-1) = 2$

23. $y' = \dfrac{x^3 + y^3}{xy^2}, \quad y(1) = 3$

24. $xyy' + x^2 + y^2 = 0, \quad y(1) = 2$

25. $y' = \dfrac{y^2 - 3xy - 5x^2}{x^2}, \quad y(1) = -1$

26. $x^2y' = 2x^2 + y^2 + 4xy, \quad y(1) = 1$

27. $xyy' = 3x^2 + 4y^2, \quad y(1) = \sqrt{3}$

In Exercises 28–34 solve the given homogeneous equation implicitly.

28. $y' = \dfrac{x + y}{x - y}$

29. $(y'x - y)(\ln|y| - \ln|x|) = x$

30. $y' = \dfrac{y^3 + 2xy^2 + x^2y + x^3}{x(y + x)^2}$

31. $y' = \dfrac{x + 2y}{2x + y}$

32. $y' = \dfrac{y}{y - 2x}$

33. $y' = \dfrac{xy^2 + 2y^3}{x^3 + x^2y + xy^2}$

34. $y' = \dfrac{x^3 + x^2y + 3y^3}{x^3 + 3xy^2}$

35. L

(a) Find a solution of the initial value problem

$$x^2y' = y^2 + xy - 4x^2, \quad y(-1) = 0 \tag{A}$$

on the interval $(-\infty, 0)$. Verify that this solution is actually valid on $(-\infty, \infty)$.

(b) Use Theorem 2.3.1 to show that (A) has a unique solution on $(-\infty, 0)$.

(c) Plot a direction field for the differential equation in (A) on a square

$$\{-r \le x \le r, -r \le y \le r\},$$

where r is any positive number. Graph the solution you obtained in **(a)** on this field.

(d) Graph other solutions of (A) that are defined on $(-\infty, \infty)$.

(e) Graph other solutions of (A) that are defined only on intervals of the form $(-\infty, a)$, where is a finite positive number.

36. L

(a) Solve the equation

$$xyy' = x^2 - xy + y^2 \tag{A}$$

implicitly.

(b) Plot a direction field for (A) on a square

$$\{0 \le x \le r, 0 \le y \le r\}$$

where r is any positive number.

(c) Let K be a positive integer. (You may have to try several choices for K.) Graph solutions of the initial value problems

$$xyy' = x^2 - xy + y^2, \quad y(r/2) = \frac{kr}{K},$$

for $k = 1, 2, \dots, K$. Based on your observations, find conditions on the positive numbers x_0 and y_0 such that the initial value problem

$$xyy' = x^2 - xy + y^2, \quad y(x_0) = y_0, \tag{B}$$

has a unique solution (i) on $(0, \infty)$ or (ii) only on an interval (a, ∞), where $a > 0$?

(d) What can you say about the graph of the solution of (B) as $x \to \infty$? (Again, assume that $x_0 > 0$ and $y_0 > 0$.)

37. L

(a) Solve the equation

$$y' = \frac{2y^2 - xy + 2x^2}{xy + 2x^2} \tag{A}$$

implicitly.

(b) Plot a direction field for (A) on a square

$$\{-r \le x \le r, -r \le y \le r\}$$

where r is any positive number. By graphing solutions of (A), determine necessary and sufficient conditions on (x_0, y_0) such that (A) has a solution on (i) $(-\infty, 0)$ or (ii) $(0, \infty)$ such that $y(x_0) = y_0$.

38. L Follow the instructions of Exercise 37 for the equation

$$y' = \frac{xy + x^2 + y^2}{xy}.$$

39. L Pick any nonlinear homogeneous equation $y' = q(y/x)$ you like, and plot direction fields on the square $\{-r \le x \le r, \ -r \le y \le r\}$, where $r > 0$. What happens to the direction field as you vary r? Why?

40. Prove: If $ad - bc \neq 0$, the equation

$$y' = \frac{ax + by + \alpha}{cx + dy + \beta}$$

can be transformed into the homogeneous nonlinear equation

$$\frac{dY}{dX} = \frac{aX + bY}{cX + dY}$$

by the substitution $x = X - X_0,\ y = Y - Y_0$, where X_0 and Y_0 are suitably chosen constants.

In Exercises 41-43 use a method suggested by Exercise 40 to solve the given equation implicitly.

41. $y' = \dfrac{-6x + y - 3}{2x - y - 1}$

42. $y' = \dfrac{2x + y + 1}{x + 2y - 4}$

43. $y' = \dfrac{-x + 3y - 14}{x + y - 2}$

In Exercises 44–51 find a function y_1 such that the substitution $y = uy_1$ transforms the given equation into a separable equation of the form (2.4.6). Then solve the given equation explicitly.

44. $3xy^2y' = y^3 + x$

45. $xyy' = 3x^6 + 6y^2$

46. $x^3y' = 2(y^2 + x^2y - x^4)$

47. $y' = y^2e^{-x} + 4y + 2e^x$

48. $y' = \dfrac{y^2 + y\tan x + \tan^2 x}{\sin^2 x}$

49. $x(\ln x)^2y' = -4(\ln x)^2 + y\ln x + y^2$

50. $2x(y + 2\sqrt{x})y' = (y + \sqrt{x})^2$

51. $(y + e^{x^2})y' = 2x(y^2 + ye^{x^2} + e^{2x^2})$

52. Solve the initial value problem

$$y' + \frac{2}{x}y = \frac{3x^2y^2 + 6xy + 2}{x^2(2xy + 3)}, \quad y(2) = 2.$$

53. Solve the initial value problem

$$y' + \frac{3}{x}y = \frac{3x^4y^2 + 10x^2y + 6}{x^3(2x^2y + 5)}, \quad y(1) = 1.$$

54. Prove: If y is a solution of a homogeneous nonlinear equation $y' = q(y/x)$, so is $y_1 = y(ax)/a$, where a is any nonzero constant.

55. A *generalized Riccati equation* is of the form

$$y' = P(x) + Q(x)y + R(x)y^2. \tag{A}$$

(If $R \equiv -1$, (A) is a *Riccati equation*.) Let y_1 be a known solution and y an arbitrary solution of (A). Let $z = y - y_1$. Show that z is a solution of a Bernoulli equation with $n = 2$.

In Exercises 56–59, given that y_1 is a solution of the given equation, use the method suggested by Exercise 55 to find other solutions.

56. $y' = 1 + x - (1 + 2x)y + xy^2; \quad y_1 = 1$

57. $y' = e^{2x} + (1 - 2e^x)y + y^2; \quad y_1 = e^x$

58. $xy' = 2 - x + (2x - 2)y - xy^2; \quad y_1 = 1$

59. $xy' = x^3 + (1 - 2x^2)y + xy^2; \quad y_1 = x$

2.5 EXACT EQUATIONS

In this section it's convenient to write first order differential equations in the form

$$M(x, y)\, dx + N(x, y)\, dy = 0. \tag{2.5.1}$$

This equation can be interpreted as

$$M(x, y) + N(x, y)\, \frac{dy}{dx} = 0, \tag{2.5.2}$$

where x is the independent variable and y is the dependent variable, or as

$$M(x, y)\, \frac{dx}{dy} + N(x, y) = 0, \tag{2.5.3}$$

where y is the independent variable and x is the dependent variable. Since the solutions of (2.5.2) and (2.5.3) will often have to be left in implicit, form we'll say that $F(x, y) = c$ is an implicit solution of (2.5.1) if every differentiable function $y = y(x)$ that satisfies $F(x, y) = c$ is a solution of (2.5.2) and every differentiable function $x = x(y)$ that satisfies $F(x, y) = c$ is a solution of (2.5.3).

