CHAPTER 3

Classifications of First-Order Differential Equations

STANDARD FORM AND DIFFERENTIAL FORM

Standard form for a first-order differential equation in the unknown function $y(x)$ is

$$y' = f(x, y) \tag{3.1}$$

where the derivative y' appears only on the left side of (*3.1*). Many, but not all, first-order differential equations can be written in standard form by algebraically solving for y' and then setting $f(x, y)$ equal to the right side of the resulting equation.

The right side of (*3.1*) can always be written as a quotient of two other functions $M(x, y)$ and $-N(x, y)$. Then (*3.1*) becomes $dy/dx = M(x, y)/-N(x, y)$, which is equivalent to the *differential form*

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{3.2}$$

LINEAR EQUATIONS

Consider a differential equation in standard form (*3.1*). If $f(x, y)$ can be written as $f(x, y) = -p(x)y + q(x)$ (that is, as a function of x times y, plus another function of x), the differential equation is *linear*. First-order linear differential equations can always be expressed as

$$y' + p(x)y = q(x) \tag{3.3}$$

Linear equations are solved in Chapter 6.

BERNOULLI EQUATIONS

A *Bernoulli* differential equation is an equation of the form

$$y' + p(x)y = q(x)y^n \tag{3.4}$$

where n denotes a real number. When $n = 1$ or $n = 0$, a Bernoulli equation reduces to a linear equation. Bernoulli equations are solved in Chapter 6.

HOMOGENEOUS EQUATIONS

A differential equation in standard form (*3.1*) is *homogeneous* if

$$f(tx, ty) = f(x, y) \tag{3.5}$$

for every real number t. Homogeneous equations are solved in Chapter 4.

Note: In the general framework of differential equations, the word "homogeneous" has an entirely different meaning (see Chapter 8). Only in the context of first-order differential equations does "homogeneous" have the meaning defined above.

SEPARABLE EQUATIONS

Consider a differential equation in differential form (*3.2*). If $M(x, y) = A(x)$ (a function only of x) and $N(x, y) = B(y)$ (a function only of y), the differential equation is *separable*, or has its *variables separated.* Separable equations are solved in Chapter 4.

EXACT EQUATIONS

A differential equation in differential form (*3.2*) is *exact* if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \tag{3.6}$$

Exact equations are solved in Chapter 5 (where a more precise definition of exactness is given).

Solved Problems

3.1. Write the differential equation $xy' - y^2 = 0$ in standard form.

Solving for y', we obtain $y' = y^2/x$ which has form (*3.1*) with $f(x, y) = y^2/x$.

3.2. Write the differential equation $e^x y' + e^{2x} y = \sin x$ in standard form.

Solving for y', we obtain

$$e^x y' = -e^{2x} y + \sin x$$

or

$$y' = -e^x y + e^{-x} \sin x$$

which has form (*3.1*) with $f(x, y) = -e^x y + e^{-x} \sin x$.

3.3. Write the differential equation $(y' + y)^5 = \sin (y'/x)$ in standard form.

This equation cannot be solved algebraically for y', and *cannot* be written in standard form.

3.4. Write the differential equation $y(yy' - 1) = x$ in differential form.

Solving for y', we have

$$y^2 y' - y = x$$

$$y^2 y' = x + y$$

or

$$y' = \frac{x+y}{y^2} \tag{1}$$

which is in standard form with $f(x, y) = (x + y)/y^2$. There are infinitely many different differential forms associated with (*1*). Four such forms are:

(*a*) Take $M(x, y) = x + y$, $N(x, y) = -y^2$. Then

$$\frac{M(x,y)}{-N(x,y)} = \frac{x+y}{-(-y^2)} = \frac{x+y}{y^2}$$

and (*1*) is equivalent to the differential form

$$(x + y)dx + (-y^2)dy = 0$$

(*b*) Take $M(x, y) = -1$, $N(x,y) = \dfrac{y^2}{x+y}$. Then

$$\frac{M(x,y)}{-N(x,y)} = \frac{-1}{-y^2/(x+y)} = \frac{x+y}{y^2}$$

and (*1*) is equivalent to the differential form

$$(-1)dx + \left(\frac{y^2}{x+y}\right)dy = 0$$

(*c*) Take $M(x,y) = \dfrac{x+y}{2}$, $N(x,y) = \dfrac{-y^2}{2}$. Then

$$\frac{M(x,y)}{-N(x,y)} = \frac{(x+y)/2}{-(-y^2/2)} = \frac{x+y}{y^2}$$

and (*1*) is equivalent to the differential form

$$\left(\frac{x+y}{2}\right)dx + \left(\frac{-y^2}{2}\right)dy = 0$$

(*d*) Take $M(x,y) = \dfrac{-x-y}{x^2}$, $N(x,y) = \dfrac{y^2}{x^2}$. Then

$$\frac{M(x,y)}{-N(x,y)} = \frac{(-x-y)/x^2}{-y^2/x^2} = \frac{x+y}{y^2}$$

and (*1*) is equivalent to the differential form

$$\left(\frac{-x-y}{x^2}\right)dx + \left(\frac{y^2}{x^2}\right)dy = 0$$

3.5. Write the differential equation $dy/dx = y/x$ in differential form

This equation has infinitely many differential forms. One is

$$dy = \frac{y}{x}dx$$

which can be written in form (*3.2*) as

$$\frac{y}{x}dx + (-1)dy = 0 \tag{1}$$

Multiplying (*1*) through by x, we obtain

$$y\,dx + (-x)dy = 0 \tag{2}$$

as a second differential form. Multiplying (*1*) through by $1/y$, we obtain

$$\frac{1}{x}dx + \frac{-1}{y}dy = 0 \tag{3}$$

as a third differential form. Still other differential forms are derived from (*1*) by multiplying that equation through by any other function of x and y.

3.6. Write the differential equation $(xy + 3)dx + (2x - y^2 + 1)dy = 0$ in standard form.

This equation is in differential form. We rewrite it as

$$(2x - y^2 + 1)\,dy = -(xy + 3)\,dx$$

which has the standard form

$$\frac{dy}{dx} = \frac{-(xy+3)}{2x - y^2 + 1}$$

or

$$y' = \frac{xy+3}{y^2 - 2x - 1}$$

3.7. Determine if the following differential equations are linear:

(*a*) $y' = (\sin x)y + e^x$ (*b*) $y' = x \sin y + e^x$ (*c*) $y' = 5$ (*d*) $y' = y^2 + x$

(*e*) $y' + xy^5 = 0$ (*f*) $xy' + y = \sqrt{y}$ (*g*) $y' + xy = e^x y$ (*h*) $y' + \dfrac{x}{y} = 0$

(*a*) The equation is linear; here $p(x) = -\sin x$ and $q(x) = e^x$.
(*b*) The equation is not linear because of the term $\sin y$.
(*c*) The equation is linear; here $p(x) = 0$ and $q(x) = 5$.
(*d*) The equation is not linear because of the term y^2.
(*e*) The equation is not linear because of the y^5 term.
(*f*) The equation is not linear because of the $y^{1/2}$ term.
(*g*) The equation is linear. Rewrite it as $y' + (x - e^x)y = 0$ with $p(x) = x - e^x$ and $q(x) = 0$.
(*h*) The equation is not linear because of the $1/y$ term.

3.8. Determine whether any of the differential equations in Problem 3.7 are Bernoulli equations.

All of the linear equations are Bernoulli equations with $n = 0$. In addition, three of the nonlinear equations, (*e*), (*f*) and (*h*), are as well. Rewrite (*e*) as $y' = -xy^5$; it has form (*3.4*) with $p(x) = 0$, $q(x) = -x$, and $n = 5$. Rewrite (*f*) as

$$y' + \frac{1}{x}y = \frac{1}{x}y^{1/2}$$

It has form (*3.4*) with $p(x) = q(x) = 1/x$ and $n = 1/2$. Rewrite (*h*) as $y' = -xy^{-1}$ with $p(x) = 0$, $q(x) = -x$, and $n = -1$.

3.9. Determine if the following differential equations are homogeneous:

(*a*) $y' = \dfrac{y+x}{x}$ (*b*) $y' = \dfrac{y^2}{x}$ (*c*) $y' = \dfrac{2xye^{x/y}}{x^2 + y^2 \sin\dfrac{x}{y}}$ (*d*) $y' = \dfrac{x^2 + y}{x^3}$

(*a*) The equation is homogeneous, since

$$f(tx, ty) = \frac{ty + tx}{tx} = \frac{t(y + x)}{tx} = \frac{y + x}{x} = f(x, y)$$

(*b*) The equation is not homogeneous, since

$$f(tx, ty) = \frac{(ty)^2}{tx} = \frac{t^2y^2}{tx} = t\frac{y^2}{x} \neq f(x, y)$$

(*c*) The equation is homogeneous, since

$$\begin{aligned} f(tx, ty) &= \frac{2(tx)(ty)e^{tx/ty}}{(tx)^2 + (ty)^2 \sin\dfrac{tx}{ty}} = \frac{t^2 2xye^{x/y}}{t^2x^2 + t^2y^2 \sin\dfrac{x}{y}} \\ &= \frac{2xye^{x/y}}{x^2 + y^2 \sin\dfrac{x}{y}} = f(x, y) \end{aligned}$$

(*d*) The equation is not homogeneous, since

$$f(tx, ty) = \frac{(tx)^2 + ty}{(tx)^3} = \frac{t^2x^2 + ty}{t^3x^3} = \frac{tx^2 + y}{t^2x^3} \neq f(x, y)$$

3.10. Determine if the following differential equations are separable:

(*a*) $\sin x\, dx + y^2 dy = 0$ (*b*) $xy^2 dx - x^2y^2 dy = 0$ (*c*) $(1 + xy)dx + y\, dy = 0$

(*a*) The differential equation is separable; here $M(x, y) = A(x) = \sin x$ and $N(x, y) = B(y) = y^2$.
(*b*) The equation is not separable in its present form, since $M(x, y) = xy^2$ is not a function of x alone. But if we divide both sides of the equation by x^2y^2, we obtain the equation $(1/x)dx + (-1)dy = 0$, which is separable. Here, $A(x) = 1/x$ and $B(y) = -1$.
(*c*) The equation is not separable, since $M(x, y) = 1 + xy$, which is not a function of x alone.