Here are some examples:

Equation (2.5.1)	Equation (2.5.2)	Equation (2.5.3)
$3x^2y^2\, dx + 2x^3y\, dy = 0$	$3x^2y^2 + 2x^3y\, \frac{dy}{dx} = 0$	$3x^2y^2\, \frac{dx}{dy} + 2x^3y = 0$
$(x^2 + y^2)\, dx + 2xy\, dy = 0$	$(x^2 + y^2) + 2xy\, \frac{dy}{dx} = 0$	$(x^2 + y^2)\, \frac{dx}{dy} + 2xy = 0$
$3y \sin x\, dx - 2xy \cos x\, dy = 0$	$3y \sin x - 2xy \cos x\, \frac{dy}{dx} = 0$	$3y \sin x\, \frac{dx}{dy} - 2xy \cos x = 0$

Note that a separable equation can be written as (2.5.1) as

$$M(x)\, dx + N(y)\, dy = 0.$$

We'll develop a method for solving (2.5.1) under appropriate assumptions on M and N. This method is an extension of the method of separation of variables (Exercise 41). Before stating it we consider an example.

Example 2.5.1 Show that

$$x^4y^3 + x^2y^5 + 2xy = c \tag{2.5.4}$$

is an implicit solution of

$$(4x^3y^3 + 2xy^5 + 2y)\,dx + (3x^4y^2 + 5x^2y^4 + 2x)\,dy = 0. \tag{2.5.5}$$

Solution Regarding y as a function of x and differentiating (2.5.4) implicitly with respect to x yields

$$(4x^3y^3 + 2xy^5 + 2y) + (3x^4y^2 + 5x^2y^4 + 2x)\,\frac{dy}{dx} = 0.$$

Similarly, regarding x as a function of y and differentiating (2.5.4) implicitly with respect to y yields

$$(4x^3y^3 + 2xy^5 + 2y)\frac{dx}{dy} + (3x^4y^2 + 5x^2y^4 + 2x) = 0.$$

Therefore (2.5.4) is an implicit solution of (2.5.5) in either of its two possible interpretations. ■

You may think this example is pointless, since concocting a differential equation that has a given implicit solution isn't particularly interesting. However, it illustrates the next important theorem, which we'll prove by using implicit differentiation, as in Example 2.5.1.

Theorem 2.5.1 *If $F = F(x, y)$ has continuous partial derivatives F_x and F_y, then*

$$F(x, y) = c \qquad (c\text{=constant}), \tag{2.5.6}$$

is an implicit solution of the differential equation

$$F_x(x, y)\,dx + F_y(x, y)\,dy = 0. \tag{2.5.7}$$

Proof Regarding y as a function of x and differentiating (2.5.6) implicitly with respect to x yields

$$F_x(x, y) + F_y(x, y)\,\frac{dy}{dx} = 0.$$

On the other hand, regarding x as a function of y and differentiating (2.5.6) implicitly with respect to y yields

$$F_x(x, y)\,\frac{dx}{dy} + F_y(x, y) = 0.$$

Thus, (2.5.6) is an implicit solution of (2.5.7) in either of its two possible interpretations. ■

We'll say that the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{2.5.8}$$

is *exact* on an an open rectangle R if there's a function $F = F(x, y)$ such F_x and F_y are continuous, and

$$F_x(x, y) = M(x, y) \quad \text{and} \quad F_y(x, y) = N(x, y) \tag{2.5.9}$$

for all (x, y) in R. This usage of "exact" is related to its usage in calculus, where the expression

$$F_x(x, y)\,dx + F_y(x, y)\,dy$$

(obtained by substituting (2.5.9) into the left side of (2.5.8)) is the *exact differential of* F.

Example 2.5.1 shows that it's easy to solve (2.5.8) if it's exact *and* we know a function F that satisfies (2.5.9). The important questions are:

QUESTION 1. Given an equation (2.5.8), how can we determine whether it's exact?

QUESTION 2. If (2.5.8) is exact, how do we find a function F satisfying (2.5.9)?

To discover the answer to Question 1, assume that there's a function F that satisfies (2.5.9) on some open rectangle R, and in addition that F has continuous mixed partial derivatives F_{xy} and F_{yx}. Then a theorem from calculus implies that

$$F_{xy} = F_{yx}. \tag{2.5.10}$$

If $F_x = M$ and $F_y = N$, differentiating the first of these equations with respect to y and the second with respect to x yields

$$F_{xy} = M_y \quad \text{and} \quad F_{yx} = N_x. \tag{2.5.11}$$

From (2.5.10) and (2.5.11), we conclude that a necessary condition for exactness is that $M_y = N_x$. This motivates the next theorem, which we state without proof.

Theorem 2.5.2 *[The Exactness Condition] Suppose M and N are continuous and have continuous partial derivatives M_y and N_x on an open rectangle R. Then*

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is exact on R if and only if

$$M_y(x, y) = N_x(x, y) \tag{2.5.12}$$

for all (x, y) in R..

To help you remember the exactness condition, observe that the coefficients of dx and dy are differentiated in (2.5.12) with respect to the "opposite" variables; that is, the coefficient of dx is differentiated with respect to y, while the coefficient of dy is differentiated with respect to x.

Example 2.5.2 Show that the equation

$$3x^2y\, dx + 4x^3\, dy = 0$$

is not exact on any open rectangle.

Solution Here

$$M(x, y) = 3x^2y \quad \text{and} \quad N(x, y) = 4x^3$$

so

$$M_y(x, y) = 3x^2 \quad \text{and} \quad N_x(x, y) = 12x^2.$$

Therefore $M_y = N_x$ on the line $x = 0$, but not on any open rectangle, so there's no function F such that $F_x(x, y) = M(x, y)$ and $F_y(x, y) = N(x, y)$ for all (x, y) on any open rectangle. ■

The next example illustrates two possible methods for finding a function F that satisfies the condition $F_x = M$ and $F_y = N$ if $M\, dx + N\, dy = 0$ is exact.

Example 2.5.3 Solve

$$(4x^3y^3 + 3x^2)\, dx + (3x^4y^2 + 6y^2)\, dy = 0. \tag{2.5.13}$$

Solution (Method 1) Here

$$M(x, y) = 4x^3y^3 + 3x^2, \quad N(x, y) = 3x^4y^2 + 6y^2,$$

and

$$M_y(x,y) = N_x(x,y) = 12x^3y^2$$

for all (x,y). Therefore Theorem 2.5.2 implies that there's a function F such that

$$F_x(x,y) = M(x,y) = 4x^3y^3 + 3x^2 \tag{2.5.14}$$

and

$$F_y(x,y) = N(x,y) = 3x^4y^2 + 6y^2 \tag{2.5.15}$$

for all (x,y). To find F, we integrate (2.5.14) with respect to x to obtain

$$F(x,y) = x^4y^3 + x^3 + \phi(y), \tag{2.5.16}$$

where $\phi(y)$ is the "constant" of integration. (Here ϕ is "constant" in that it's independent of x, the variable of integration.) If ϕ is any differentiable function of y then F satisfies (2.5.14). To determine ϕ so that F also satisfies (2.5.15), assume that ϕ is differentiable and differentiate F with respect to y. This yields

$$F_y(x,y) = 3x^4y^2 + \phi'(y).$$

Comparing this with (2.5.15) shows that

$$\phi'(y) = 6y^2.$$

We integrate this with respect to y and take the constant of integration to be zero because we're interested only in finding *some* F that satisfies (2.5.14) and (2.5.15). This yields

$$\phi(y) = 2y^3.$$

Substituting this into (2.5.16) yields

$$F(x,y) = x^4y^3 + x^3 + 2y^3. \tag{2.5.17}$$

Now Theorem 2.5.1 implies that

$$x^4y^3 + x^3 + 2y^3 = c$$

is an implicit solution of (2.5.13). Solving this for y yields the explicit solution

$$y = \left(\frac{c - x^3}{2 + x^4}\right)^{1/3}.$$

Solution (Method 2) Instead of first integrating (2.5.14) with respect to x, we could begin by integrating (2.5.15) with respect to y to obtain

$$F(x,y) = x^4y^3 + 2y^3 + \psi(x), \tag{2.5.18}$$

where ψ is an arbitrary function of x. To determine ψ, we assume that ψ is differentiable and differentiate F with respect to x, which yields

$$F_x(x,y) = 4x^3y^3 + \psi'(x).$$

Comparing this with (2.5.14) shows that

$$\psi'(x) = 3x^2.$$

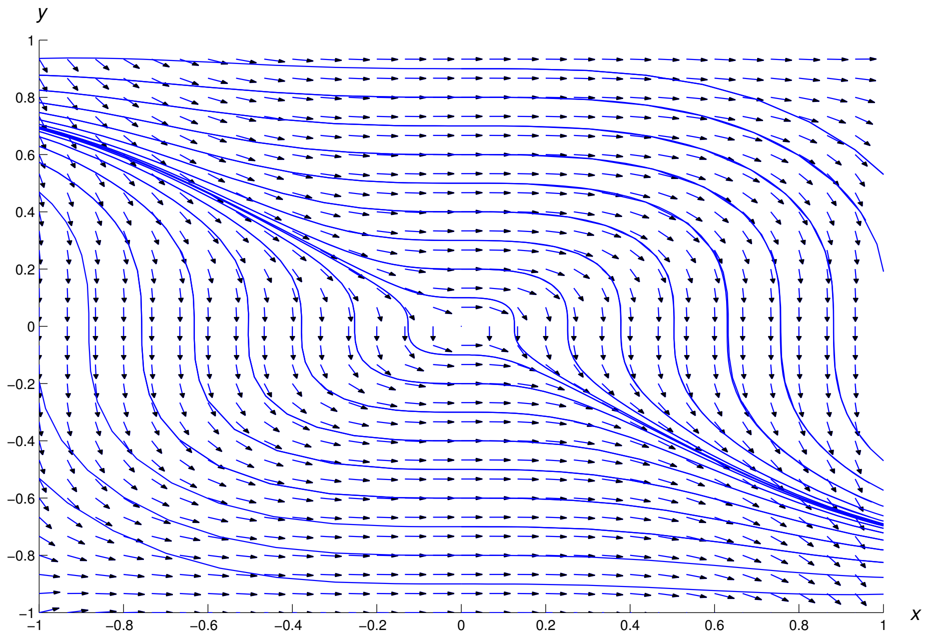

Figure 2.5.1 A direction field and integral curves for $(4x^3y^3 + 3x^2)\,dx + (3x^4y^2 + 6y^2)\,dy = 0$

Integrating this and again taking the constant of integration to be zero yields

$$\psi(x) = x^3.$$

Substituting this into (2.5.18) yields (2.5.17).

Figure 2.5.1 shows a direction field and some integral curves of (2.5.13),

Here's a summary of the procedure used in Method 1 of this example. You should summarize procedure used in Method 2.

Procedure For Solving An Exact Equation

Step 1. Check that the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{2.5.19}$$

satisfies the exactness condition $M_y = N_x$. If not, don't go further with this procedure.

Step 2. Integrate

$$\frac{\partial F(x, y)}{\partial x} = M(x, y)$$

with respect to x to obtain

$$F(x, y) = G(x, y) + \phi(y), \tag{2.5.20}$$

where G is an antiderivative of M with respect to x, and ϕ is an unknown function of y.

Step 3. Differentiate (2.5.20) with respect to y to obtain

$$\frac{\partial F(x, y)}{\partial y} = \frac{\partial G(x, y)}{\partial y} + \phi'(y).$$

Step 4. Equate the right side of this equation to N and solve for ϕ'; thus,

$$\frac{\partial G(x,y)}{\partial y}+\phi'(y)=N(x,y), \quad \text{so} \quad \phi'(y)=N(x,y)-\frac{\partial G(x,y)}{\partial y}.$$

Step 5. Integrate ϕ' with respect to y, taking the constant of integration to be zero, and substitute the result in (2.5.20) to obtain $F(x,y)$.

Step 6. Set $F(x,y)=c$ to obtain an implicit solution of (2.5.19). If possible, solve for y explicitly as a function of x.

It's a common mistake to omit Step 6. However, it's important to include this step, since F isn't itself a solution of (2.5.19).

Many equations can be conveniently solved by either of the two methods used in Example 2.5.3. However, sometimes the integration required in one approach is more difficult than in the other. In such cases we choose the approach that requires the easier integration.

Example 2.5.4 Solve the equation

$$(ye^{xy}\tan x+e^{xy}\sec^2 x)\,dx+xe^{xy}\tan x\,dy=0. \tag{2.5.21}$$

Solution We leave it to you to check that $M_y = N_x$ on any open rectangle where $\tan x$ and $\sec x$ are defined. Here we must find a function F such that

$$F_x(x,y)=ye^{xy}\tan x+e^{xy}\sec^2 x \tag{2.5.22}$$

and

$$F_y(x,y)=xe^{xy}\tan x. \tag{2.5.23}$$

It's difficult to integrate (2.5.22) with respect to x, but easy to integrate (2.5.23) with respect to y. This yields

$$F(x,y)=e^{xy}\tan x+\psi(x). \tag{2.5.24}$$

Differentiating this with respect to x yields

$$F_x(x,y)=ye^{xy}\tan x+e^{xy}\sec^2 x+\psi'(x).$$

Comparing this with (2.5.22) shows that $\psi'(x)=0$. Hence, ψ is a constant, which we can take to be zero in (2.5.24), and

$$e^{xy}\tan x=c$$

is an implicit solution of (2.5.21). ■

Attempting to apply our procedure to an equation that isn't exact will lead to failure in Step 4, since the function

$$N-\frac{\partial G}{\partial y}$$

won't be independent of x if $M_y \neq N_x$ (Exercise 31), and therefore can't be the derivative of a function of y alone. Here's an example that illustrates this.

Example 2.5.5 Verify that the equation

$$3x^2y^2\,dx+6x^3y\,dy=0 \tag{2.5.25}$$

is not exact, and show that the procedure for solving exact equations fails when applied to (2.5.25).

Solution Here

$$M_y(x, y) = 6x^2y \quad \text{and} \quad N_x(x, y) = 18x^2y,$$

so (2.5.25) isn't exact. Nevertheless, let's try to find a function F such that

$$F_x(x, y) = 3x^2y^2 \tag{2.5.26}$$

and

$$F_y(x, y) = 6x^3y. \tag{2.5.27}$$

Integrating (2.5.26) with respect to x yields

$$F(x, y) = x^3y^2 + \phi(y),$$

and differentiating this with respect to y yields

$$F_y(x, y) = 2x^3y + \phi'(y).$$

For this equation to be consistent with (2.5.27),

$$6x^3y = 2x^3y + \phi'(y),$$

or

$$\phi'(y) = 4x^3y.$$

This is a contradiction, since ϕ' must be independent of x. Therefore the procedure fails.

2.5 Exercises

In Exercises 1–17 determine which equations are exact and solve them.