3.11. Determine whether the following differential equations are exact:

(*a*) $3x^2y\, dx + (y + x^3)dy = 0$ (*b*) $xy\, dx + y^2 dy = 0$

(*a*) The equation is exact; here $M(x, y) = 3x^2y$, $N(x, y) = y + x^3$, and $\partial M/\partial y = \partial N/\partial x = 3x^2$.
(*b*) The equation is not exact. Here $M(x, y) = xy$ and $N(x, y) = y^2$; hence $\partial M/\partial y = x$, $\partial N/\partial x = 0$, and $\partial M/\partial y \neq \partial N/\partial x$.

3.12. Determine whether the differential equation $y' = y/x$ is exact.

Exactness is only defined for equations in differential form, not standard form. The given differential equation has many differential forms. One such form is given in Problem 3.5, Eq. (*1*), as

$$\frac{y}{x}dx + (-1)dy = 0$$

Here $M(x, y) = y/x$, $N(x, y) = -1$,

$$\frac{\partial M}{\partial y} = \frac{1}{x} \neq 0 = \frac{\partial N}{\partial x}$$

and the equation is not exact. A second differential form for the same differential equation is given in Eq. (*3*) of Problem 3.5 as

$$\frac{1}{x}dx + \frac{-1}{y}dy = 0$$

Here $M(x, y) = 1/x$, $N(x, y) = -1/y$,

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$

and the equation is exact. Thus, a given differential equation has many differential forms, some of which may be exact.

3.13. Prove that a separable equation is always exact.

For a separable differential equation, $M(x, y) = A(x)$ and $N(x, y) = B(y)$. Thus,

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial A(x)}{\partial y} = 0 \quad \text{and} \quad \frac{\partial N(x,y)}{\partial x} = \frac{\partial B(y)}{\partial x} = 0$$

Since $\partial M/\partial y = \partial N/\partial x$, the differential equation is exact.

3.14. A theorem of first-order differential equations states that if $f(x, y)$ and $\partial f(x, y)/\partial y$ are continuous in a rectangle $\mathcal{R}$: $|x - x_0| \le a$, $|y - y_0| \le b$, then there exists an interval about x_0 in which the initial-value problem $y' = f(x, y)$; $y(x_0) = y_0$ has a unique solution. The initial-value problem $y' = 2\sqrt{|y|}$; $y(0) = 0$ has the two solutions $y = x\,|x|$ and $y \equiv 0$. Does this result violate the theorem?

No. Here, $f(x,y) = 2\sqrt{|y|}$ and, therefore, $\partial f/\partial y$ does not exist at the origin.

Supplementary Problems

In Problems 3.15 through 3.25, write the given differential equations in standard form.

3.15. $xy' + y^2 = 0$

3.16. $e^x y' - x = y'$

3.17. $(y')^3 + y^2 + y = \sin x$

3.18. $xy' + \cos(y' + y) = 1$

3.19. $e^{(y' + y)} = x$

3.20. $(y')^2 - 5y' + 6 = (x + y)(y' - 2)$

3.21. $(x - y)dx + y^2 dy = 0$

3.22. $\dfrac{x + y}{x - y}dx - dy = 0$

3.23. $dx + \dfrac{x + y}{x - y}dy = 0$

3.24. $(e^{2x} - y)dx + e^x dy = 0$

3.25. $dy + dx = 0$

In Problems 3.26 through 3.35, differential equations are given in both standard and differential form. Determine whether the equations in standard form are homogeneous and/or linear, and, if not linear, whether they are Bernoulli; determine whether the equations in differential form, *as given*, are separable and/or exact.

3.26. $y' = xy$; $\quad xy\,dx - dy = 0$

3.27. $y' = xy$; $\quad x\,dx - \dfrac{1}{y}dy = 0$

3.28. $y' = xy + 1$; $\quad (xy + 1)dx - dy = 0$

3.29. $y' = \dfrac{x^2}{y^2}$; $\quad \dfrac{x^2}{y^2}dx - dy = 0$

3.30. $y' = \dfrac{x^2}{y^2}$; $\quad -x^2 dx + y^2 dy = 0$

3.31. $y' = -\dfrac{2y}{x}$; $\quad 2xy dx + x^2 dy = 0$

3.32. $y' = \dfrac{xy^2}{x^2 y + y^3}$; $\quad xy^2 dx - (x^2 y + y^3) dy = 0$

3.33. $y' = \dfrac{-xy^2}{x^2 y + y^2}$; $\quad xy^2 dx + (x^2 y + y^2) dy = 0$

3.34. $y' = x^3 y + xy^3$; $\quad (x^2 + y^2) dx - \dfrac{1}{xy} dy = 0$

3.35. $y' = 2xy + x$; $\quad (2xye^{-x^2} + xe^{-x^2}) dx - e^{-x^2} dy = 0$

CHAPTER 4

Separable First-Order Differential Equations

GENERAL SOLUTION

The solution to the first-order separable differential equation (see Chapter 3)

$$A(x)\,dx + B(y)\,dy = 0 \tag{4.1}$$

is

$$\int A(x)\,dx + \int B(y)\,dy = c \tag{4.2}$$

where c represents an arbitrary constant.