1. $6x^2y^2\,dx + 4x^3y\,dy = 0$

2. $(3y\cos x + 4xe^x + 2x^2e^x)\,dx + (3\sin x + 3)\,dy = 0$

3. $14x^2y^3\,dx + 21x^2y^2\,dy = 0$

4. $(2x - 2y^2)\,dx + (12y^2 - 4xy)\,dy = 0$

5. $(x + y)^2\,dx + (x + y)^2\,dy = 0$ **6.** $(4x + 7y)\,dx + (3x + 4y)\,dy = 0$

7. $(-2y^2\sin x + 3y^3 - 2x)\,dx + (4y\cos x + 9xy^2)\,dy = 0$

8. $(2x + y)\,dx + (2y + 2x)\,dy = 0$

9. $(3x^2 + 2xy + 4y^2)\,dx + (x^2 + 8xy + 18y)\,dy = 0$

10. $(2x^2 + 8xy + y^2)\,dx + (2x^2 + xy^3/3)\,dy = 0$

11. $\left(\frac{1}{x} + 2x\right)dx + \left(\frac{1}{y} + 2y\right)dy = 0$

12. $(y\sin xy + xy^2\cos xy)\,dx + (x\sin xy + xy^2\cos xy)\,dy = 0$

13. $\frac{x\,dx}{(x^2 + y^2)^{3/2}} + \frac{y\,dy}{(x^2 + y^2)^{3/2}} = 0$

14. $\left(e^x(x^2y^2 + 2xy^2) + 6x\right)dx + (2x^2ye^x + 2)\,dy = 0$

15. $\left(x^2e^{x^2+y}(2x^2 + 3) + 4x\right)dx + (x^3e^{x^2+y} - 12y^2)\,dy = 0$

16. $\left(e^{xy}(x^4y+4x^3)+3y\right)\,dx+(x^5e^{xy}+3x)\,dy=0$

17. $(3x^2\cos xy-x^3y\sin xy+4x)\,dx+(8y-x^4\sin xy)\,dy=0$

In Exercises 18–22 *solve the initial value problem.*

18. $(4x^3y^2-6x^2y-2x-3)\,dx+(2x^4y-2x^3)\,dy=0, \quad y(1)=3$

19. $(-4y\cos x+4\sin x\cos x+\sec^2 x)\,dx+(4y-4\sin x)\,dy=0, \quad y(\pi/4)=0$

20. $(y^3-1)e^x\,dx+3y^2(e^x+1)\,dy=0, \quad y(0)=0$

21. $(\sin x-y\sin x-2\cos x)\,dx+\cos x\,dy=0, \quad y(0)=1$

22. $(2x-1)(y-1)\,dx+(x+2)(x-3)\,dy=0, \quad y(1)=-1$

23. C/G Solve the exact equation

$$(7x+4y)\,dx+(4x+3y)\,dy=0.$$

Plot a direction field and some integral curves for this equation on the rectangle

$$\{-1\le x\le 1,-1\le y\le 1\}.$$

24. C/G Solve the exact equation

$$e^x(x^4y^2+4x^3y^2+1)\,dx+(2x^4ye^x+2y)\,dy=0.$$

Plot a direction field and some integral curves for this equation on the rectangle

$$\{-2\le x\le 2,-1\le y\le 1\}.$$

25. C/G Plot a direction field and some integral curves for the exact equation

$$(x^3y^4+x)\,dx+(x^4y^3+y)\,dy=0$$

on the rectangle $\{-1\le x\le 1,-1\le y\le 1\}$. (See Exercise 37**(a)**).

26. C/G Plot a direction field and some integral curves for the exact equation

$$(3x^2+2y)\,dx+(2y+2x)\,dy=0$$

on the rectangle $\{-2\le x\le 2,-2\le y\le 2\}$. (See Exercise 37(b)).

27. L

(a) Solve the exact equation

$$(x^3y^4+2x)\,dx+(x^4y^3+3y)\,dy=0 \qquad \text{(A)}$$

implicitly.

(b) For what choices of (x_0,y_0) does Theorem 2.3.1 imply that the initial value problem

$$(x^3y^4+2x)\,dx+(x^4y^3+3y)\,dy=0, \quad y(x_0)=y_0, \qquad \text{(B)}$$

has a unique solution on an open interval (a,b) that contains x_0?

(c) Plot a direction field and some integral curves for (A) on a rectangular region centered at the origin. What is the interval of validity of the solution of (B)?

28. L

(a) Solve the exact equation

$$(x^2 + y^2)\,dx + 2xy\,dy = 0 \tag{A}$$

implicitly.

(b) For what choices of (x_0, y_0) does Theorem 2.3.1 imply that the initial value problem

$$(x^2 + y^2)\,dx + 2xy\,dy = 0, \quad y(x_0) = y_0, \tag{B}$$

has a unique solution $y = y(x)$ on some open interval (a, b) that contains x_0?

(c) Plot a direction field and some integral curves for (A). From the plot determine, the interval (a, b) of **(b)**, the monotonicity properties (if any) of the solution of (B), and $\lim_{x\to a+} y(x)$ and $\lim_{x\to b-} y(x)$. HINT: *Your answers will depend upon which quadrant contains* (x_0, y_0).

29. Find all functions M such that the equation is exact.

(a) $M(x, y)\,dx + (x^2 - y^2)\,dy = 0$

(b) $M(x, y)\,dx + 2xy \sin x \cos y\,dy = 0$

(c) $M(x, y)\,dx + (e^x - e^y \sin x)\,dy = 0$

30. Find all functions N such that the equation is exact.

(a) $(x^3y^2 + 2xy + 3y^2)\,dx + N(x, y)\,dy = 0$

(b) $(\ln xy + 2y \sin x)\,dx + N(x, y)\,dy = 0$

(c) $(x \sin x + y \sin y)\,dx + N(x, y)\,dy = 0$

31. Suppose M, N, and their partial derivatives are continuous on an open rectangle R, and G is an antiderivative of M with respect to x; that is,

$$\frac{\partial G}{\partial x} = M.$$

Show that if $M_y \neq N_x$ in R then the function

$$N - \frac{\partial G}{\partial y}$$

is not independent of x.

32. Prove: If the equations $M_1\,dx + N_1\,dy = 0$ and $M_2\,dx + N_2\,dy = 0$ are exact on an open rectangle R, so is the equation

$$(M_1 + M_2)\,dx + (N_1 + N_2)\,dy = 0.$$

33. Find conditions on the constants A, B, C, and D such that the equation

$$(Ax + By)\,dx + (Cx + Dy)\,dy = 0$$

is exact.

34. Find conditions on the constants A, B, C, D, E, and F such that the equation

$$(Ax^2 + Bxy + Cy^2)\,dx + (Dx^2 + Exy + Fy^2)\,dy = 0$$

is exact.

35. Suppose M and N are continuous and have continuous partial derivatives M_y and N_x that satisfy the exactness condition $M_y = N_x$ on an open rectangle R. Show that if (x, y) is in R and

$$F(x, y) = \int_{x_0}^{x} M(s, y_0)\, ds + \int_{y_0}^{y} N(x, t)\, dt,$$

then $F_x = M$ and $F_y = N$.

36. Under the assumptions of Exercise 35, show that

$$F(x, y) = \int_{y_0}^{y} N(x_0, s)\, ds + \int_{x_0}^{x} M(t, y)\, dt.$$

37. Use the method suggested by Exercise 35, with $(x_0, y_0) = (0, 0)$, to solve the these exact equations:

(a) $(x^3y^4 + x)\, dx + (x^4y^3 + y)\, dy = 0$
(b) $(x^2 + y^2)\, dx + 2xy\, dy = 0$
(c) $(3x^2 + 2y)\, dx + (2y + 2x)\, dy = 0$

38. Solve the initial value problem

$$y' + \frac{2}{x}y = -\frac{2xy}{x^2 + 2x^2y + 1}, \quad y(1) = -2.$$

39. Solve the initial value problem

$$y' - \frac{3}{x}y = \frac{2x^4(4x^3 - 3y)}{3x^5 + 3x^3 + 2y}, \quad y(1) = 1.$$

40. Solve the initial value problem

$$y' + 2xy = -e^{-x^2}\left(\frac{3x + 2ye^{x^2}}{2x + 3ye^{x^2}}\right), \quad y(0) = -1.$$

41. Rewrite the separable equation

$$h(y)y' = g(x) \tag{A}$$

as an exact equation

$$M(x, y)\, dx + N(x, y)\, dy = 0. \tag{B}$$

Show that applying the method of this section to (B) yields the same solutions that would be obtained by applying the method of separation of variables to (A)

42. Suppose all second partial derivatives of $M = M(x, y)$ and $N = N(x, y)$ are continuous and $M\, dx + N\, dy = 0$ and $-N\, dx + M\, dy = 0$ are exact on an open rectangle R. Show that $M_{xx} + M_{yy} = N_{xx} + N_{yy} = 0$ on R.

43. Suppose all second partial derivatives of $F = F(x, y)$ are continuous and $F_{xx} + F_{yy} = 0$ on an open rectangle R. (A function with these properties is said to be *harmonic*; see also Exercise 42.) Show that $-F_y\, dx + F_x\, dy = 0$ is exact on R, and therefore there's a function G such that $G_x = -F_y$ and $G_y = F_x$ in R. (A function G with this property is said to be a *harmonic conjugate* of F.)

44. Verify that the following functions are harmonic, and find all their harmonic conjugates. (See Exercise 43.)

(a) $x^2 - y^2$ **(b)** $e^x \cos y$ **(c)** $x^3 - 3xy^2$
(d) $\cos x \cosh y$ **(e)** $\sin x \cosh y$

2.6 INTEGRATING FACTORS

In Section 2.5 we saw that if M, N, M_y and N_x are continuous and $M_y = N_x$ on an open rectangle R then

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{2.6.1}$$

is exact on R. Sometimes an equation that isn't exact can be made exact by multiplying it by an appropriate function. For example,

$$(3x + 2y^2)\,dx + 2xy\,dy = 0 \tag{2.6.2}$$

is not exact, since $M_y(x, y) = 4y \neq N_x(x, y) = 2y$ in (2.6.2). However, multiplying (2.6.2) by x yields

$$(3x^2 + 2xy^2)\,dx + 2x^2y\,dy = 0, \tag{2.6.3}$$

which is exact, since $M_y(x, y) = N_x(x, y) = 4xy$ in (2.6.3). Solving (2.6.3) by the procedure given in Section 2.5 yields the implicit solution

$$x^3 + x^2y^2 = c.$$

A function $\mu = \mu(x, y)$ is an *integrating factor* for (2.6.1) if

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0 \tag{2.6.4}$$

is exact. If we know an integrating factor μ for (2.6.1), we can solve the exact equation (2.6.4) by the method of Section 2.5. It would be nice if we could say that (2.6.1) and (2.6.4) always have the same solutions, but this isn't so. For example, a solution $y = y(x)$ of (2.6.4) such that $\mu(x, y(x)) = 0$ on some interval $a < x < b$ could fail to be a solution of (2.6.1) (Exercise 1), while (2.6.1) may have a solution $y = y(x)$ such that $\mu(x, y(x))$ isn't even defined (Exercise 2). Similar comments apply if y is the independent variable and x is the dependent variable in (2.6.1) and (2.6.4). However, if $\mu(x, y)$ is defined and nonzero for all (x, y), (2.6.1) and (2.6.4) are equivalent; that is, they have the same solutions.

Finding Integrating Factors

By applying Theorem 2.5.2 (with M and N replaced by μM and μN), we see that (2.6.4) is exact on an open rectangle R if μM, μN, $(\mu M)_y$, and $(\mu N)_x$ are continuous and

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \quad \text{or, equivalently,} \quad \mu_y M + \mu M_y = \mu_x N + \mu N_x$$

on R. It's better to rewrite the last equation as

$$\mu(M_y - N_x) = \mu_x N - \mu_y M, \tag{2.6.5}$$

which reduces to the known result for exact equations; that is, if $M_y = N_x$ then (2.6.5) holds with $\mu = 1$, so (2.6.1) is exact.

You may think (2.6.5) is of little value, since it involves *partial* derivatives of the unknown integrating factor μ, and we haven't studied methods for solving such equations. However, we'll now show that

(2.6.5) is useful if we restrict our search to integrating factors that are products of a function of x and a function of y; that is, $\mu(x, y) = P(x)Q(y)$. We're not saying that *every* equation $M\,dx + N\,dy = 0$ has an integrating factor of this form; rather, we're saying that *some* equations have such integrating factors.We'llnow develop a way to determine whether a given equation has such an integrating factor, and a method for finding the integrating factor in this case.

If $\mu(x, y) = P(x)Q(y)$, then $\mu_x(x, y) = P'(x)Q(y)$ and $\mu_y(x, y) = P(x)Q'(y)$, so (2.6.5) becomes

$$P(x)Q(y)(M_y - N_x) = P'(x)Q(y)N - P(x)Q'(y)M, \tag{2.6.6}$$

or, after dividing through by $P(x)Q(y)$,

$$M_y - N_x = \frac{P'(x)}{P(x)}N - \frac{Q'(y)}{Q(y)}M. \tag{2.6.7}$$

Now let

$$p(x) = \frac{P'(x)}{P(x)} \quad \text{and} \quad q(y) = \frac{Q'(y)}{Q(y)},$$

so (2.6.7) becomes

$$M_y - N_x = p(x)N - q(y)M. \tag{2.6.8}$$

We obtained (2.6.8) by *assuming* that $M\,dx + N\,dy = 0$ has an integrating factor $\mu(x, y) = P(x)Q(y)$. However, we can now view (2.6.7) differently: If there are functions $p = p(x)$ and $q = q(y)$ that satisfy (2.6.8) and we define

$$P(x) = \pm e^{\int p(x)\,dx} \quad \text{and} \quad Q(y) = \pm e^{\int q(y)\,dy}, \tag{2.6.9}$$

then reversing the steps that led from (2.6.6) to (2.6.8) shows that $\mu(x, y) = P(x)Q(y)$ is an integrating factor for $M\,dx + N\,dy = 0$. In using this result, we take the constants of integration in (2.6.9) to be zero and choose the signs conveniently so the integrating factor has the simplest form.

There's no simple general method for ascertaining whether functions $p = p(x)$ and $q = q(y)$ satisfying (2.6.8) exist. However, the next theorem gives simple sufficient conditions for the given equation to have an integrating factor that depends on only one of the independent variables x and y, and for finding an integrating factor in this case.

Theorem 2.6.1 *Let M, N, M_y, and N_x be continuous on an open rectangle R. Then:*

(a) *If $(M_y - N_x)/N$ is independent of y on R and we define*

$$p(x) = \frac{M_y - N_x}{N}$$

then

$$\mu(x) = \pm e^{\int p(x)\,dx} \tag{2.6.10}$$

is an integrating factor for

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{2.6.11}$$

on R.

(b) *If $(N_x - M_y)/M$ is independent of x on R and we define*

$$q(y) = \frac{N_x - M_y}{M},$$

then

$$\mu(y) = \pm e^{\int q(y)\,dy} \tag{2.6.12}$$

is an integrating factor for (2.6.11) on R.

Proof (a) If $(M_y - N_x)/N$ is independent of y, then (2.6.8) holds with $p = (M_y - N_x)/N$ and $q \equiv 0$. Therefore

$$P(x) = \pm e^{\int p(x)\,dx} \quad \text{and} \quad Q(y) = \pm e^{\int q(y)\,dy} = \pm e^0 = \pm 1,$$

so (2.6.10) is an integrating factor for (2.6.11) on R.

(b) If $(N_x - M_y)/M$ is independent of x then eqrefeq:2.6.8 holds with $p \equiv 0$ and $q = (N_x - M_y)/M$, and a similar argument shows that (2.6.12) is an integrating factor for (2.6.11) on R. ■

The next two examples show how to apply Theorem 2.6.1.

Example 2.6.1 Find an integrating factor for the equation

$$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + (3x^2y^2 + 4y)\,dy = 0 \tag{2.6.13}$$

and solve the equation.