The integrals obtained in Eq. (*4.2*) may be, for all practical purposes, impossible to evaluate. In such cases, numerical techniques (see Chapters 18, 19, 20) are used to obtain an approximate solution. Even if the indicated integrations in (*4.2*) can be performed, it may not be algebraically possible to solve for y explicitly in terms of x. In that case, the solution is left in implicit form.

SOLUTIONS TO THE INITIAL-VALUE PROBLEM

The solution to the initial-value problem

$$A(x)\,dx + B(y)\,dy = 0; \quad y(x_0) = y_0 \tag{4.3}$$

can be obtained, as usual, by first using Eq. (*4.2*) to solve the differential equation and then applying the initial condition directly to evaluate c.

Alternatively, the solution to Eq. (*4.3*) can be obtained from

$$\int_{x_0}^{x} A(x)\,dx + \int_{y_0}^{y} B(y)\,dy = 0 \tag{4.4}$$

Equation (*4.4*), however, may not determine the solution of (*4.3*) *uniquely*; that is, (*4.4*) may have many solutions, of which only one will satisfy the initial-value problem.

REDUCTION OF HOMOGENEOUS EQUATIONS

The homogeneous differential equation

$$\frac{dy}{dx} = f(x, y) \tag{4.5}$$

having the property that $f(tx, ty) = f(x, y)$ (see Chapter 3) can be transformed into a separable equation by making the substitution

$$y = xv \tag{4.6}$$

along with its corresponding derivative

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \tag{4.7}$$

The resulting equation in the variables v and x is solved as a separable differential equation; the required solution to Eq. (*4.5*) is obtained by back substitution.

Alternatively, the solution to (*4.5*) can be obtained by rewriting the differential equation as

$$\frac{dx}{dy} = \frac{1}{f(x, y)} \tag{4.8}$$

and then substituting

$$x = yu \tag{4.9}$$

and the corresponding derivative

$$\frac{dx}{dy} = u + y\frac{du}{dy} \tag{4.10}$$

into Eq. (*4.8*). After simplifying, the resulting differential equation will be one with variables (this time, u and y) separable.

Ordinarily, it is immaterial which method of solution is used (see Problems 4.12 and 4.13). Occasionally, however, one of the substitutions (*4.6*) or (*4.9*) is definitely superior to the other one. In such cases, the better substitution is usually apparent from the form of the differential equation itself. (See Problem 4.17.)

Solved Problems

4.1. Solve $x\,dx - y^2\,dy = 0$.

For this differential equation, $A(x) = x$ and $B(y) = -y^2$. Substituting these values into Eq. (*4.2*), we have

$$\int x\,dx + \int (-y^2)\,dy = c$$

which, after the indicated integrations are performed, becomes $x^2/2 - y^3/3 = c$. Solving for y explicitly, we obtain the solution as

$$y = \left(\frac{3}{2}x^2 + k\right)^{1/3}; \quad k = -3c$$

4.2. Solve $y' = y^2x^3$.

We first rewrite this equation in the differential form (see Chapter 3) $x^3\,dx - (1/y^2)\,dy = 0$. Then $A(x) = x^3$ and $B(y) = -1/y^2$. Substituting these values into Eq. (*4.2*), we have

$$\int x^3\,dx + \int (-1/y^2)\,dy = c$$

or, by performing the indicated integrations, $x^4/4 + 1/y = c$. Solving explicitly for y, we obtain the solution as

$$y = \frac{-4}{x^4 + k}, \quad k = -4c$$

4.3. Solve $\dfrac{dy}{dx} = \dfrac{x^2 + 2}{y}$

This equation may be rewritten in the differential form

$$(x^2 + 2)\,dx - y\,dy = 0$$

which is separable with $A(x) = x^2 + 2$ and $B(y) = -y$. Its solution is

$$\int (x^2 + 2)\,dx - \int y\,dy = c$$

or

$$\frac{1}{3}x^3 + 2x - \frac{1}{2}y^2 = c$$

Solving for y, we obtain the solution in implicit form as

$$y^2 = \frac{2}{3}x^3 + 4x + k$$

with $k = -2c$. Solving for y implicitly, we obtain the two solutions

$$y = \sqrt{\frac{2}{3}x^3 + 4x + k} \quad \text{and} \quad y = -\sqrt{\frac{2}{3}x^3 + 4x + k}$$

4.4. Solve $y' = 5y$.

First rewrite this equation in the differential form $5\,dx - (1/y)\,dy = 0$, which is separable. Its solution is

$$\int 5\,dx + \int (-1/y)\,dy = c$$

or, by evaluating, $5x - \ln|y| = c$.

To solve for y explicitly, we first rewrite the solution as $\ln|y| = 5x - c$ and then take the exponential of both sides. Thus, $e^{\ln|y|} = e^{5x-c}$. Noting that $e^{\ln|y|} = |y|$, we obtain $|y| = e^{5x}e^{-c}$, or $y = \pm e^{-c}e^{5x}$. The solution is given explicitly by $y = ke^{5x}$, $k = \pm e^{-c}$.

Note that the presence of the term $(-1/y)$ in the differential form of the differential equation requires the restriction $y \neq 0$ in our derivation of the solution. This restriction is equivalent to the restriction $k \neq 0$, since $y = ke^{5x}$. However, by inspection, $y \equiv 0$ is a solution of the differential equation as originally given. Thus, $y = ke^{5x}$ is the solution for all k.

The differential equation as originally given is also linear. See Problem 6.9 for an alternate method of solution.

4.5. Solve $y' = \dfrac{x+1}{y^4+1}$.

This equation, in differential form, is $(x + 1)\,dx + (-y^4 - 1)\,dy = 0$, which is separable. Its solution is

$$\int (x+1)\,dx + \int (-y^4 - 1)\,dy = c$$

or, by evaluating,

$$\frac{x^2}{2} + x - \frac{y^5}{5} - y = c$$

Since it is impossible algebraically to solve this equation explicitly for y, the solution must be left in its present implicit form.

4.6. Solve $dy = 2t(y^2 + 9)\, dt$.

This equation may be rewritten as

$$\frac{dy}{y^2 + 9} - 2t\, dt = 0$$

which is separable in variables y and t. Its solution is

$$\int \frac{dy}{y^2 + 9} - \int 2t\, dt = c$$

or, upon evaluating the given integrals,

$$\frac{1}{3}\arctan\left(\frac{y}{3}\right) - t^2 = c$$

Solving for y, we obtain

$$\arctan\left(\frac{y}{3}\right) = 3(t^2 + c)$$

$$\frac{y}{3} = \tan\,(3t^2 + 3c)$$

or

$$y = 3\,\tan\,(3t^2 + \mathrm{k})$$

with $k = 3c$.

4.7. Solve $\dfrac{dx}{dt} = x^2 - 2x + 2$.

This equation may be rewritten in differential form

$$\frac{dx}{x^3 - 2x + 2} - dt = 0$$

which is separable in the variables x and t. Its solution is

$$\int \frac{dx}{x^3 - 2x + 2} - \int dt = c$$

Evaluating the first integral by first completing the square, we obtain

$$\int \frac{dx}{(x-1)^2 + 1} - \int dt = c$$

or

$$\arctan\,(x - 1) - t = c$$

Solving for x as a function of t, we obtain

$$\arctan\,(x - 1) = t + c$$

$$x - 1 = \tan\,(t + c)$$

or

$$x = 1 + \tan\,(t + c)$$

4.8. Solve $e^x\,dx - y\,dy = 0$; $y(0) = 1$.

The solution to the differential equation is given by Eq. (*4.2*) as

$$\int e^x\,dx + \int(-y)\,dy = c$$

or, by evaluating, as $y^2 = 2e^x + k$, $k = -2c$. Applying the initial condition, we obtain $(1)^2 = 2e^0 + k$, $1 = 2 + k$, or $k = -1$. Thus, the solution to the initial-value problem is

$$y^2 = 2e^x - 1 \quad \text{or} \quad y = \sqrt{2e^x - 1}$$

[Note that we cannot choose the negative square root, since then $y(0) = -1$, which violates the initial condition.]

To ensure that y remains real, we must restrict x so that $2e^x - 1 \geq 0$. To guarantee that y' exists [note that $y'(x) = dy/dx = e^x/y$], we must restrict x so that $2e^x - 1 \neq 0$. Together these conditions imply that $2e^x - 1 > 0$, or $x > \ln\frac{1}{2}$.

4.9. Use Eq. (*4.4*) to solve Problem 4.8.

For this problem, $x_0 = 0$, $y_0 = 1$, $A(x) = e^x$, and $B(y) = -y$. Substituting these values into Eq. (*4.4*), we obtain

$$\int_0^x e^x\,dx + \int_1^y (-y)\,dy = 0$$

Evaluating these integrals, we have

$$e^x\Big|_0^x + \left(\frac{-y^2}{2}\right)\Bigg|_1^y = 0 \quad \text{or} \quad e^x - e^0 + \left(\frac{-y^2}{2}\right) - \left(-\frac{1}{2}\right) = 0$$

Thus, $y^2 = 2e^x - 1$, and, as in Problem 4.8, $y = \sqrt{2e^x - 1}$, $\quad x > \ln\frac{1}{2}$.