Solution In (2.6.13)

$$M = 2xy^3 - 2x^3y^3 - 4xy^2 + 2x, \; N = 3x^2y^2 + 4y,$$

and

$$M_y - N_x = (6xy^2 - 6x^3y^2 - 8xy) - 6xy^2 = -6x^3y^2 - 8xy,$$

so (2.6.13) isn't exact. However,

$$\frac{M_y - N_x}{N} = -\frac{6x^3y^2 + 8xy}{3x^2y^2 + 4y} = -2x$$

is independent of y, so Theorem 2.6.1**(a)** applies with $p(x) = -2x$. Since

$$\int p(x)\,dx = -\int 2x\,dx = -x^2,$$

$\mu(x) = e^{-x^2}$ is an integrating factor. Multiplying (2.6.13) by μ yields the exact equation

$$e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + e^{-x^2}(3x^2y^2 + 4y)\,dy = 0. \tag{2.6.14}$$

To solve this equation, we must find a function F such that

$$F_x(x,y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) \tag{2.6.15}$$

and

$$F_y(x,y) = e^{-x^2}(3x^2y^2 + 4y). \tag{2.6.16}$$

Integrating (2.6.16) with respect to y yields

$$F(x,y) = e^{-x^2}(x^2y^3 + 2y^2) + \psi(x). \tag{2.6.17}$$

Differentiating this with respect to x yields

$$F_x(x,y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2) + \psi'(x).$$

Comparing this with (2.6.15) shows that $\psi'(x) = 2xe^{-x^2}$; therefore, we can let $\psi(x) = -e^{-x^2}$ in (2.6.17) and conclude that

$$e^{-x^2}\left(y^2(x^2y + 2) - 1\right) = c$$

is an implicit solution of (2.6.14). It is also an implicit solution of (2.6.13).

Figure 2.6.1 shows a direction field and some integal curves for (2.6.13)

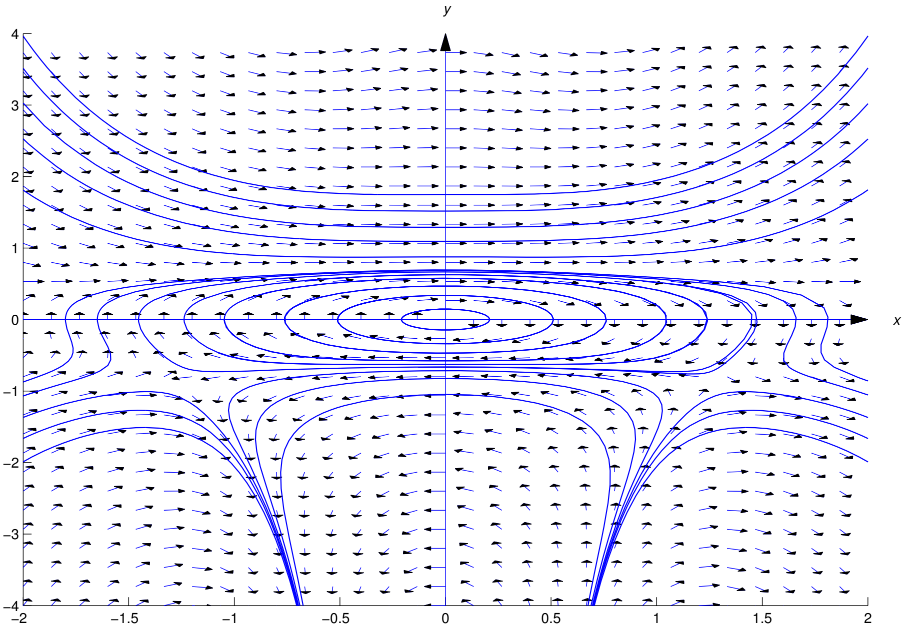

Figure 2.6.1 A direction field and integral curves for
$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + (3x^2y^2 + 4y)\,dy = 0$

Example 2.6.2 Find an integrating factor for

$$2xy^3\,dx + (3x^2y^2 + x^2y^3 + 1)\,dy = 0 \tag{2.6.18}$$

and solve the equation.

Solution In (2.6.18),

$$M = 2xy^3, \quad N = 3x^2y^2 + x^2y^3 + 1,$$

and

$$M_y - N_x = 6xy^2 - (6xy^2 + 2xy^3) = -2xy^3,$$

so (2.6.18) isn't exact. Moreover,

$$\frac{M_y - N_x}{N} = -\frac{2xy^3}{3x^2y^2 + x^2y^2 + 1}$$

is not independent of y, so Theorem 2.6.1(a) does not apply. However, Theorem 2.6.1(b) does apply, since

$$\frac{N_x - M_y}{M} = \frac{2xy^3}{2xy^3} = 1$$

is independent of x, so we can take $q(y) = 1$. Since

$$\int q(y)\,dy = \int dy = y,$$

$\mu(y) = e^y$ is an integrating factor. Multiplying (2.6.18) by μ yields the exact equation

$$2xy^3e^y\,dx + (3x^2y^2 + x^2y^3 + 1)e^y\,dy = 0. \tag{2.6.19}$$

To solve this equation, we must find a function F such that

$$F_x(x, y) = 2xy^3e^y \tag{2.6.20}$$

and

$$F_y(x, y) = (3x^2y^2 + x^2y^3 + 1)e^y. \tag{2.6.21}$$

Integrating (2.6.20) with respect to x yields

$$F(x, y) = x^2y^3e^y + \phi(y). \tag{2.6.22}$$

Differentiating this with respect to y yields

$$F_y = (3x^2y^2 + x^2y^3)e^y + \phi'(y),$$

and comparing this with (2.6.21) shows that $\phi'(y) = e^y$. Therefore we set $\phi(y) = e^y$ in (2.6.22) and conclude that

$$(x^2y^3 + 1)e^y = c$$

is an implicit solution of (2.6.19). It is also an implicit solution of (2.6.18). Figure 2.6.2 shows a direction field and some integral curves for (2.6.18). ■

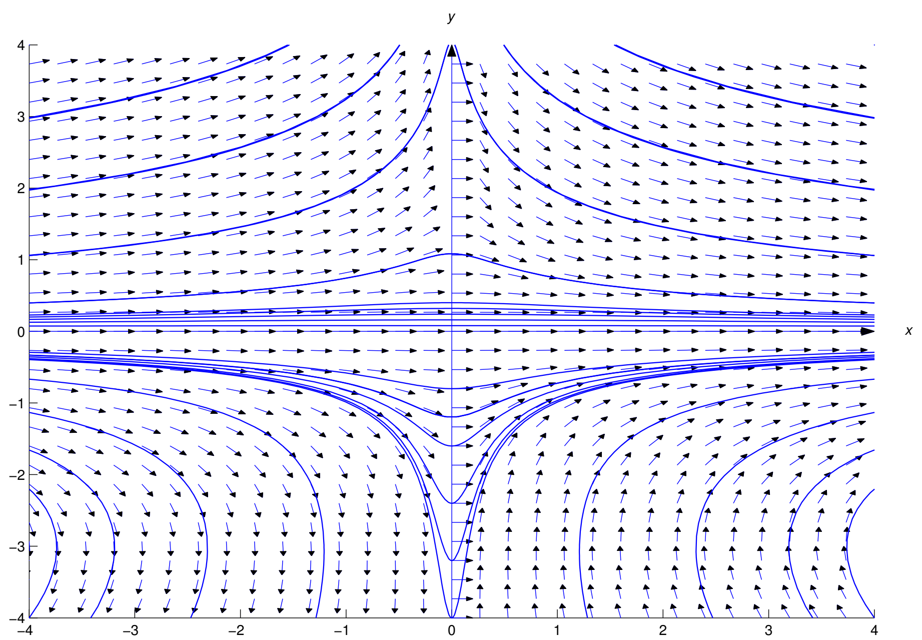

Figure 2.6.2 A direction field and integral curves for $2xy^3e^y\,dx + (3x^2y^2 + x^2y^3 + 1)e^y\,dy = 0$

Theorem 2.6.1 does not apply in the next example, but the more general argument that led to Theorem 2.6.1 provides an integrating factor.