4.10. Solve $x\cos x\,dx + (1 - 6y^5)\,dy = 0$; $y(\pi) = 0$.

Here, $x_0 = \pi$, $y_0 = 0$, $A(x) = x\cos x$, and $B(y) = 1 - 6y^5$. Substituting these values into Eq. (*4.4*), we obtain

$$\int_\pi^x x\cos x\,dx + \int_0^y (1 - 6y^5)\,dy = 0$$

Evaluating these integrals (the first one by integration by parts), we find

$$x\sin x\Big|_\pi^x + \cos x\Big|_\pi^x + (y - y^6)\Big|_0^y = 0$$

or

$$x\sin x + \cos x + 1 = y^6 - y$$

Since we cannot solve this last equation for y explicitly, we must be content with the solution in its present implicit form.

4.11. Solve $y' = \dfrac{y + x}{x}$.

This differential equation is not separable, but it is homogeneous as shown in Problem 3.9(*a*). Substituting Eqs. (*4.6*) and (*4.7*) into the equation, we obtain

$$v + x\frac{dv}{dx} = \frac{xv + x}{x}$$

which can be algebraically simplified to

$$x\frac{dv}{dx} = 1 \quad \text{or} \quad \frac{1}{x}dx - dv = 0$$

This last equation is separable; its solution is

$$\int \frac{1}{x}\,dx - \int dv = c$$

which, when evaluated, yields $v = \ln|x| - c$, or

$$v = \ln|kx| \tag{1}$$

where we have set $c = -\ln|k|$ and have noted that $\ln|x| + \ln|k| = \ln|kx|$. Finally, substituting $v = y/x$ back into (*1*), we obtain the solution to the given differential equation as $y = x\ln|kx|$.

4.12. Solve $y' = \dfrac{2y^4 + x^4}{xy^3}$.

This differential equation is not separable. Instead it has the form $y' = f(x, y)$, with

$$f(x,y) = \frac{2y^4 + x^4}{xy^3}$$

where
$$f(tx, ty) = \frac{2(ty)^4 + (tx)^4}{(tx)(ty)^3} = \frac{t^4(2y^4 + x^4)}{t^4(xy^3)} = \frac{2y^4 + x^4}{xy^3} = f(x,y)$$

so it is homogeneous. Substituting Eqs. (*4.6*) and (*4.7*) into the differential equation as originally given, we obtain

$$v + x\frac{dv}{dx} = \frac{2(xv)^4 + x^4}{x(xv)^3}$$

which can be algebraically simplified to

$$x\frac{dv}{dx} = \frac{v^4 + 1}{v^3} \quad \text{or} \quad \frac{1}{x}dx - \frac{v^3}{v^4 + 1}dv = 0$$

This last equation is separable; its solution is

$$\int \frac{1}{x}\,dx - \int \frac{v^3}{v^4 + 1}\,dv = c$$

Integrating, we obtain in $\ln|x| - \frac{1}{4}\ln(v^4 + 1) = c$, or

$$v^4 + 1 = (kx)^4 \tag{1}$$

where we have set $c = -\ln|k|$ and then used the identities

$$\ln|x| + \ln|k| = \ln|kx| \quad \text{and} \quad 4\ln|kx| = \ln(kx)^4$$

Finally, substituting $v = y/x$ back into (*1*), we obtain

$$y^4 = c_1x^8 - x^4 \quad (c_1 = k^4) \tag{2}$$

4.13. **Solve the differential equation of Problem 4.12 by using Eqs. (*4.9*) and (*4.10*).**

We first rewrite the differential equation as

$$\frac{dx}{dy} = \frac{xy^3}{2y^4 + x^4}$$

Then substituting (*4.9*) and (*4.10*) into this new differential equation, we obtain

$$u + y\frac{du}{dy} = \frac{(yu)y^3}{2y^4 + (yu)^4}$$

which can be algebraically simplified to

$$y\frac{du}{dy} = -\frac{u+u^5}{2+u^4}$$

or

$$\frac{1}{y}dy + \frac{2+u^4}{u+u^5}du = 0 \tag{1}$$

Equation (*1*) is separable; its solution is

$$\int \frac{1}{y}dy + \int \frac{2+u^4}{u+u^5}du = c$$

The first integral is in ln |y|. To evaluate the second integral, we use partial fractions on the integrand to obtain

$$\frac{2+u^4}{u+u^5} = \frac{2+u^4}{u(1+u^4)} = \frac{2}{u} - \frac{u^3}{1+u^4}$$

Therefore,

$$\int \frac{2+u^4}{u+u^5}du = \int \frac{2}{u}du - \int \frac{u^3}{1+u^4}du = 2\ln|u| - \frac{1}{4}\ln(1+u^4)$$

The solution to (*1*) is in $\ln|y| + 2\ln|u| - \frac{1}{4}\ln(1+u^4) = c$, which can be rewritten as

$$ky^4u^8 = 1 + u^4 \tag{2}$$

where $c = -\frac{1}{4}\ln|k|$. Substituting $u = x/y$ back into (*2*), we once again have (*2*) of Problem 4.12.

4.14. Solve $y' = \dfrac{2xy}{x^2 - y^2}$.

This differential equation is not separable. Instead it has the form $y' = f(x, y)$, with

$$f(x,y) = \frac{2xy}{x^2 - y^2}$$

where

$$f(tx, ty) = \frac{2(tx)(ty)}{(tx)^2 - (ty)^2} = \frac{t^2(2xy)}{t^2(x^2 - y^2)} = \frac{2xy}{x^2 - y^2} = f(x, y)$$

so it is homogenous. Substituting Eqs. (*4.6*) and (*4.7*) into the differential equation as originally given, we obtain

$$v + x\frac{dv}{dx} = \frac{2x(xv)}{x^2 - (xv)^2}$$

which can be algebraically simplified to

$$x\frac{dv}{dx} = -\frac{v(v^2+1)}{v^2-1}$$

or

$$\frac{1}{x}dx + \frac{v^2-1}{v(v^2+1)}dv = 0 \tag{1}$$

Using partial fractions, we can expand (*1*) to

$$\frac{1}{x}dx + \left(-\frac{1}{v} + \frac{2v}{v^2+1}\right)dv = 0 \tag{2}$$

The solution to this separable equation is found by integrating both sides of (*2*). Doing so, we obtain $\ln|x| - \ln|v| + \ln(v^2+1) = c$, which can be simplified to

$$x(v^2+1) = kv \quad (c = \ln|k|) \tag{3}$$

Substituting $v = y/x$ into (*3*), we find the solution of the given differential equation is $x^2 + y^2 = ky$.

4.15. Solve $y' = \dfrac{x^2 + y^2}{xy}$.

This differential equation is homogeneous. Substituting Eqs. (*4.6*) and (*4.7*) into it, we obtain

$$v + x\frac{dv}{dx} = \frac{x^2 + (xv)^2}{x(xv)}$$

which can be algebraically simplified to

$$x\frac{dv}{dx} = \frac{1}{v} \quad \text{or} \quad \frac{1}{x}dx - v\,dv = 0$$

The solution to this separable equation is $\ln |x| - v^2/2 = c$, or equivalently

$$v^2 = \ln x^2 + k \qquad (k = -2c) \tag{1}$$

Substituting $v = y/x$ into (*1*), we find that the solution to the given differential equation is

$$y^2 = x^2 \ln x^2 + kx^2$$

4.16. Solve $y' = \dfrac{x^2 + y^2}{xy}$; $y(1) = -2$.

The solution to the differential equation is given in Problem 3.15 as $y^2 = x^2 \ln x^2 + kx^2$. Applying the initial condition, we obtain $(-2)^2 = (1)^2 \ln (1)^2 + k(1)^2$, or $k = 4$. (Recall that $\ln 1 = 0$.) Thus, the solution to the initial-value problem is

$$y^2 = x^2 \ln x^2 + 4x^2 \quad \text{or} \quad y = -\sqrt{x^2 \ln x^2 + 4x^2}$$

The negative square root is taken, to be consistent with the initial condition.

4.17. Solve $y' = \dfrac{2xye^{(x/y)^2}}{y^2 + y^2e^{(x/y)^2} + 2x^2e^{(x/y)^2}}$.

This differential equation is not separable, but it is homogeneous. Noting the (x/y)-term in the exponential, we try the substitution $u = x/y$, which is an equivalent form of (*4.9*). Rewriting the differential equation as

$$\frac{dx}{dy} = \frac{y^2 + y^2e^{(x/y)^2} + 2x^2e^{(x/y)^2}}{2xye^{(x/y)^2}}$$

we have upon using substitutions (*4.9*) and (*4.10*) and simplifying,

$$y\frac{du}{dy} = \frac{1 + e^{u^2}}{2ue^{u^2}} \quad \text{or} \quad \frac{1}{y}dy - \frac{2ue^{u^2}}{1 + e^{u^2}}du = 0$$

This equation is separable; its solution is

$$\ln |y| - \ln (1 + e^{u^2}) = c$$

which can be rewritten as

$$y = k(1 + e^{u^2}) \quad (c = \ln |k|) \tag{1}$$

Substituting $u = x/y$ into (*1*), we obtain the solution of the given differential equation as

$$y = k[1 + e^{(x/y)^2}]$$

4.18. Prove that every solution of Eq. (*4.2*) satisfies Eq. (*4.1*).

Rewrite (*4.1*) as $A(x) + B(y)y' = 0$. If $y(x)$ is a solution, it must satisfy this equation identically in x; hence,

$$A(x) + B[y(x)]y'(x) = 0$$

Integrating both sides of this last equation with respect to x, we obtain

$$\int A(x)\,dx + \int B[y(x)]\,y'(x)\,dx = c$$

In the second integral, make the change of variables $y = y(x)$, hence $dy = y'(x)\,dx$. The result of this substitution is (*4.2*).

4.19. Prove that every solution of system (*4.3*) is a solution of (*4.4*).

Following the same reasoning as in Problem 4.18, except now integrating from $x = x_0$ to $x = x$, we obtain

$$\int_{x_0}^{x} A(x)\,dx + \int_{x_0}^{x} B[y(x)]y'(x)\,dx = 0$$

The substitution $y = y(x)$ again gives the desired result. Note that as x varies from x_0 to x, y will vary from $y(x_0) = y_0$ to $y(x) = y$.