Example 2.6.3 Find an integrating factor for

$$(3xy + 6y^2)\,dx + (2x^2 + 9xy)\,dy = 0 \tag{2.6.23}$$

and solve the equation.

Solution In (2.6.23)

$$M = 3xy + 6y^2, \quad N = 2x^2 + 9xy,$$

and

$$M_y - N_x = (3x + 12y) - (4x + 9y) = -x + 3y.$$

Therefore

$$\frac{M_y - N_x}{M} = \frac{-x + 3y}{3xy + 6y^2} \quad \text{and} \quad \frac{N_x - M_y}{N} = \frac{x - 3y}{2x^2 + 9xy},$$

so Theorem 2.6.1 does not apply. Following the more general argument that led to Theorem 2.6.1, we look for functions $p = p(x)$ and $q = q(y)$ such that

$$M_y - N_x = p(x)N - q(y)M;$$

that is,

$$-x + 3y = p(x)(2x^2 + 9xy) - q(y)(3xy + 6y^2).$$

Since the left side contains only first degree terms in x and y, we rewrite this equation as

$$xp(x)(2x + 9y) - yq(y)(3x + 6y) = -x + 3y.$$

This will be an identity if

$$xp(x) = A \quad \text{and} \quad yq(y) = B, \tag{2.6.24}$$

where A and B are constants such that

$$-x + 3y = A(2x + 9y) - B(3x + 6y),$$

or, equivalently,

$$-x + 3y = (2A - 3B)x + (9A - 6B)y.$$

Equating the coefficients of x and y on both sides shows that the last equation holds for all (x, y) if

$$\begin{aligned} 2A - 3B &= -1 \\ 9A - 6B &= 3, \end{aligned}$$

which has the solution $A = 1$, $B = 1$. Therefore (2.6.24) implies that

$$p(x) = \frac{1}{x} \quad \text{and} \quad q(y) = \frac{1}{y}.$$

Since

$$\int p(x)\,dx = \ln|x| \quad \text{and} \quad \int q(y)\,dy = \ln|y|,$$

we can let $P(x) = x$ and $Q(y) = y$; hence, $\mu(x, y) = xy$ is an integrating factor. Multiplying (2.6.23) by μ yields the exact equation

$$(3x^2y^2 + 6xy^3)\,dx + (2x^3y + 9x^2y^2)\,dy = 0.$$

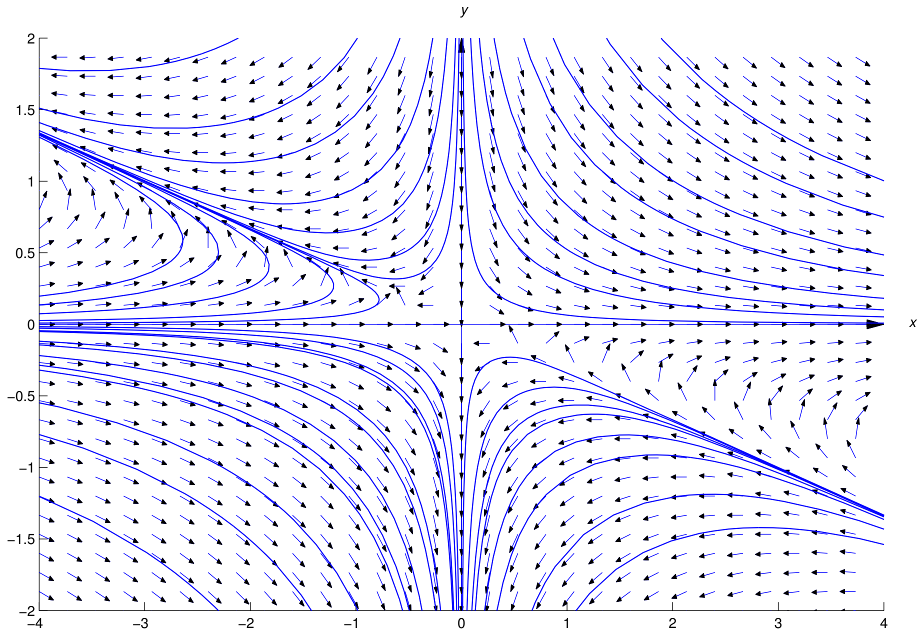

Figure 2.6.3 A direction field and integral curves for $(3xy + 6y^2)\,dx + (2x^2 + 9xy)\,dy = 0$

We leave it to you to use the method of Section 2.5 to show that this equation has the implicit solution

$$x^3y^2 + 3x^2y^3 = c. \tag{2.6.25}$$

This is also an implicit solution of (2.6.23). Since $x \equiv 0$ and $y \equiv 0$ satisfy (2.6.25), you should check to see that $x \equiv 0$ and $y \equiv 0$ are also solutions of (2.6.23). (Why is it necesary to check this?)

Figure 2.6.3 shows a direction field and integral curves for (2.6.23).

See Exercise 28 for a general discussion of equations like (2.6.23).

Example 2.6.4 The separable equation

$$-y\,dx + (x + x^6)\,dy = 0 \tag{2.6.26}$$

can be converted to the exact equation

$$-\frac{dx}{x + x^6} + \frac{dy}{y} = 0 \tag{2.6.27}$$

by multiplying through by the integrating factor

$$\mu(x, y) = \frac{1}{y(x + x^6)}.$$

However, to solve (2.6.27) by the method of Section 2.5 we would have to evaluate the nasty integral

$$\int \frac{dx}{x + x^6}.$$

Instead, we solve (2.6.26) explicitly for y by finding an integrating factor of the form $\mu(x, y) = x^a y^b$.

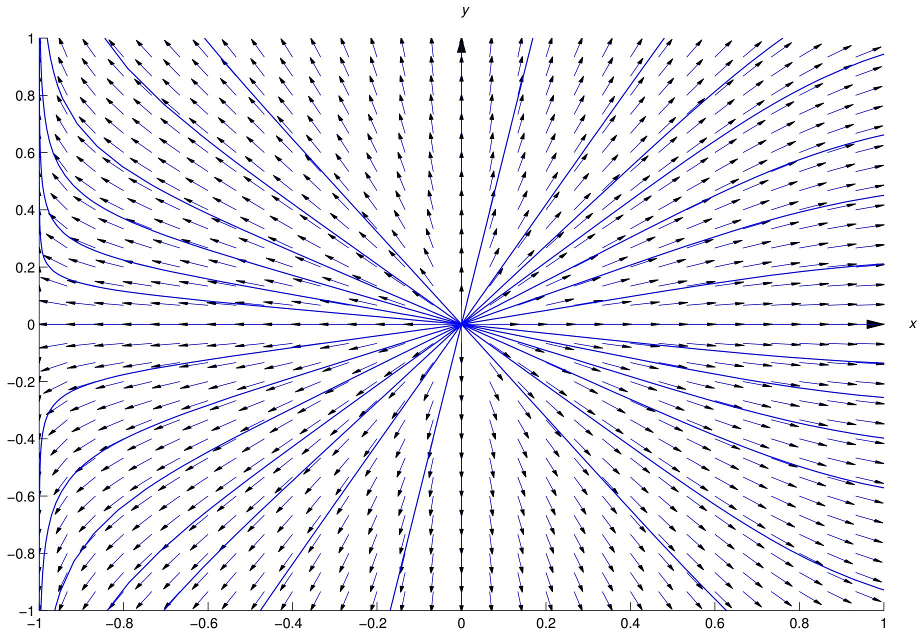

Figure 2.6.4 A direction field and integral curves for $-y\,dx + (x + x^6)\,dy = 0$