4.20. Prove that if $y' = f(x, y)$ is homogeneous, then the differential equation can be rewritten as $y' = g(y/x)$, where $g(y/x)$ depends only on the quotient y/x.

We have that $f(x, y) = f(tx, ty)$. Since this equation is valid for all t, it must be true, in particular, for $t = 1/x$. Thus, $f(x, y) = f(1, y/x)$. If we now define $g(y/x) = f(1, y/x)$, we then have $y' = f(x, y) = f(1, y/x) = g(y/x)$ as required.

Note that this form suggests the substitution $v = y/x$ which is equivalent to (*4.6*). If, in the above, we had set $t = 1/y$, then $f(x, y) = f(x/y, 1) = h(x/y)$, which suggests the alternate substitution (*4.9*).

4.21. A function $g(x, y)$ is *homogeneous of degree n* if $g(tx, ty) = t^n g(x, y)$ for all t. Determine whether the following functions are homogeneous, and, if so, find their degree:

(*a*) $xy + y^2$, (*b*) $x + y \sin (y/x)^2$, (*c*) $x^3 + xy^2 e^{x/y}$, and (*d*) $x + xy$.

(*a*) $(tx)(ty) + (ty)^2 = t^2(xy + y^2)$; homogeneous of degree two.

(*b*) $tx + ty\sin\left(\dfrac{ty}{tx}\right)^2 = t\left[x + y\sin\left(\dfrac{y}{x}\right)^2\right]$; homogeneous of degree one.

(*c*) $(tx)^3 + (tx)(ty)^2 e^{tx/ty} = t^3(x^3 + xy^2 e^{x/y})$; homogeneous of degree three.

(*d*) $tx + (tx)(ty) = tx + t^2 xy$; not homogeneous.

4.22. An alternate definition of a homogeneous differential equation is as follows: A differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is *homogenous* if both $M(x, y)$ and $N(x, y)$ are homogeneous of the same degree (see Problem 4.21). Show that this definition implies the definition given in Chapter 3.

If $M(x, y)$ and $N(x, y)$ are homogeneous of degree n, then

$$f(tx, ty) = \frac{M(tx, ty)}{-N(tx, ty)} = \frac{t^n M(x, y)}{-t^n N(x, y)} = \frac{M(x, y)}{-N(x, y)} = f(x, y)$$

Supplementary Problems

In Problems 4.23 through 4.45, solve the given differential equations or initial-value problems.

4.23. $x\,dx + y\,dy = 0$

4.24. $x\,dx - y^3\,dy = 0$

4.25. $dx + \dfrac{1}{y^4}\,dy = 0$

4.26. $(t + 1)\,dt - \dfrac{1}{y^2}\,dy = 0$

4.27. $\frac{1}{x}dx - \frac{1}{y}dy = 0$

4.28. $\frac{1}{x}dx + dy = 0$

4.29. $x\,dx + \frac{1}{y}dy = 0$

4.30. $(t^2 + 1)\,dt + (y^2 + y)\,dy = 0$

4.31. $\frac{4}{t}dt - \frac{y-3}{y}dy = 0$

4.32. $dx - \frac{1}{1+y^2}dy = 0$

4.33. $dx - \frac{1}{y^2 - 6y + 13}dy = 0$

4.34. $y' = \frac{y}{x^2}$

4.35. $y' = \frac{xe^x}{2y}$

4.36. $\frac{dy}{dx} = \frac{x+1}{y}$

4.37. $\frac{dy}{dx} = y^2$

4.38. $\frac{dx}{dt} = x^2t^2$

4.39. $\frac{dx}{dt} = \frac{x}{t}$

4.40. $\frac{dy}{dt} = 3 + 5y$

4.41. $\sin x\,dx + y\,dy = 0;\quad y(0) = -2$

4.42. $(x^2 + 1)\,dx + \frac{1}{y}dy = 0;\quad y(-1) = 1$

4.43. $xe^{x^2}\,dx + (y^5 - 1)\,dy = 0;\quad y(0) = 0$

4.44. $y' = \frac{x^2y - y}{y+1};\quad y(3) = -1$

4.45. $\frac{dx}{dt} = 8 - 3x;\quad x(0) = 4$

In Problems 4.46 through 4.54, determine whether the given differential equations are homogenous and, if so, solve them.

4.46. $y' = \frac{y-x}{x}$

4.47. $y' = \frac{2y+x}{x}$

4.48. $y' = \frac{x^2 + 2y^2}{xy}$

4.49. $y' = \frac{2x + y^2}{xy}$

4.50. $y' = \frac{x^2 + y^2}{2xy}$

4.51. $y' = \frac{2xy}{y^2 - x^2}$

4.52. $y' = \frac{y}{x + \sqrt{xy}}$

4.53. $y' = \frac{y^2}{xy + (xy^2)^{1/3}}$

4.54. $y' = \frac{x^4 + 3x^2y^2 + y^4}{x^3y}$