Solution In (2.6.26)

$$M = -y, \; N = x + x^6,$$

and

$$M_y - N_x = -1 - (1 + 6x^5) = -2 - 6x^5.$$

We look for functions $p = p(x)$ and $q = q(y)$ such that

$$M_y - N_x = p(x)N - q(y)M;$$

that is,

$$-2 - 6x^5 = p(x)(x + x^6) + q(y)y. \tag{2.6.28}$$

The right side will contain the term $-6x^5$ if $p(x) = -6/x$. Then (2.6.28) becomes

$$-2 - 6x^5 = -6 - 6x^5 + q(y)y,$$

so $q(y) = 4/y$. Since

$$\int p(x)\,dx = -\int \frac{6}{x}\,dx = -6\ln|x| = \ln\frac{1}{x^6},$$

and

$$\int q(y)\,dy = \int \frac{4}{y}\,dy = 4\ln|y| = \ln y^4,$$

we can take $P(x) = x^{-6}$ and $Q(y) = y^4$, which yields the integrating factor $\mu(x, y) = x^{-6}y^4$. Multiplying (2.6.26) by μ yields the exact equation

$$-\frac{y^5}{x^6}\,dx + \left(\frac{y^4}{x^5} + y^4\right)\,dy = 0.$$

We leave it to you to use the method of the Section 2.5 to show that this equation has the implicit solution

$$\left(\frac{y}{x}\right)^5 + y^5 = k.$$

Solving for y yields

$$y = k^{1/5}x(1 + x^5)^{-1/5},$$

which we rewrite as

$$y = cx(1 + x^5)^{-1/5}$$

by renaming the arbitrary constant. This is also a solution of (2.6.26).

Figure 2.6.4 shows a direction field and some integral curves for (2.6.26).

2.6 Exercises

1. **(a)** Verify that $\mu(x, y) = y$ is an integrating factor for

$$y\,dx + \left(2x + \frac{1}{y}\right) dy = 0 \qquad \text{(A)}$$

on any open rectangle that does not intersect the x axis or, equivalently, that

$$y^2\,dx + (2xy + 1)\,dy = 0 \qquad \text{(B)}$$

is exact on any such rectangle.

(b) Verify that $y \equiv 0$ is a solution of (B), but not of (A).

(c) Show that

$$y(xy + 1) = c \qquad \text{(C)}$$

is an implicit solution of (B), and explain why every differentiable function $y = y(x)$ other than $y \equiv 0$ that satisfies (C) is also a solution of (A).

2. **(a)** Verify that $\mu(x, y) = 1/(x - y)^2$ is an integrating factor for

$$-y^2\,dx + x^2\,dy = 0 \qquad \text{(A)}$$

on any open rectangle that does not intersect the line $y = x$ or, equivalently, that

$$-\frac{y^2}{(x-y)^2}\,dx + \frac{x^2}{(x-y)^2}\,dy = 0 \qquad \text{(B)}$$

is exact on any such rectangle.

(b) Use Theorem 2.2.1 to show that

$$\frac{xy}{(x-y)} = c \qquad \text{(C)}$$

is an implicit solution of (B), and explain why it's also an implicit solution of (A)

(c) Verify that $y = x$ is a solution of (A), even though it can't be obtained from (C).

In Exercises 3–16 find an integrating factor; that is a function of only one variable, and solve the given equation.

3. $y\,dx - x\,dy = 0$

4. $3x^2y\,dx + 2x^3\,dy = 0$

5. $2y^3\,dx + 3y^2\,dy = 0$

6. $(5xy + 2y + 5)\,dx + 2x\,dy = 0$

7. $(xy + x + 2y + 1)\,dx + (x + 1)\,dy = 0$

8. $(27xy^2 + 8y^3)\,dx + (18x^2y + 12xy^2)\,dy = 0$

9. $(6xy^2 + 2y)\,dx + (12x^2y + 6x + 3)\,dy = 0$

10. $y^2\,dx + \left(xy^2 + 3xy + \frac{1}{y}\right)\,dy = 0$

11. $(12x^3y + 24x^2y^2)\,dx + (9x^4 + 32x^3y + 4y)\,dy = 0$

12. $(x^2y + 4xy + 2y)\,dx + (x^2 + x)\,dy = 0$

13. $-y\,dx + (x^4 - x)\,dy = 0$

14. $\cos x \cos y\,dx + (\sin x \cos y - \sin x \sin y + y)\,dy = 0$

15. $(2xy + y^2)\,dx + (2xy + x^2 - 2x^2y^2 - 2xy^3)\,dy = 0$

16. $y \sin y\,dx + x(\sin y - y \cos y)\,dy = 0$

In Exercises 17–23 find an integrating factor of the form $\mu(x, y) = P(x)Q(y)$ and solve the given equation.

17. $y(1 + 5\ln|x|)\,dx + 4x\ln|x|\,dy = 0$

18. $(\alpha y + \gamma xy)\,dx + (\beta x + \delta xy)\,dy = 0$

19. $(3x^2y^3 - y^2 + y)\,dx + (-xy + 2x)\,dy = 0$

20. $2y\,dx + 3(x^2 + x^2y^3)\,dy = 0$

21. $(a \cos xy - y \sin xy)\,dx + (b \cos xy - x \sin xy)\,dy = 0$

22. $x^4y^4\,dx + x^5y^3\,dy = 0$

23. $y(x \cos x + 2 \sin x)\,dx + x(y + 1) \sin x\,dy = 0$

In Exercises 24–27 find an integrating factor and solve the equation. Plot a direction field and some integral curves for the equation in the indicated rectangular region.

24. C/G $(x^4y^3 + y)\,dx + (x^5y^2 - x)\,dy = 0;\quad \{-1 \le x \le 1, -1 \le y \le 1\}$

25. C/G $(3xy + 2y^2 + y)\,dx + (x^2 + 2xy + x + 2y)\,dy = 0;\quad \{-2 \le x \le 2, -2 \le y \le 2\}$

26. C/G $(12xy + 6y^3)\,dx + (9x^2 + 10xy^2)\,dy = 0;\quad \{-2 \le x \le 2, -2 \le y \le 2\}$

27. C/G $(3x^2y^2 + 2y)\,dx + 2x\,dy = 0;\quad \{-4 \le x \le 4, -4 \le y \le 4\}$

28. Suppose a, b, c, and d are constants such that $ad - bc \ne 0$, and let m and n be arbitrary real numbers. Show that

$$(ax^my + by^{n+1})\,dx + (cx^{m+1} + dxy^n)\,dy = 0$$

has an integrating factor $\mu(x, y) = x^\alpha y^\beta$.

29. Suppose M, N, M_x, and N_y are continuous for all (x, y), and $\mu = \mu(x, y)$ is an integrating factor for

$$M(x, y)\,dx + N(x, y)\,dy = 0. \qquad \text{(A)}$$

Assume that μ_x and μ_y are continuous for all (x, y), and suppose $y = y(x)$ is a differentiable function such that $\mu(x, y(x)) = 0$ and $\mu_x(x, y(x)) \ne 0$ for all x in some interval I. Show that y is a solution of (A) on I.

30. According to Theorem 2.1.2, the general solution of the linear nonhomogeneous equation

$$y' + p(x)y = f(x) \tag{A}$$

is

$$y = y_1(x)\left(c + \int f(x)/y_1(x)\,dx\right), \tag{B}$$

where y_1 is any nontrivial solution of the complementary equation $y' + p(x)y = 0$. In this exercise we obtain this conclusion in a different way. You may find it instructive to apply the method suggested here to solve some of the exercises in Section 2.1.

(a) Rewrite (A) as

$$[p(x)y - f(x)]\,dx + dy = 0, \tag{C}$$

and show that $\mu = \pm e^{\int p(x)\,dx}$ is an integrating factor for (C).

(b) Multiply (A) through by $\mu = \pm e^{\int p(x)\,dx}$ and verify that the resulting equation can be rewritten as

$$(\mu(x)y)' = \mu(x)f(x).$$

Then integrate both sides of this equation and solve for y to show that the general solution of (A) is

$$y = \frac{1}{\mu(x)}\left(c + \int f(x)\mu(x)\,dx\right).$$

Why is this form of the general solution equivalent to (B)?