Indefinite integral (anti-derivative)

M.W.

Mathematics Teaching and Distance Learning Centre
Gdańsk University of Technology

2012-2017

Consider functions

$$f(x) = x^3 - 2x^2 - 5x - 6$$

$$f(x) = x^3 - 2x^2 - 5x - 5$$

$$f(x) = x^3 - 2x^2 - 5x - 4$$

$$f(x) = x^3 - 2x^2 - 5x - 3$$

Find their derivatives.

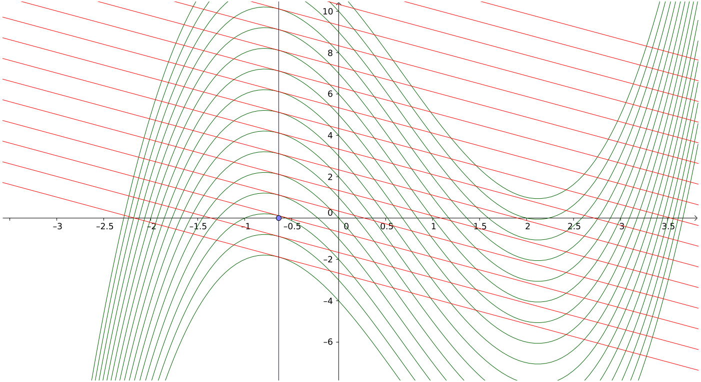
Derivatives

$$f(x) = x^3 - 2x^2 - 5x - 6 \longrightarrow f'(x) = 3x^2 - 4x - 5$$

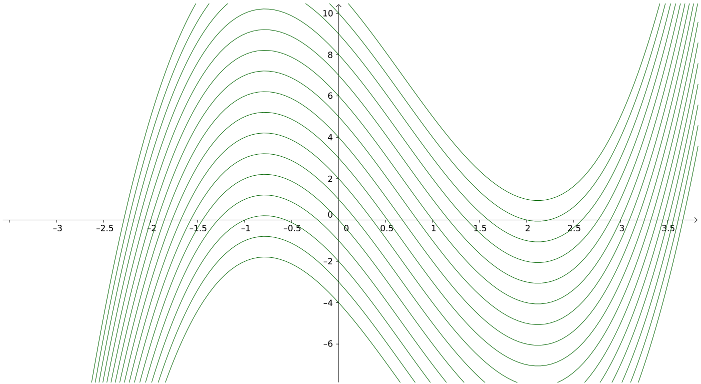
$$f(x) = x^3 - 2x^2 - 5x - 5 \longrightarrow f'(x) = 3x^2 - 4x - 5$$

$$f(x) = x^3 - 2x^2 - 5x - 4 \longrightarrow f'(x) = 3x^2 - 4x - 5$$

$$f(x) = x^3 - 2x^2 - 5x - 3 \longrightarrow f'(x) = 3x^2 - 4x - 5$$


10
8
6
4
2
0
-2
-4
-6
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5

$$f(x) = 3x^2 - 4x - 5 \longrightarrow F(x) = x^3 - 2x^2 - 5x + C$$



Indefinite integral

Definition (Indefinite integral)

Given a function $f(x)$, an indefinite integral of $f(x)$ is defined as a differentiable function $F(x)$ which satisfies the equation

$$F'(x) = \frac{d}{dx}(F(x)) = f(x)$$

The function $F(x)$ is usually denoted by

$$\int f(x)dx$$

The function $f(x)$ is called the **integrand** of the indefinite integral.

If $F(x)$ is an indefinite integral of $f(x)$, then

$$\frac{d}{dx}(F(x) + C) = \frac{d}{dx}(F(x)) = f(x)$$

Then $F(x) + C$ is another indefinite integral of $f(x)$.
If $f(x)$ has two indefinite integrals $F(x)$ and $G(x)$, then

$$F'(x) = G'(x) = f(x)$$

It follows that $G(x) = F(x) + C$ for some constant C.

Proposition

$$\left[\int f(x)dx\right]' = f(x)$$

Proposition

$$\int f'(x)dx = f(x) + C$$

Basic formulae

- $\int 0dx = C$
- $\int x^n dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1,$
- $\int \frac{1}{x}dx = \ln|x| + C$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\ln a} + C$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$
- $\int \frac{1}{\sin^2 x}dx = -\cot x + C$
- $\int \frac{1}{\cos^2 x}dx = \tan x + C$
- $\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

Examples

Example

Find the following indefinite integrals.

1. $$\int \frac{\sqrt[3]{x}x}{\sqrt{x}}dx$$

2. $$\int \frac{1}{7^x}dx$$

Properties of indefinite integral

Theorem (Linearity of indefinite integral)

- $\int cf(x)dx = c\int f(x)dx, \quad c \in \mathbb{R}$
- $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

Example

Find the following indefinite integrals.

1. $$\int (e^x - 3x^2)dx = \int e^x dx - \int 3x^2 dx = \int e^x dx - 3\int x^2 dx =$$
$$e^x - x^3 + C$$

2. $$\int \frac{1}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx =$$
$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx =$$
$$\int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx = \tan x - \cot x + C$$

Theorem (Integration by substitution)

If $u = u(x)$ is a differentiable function and g is continuous, then

$$\int g(u)\frac{du}{dx} = \int g(u)u'(x)dx = \int g(u)du$$

We can can preform the method of integration by substitution as follows:
Write the integral as the form

$$\int f(x)dx = \int g(u)\frac{du}{dx}dx$$

for some "inside function" $u = u(x)$.
Then $du = \frac{du}{dx}dx$ and so

$$\int f(x)dx = \int g(u)du.$$

Therefore, we can evaluate the latter integral instead.

Example

$$\int \cos(2x+3)dx$$

Example

$$\int x\sqrt{x-5}dx$$

Example

$$\int \frac{1}{(\arccos x)^5\sqrt{1-x^2}}dx$$

Theorem (Integration by parts)

If $u = u(x)$ and $v = v(x)$ are differentiable functions, then we have the following identity:

$$\int u dv = uv - \int v du$$

We can write also

$$\int uv' dx = uv - \int vu' dx$$

Example

$$\int x^3 \ln x dx$$

Example

$$\int e^x \sin x dx$$

Example

$$\int \ln x dx$$

Examples

Example

$$\int x \cos x dx$$

Solution: $x \sin x + \cos x + C$

Examples

Example

$$\int x^3 \ln x dx$$

Solution: $\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$

Example

$$\int (x^2 - 2x + 5)e^{-x} dx$$

Solution: $-e^{-x}(x^2 + 5) + C$

We can use tabular integration for integrals of the form $\int f(x) \cdot g(x)dx$, if $\exists_{n \in \mathbb{N}}$ such, that $f^{(n)}(x) = 0$ provided g is n-times integrable.
Then

$$\int f(x) \cdot g(x)dx =$$

$$= f(x)G_1(x) - f'(x)G_2(x) + f''(x)G_3(x) + \ldots + (-1)^{n-1}f^{(n-1)}(x)G_n(x) + C$$

Example

$$\int (x^3 - 2x^2 + 4)\sin(x)dx$$

sign	Derivatives	Integrals
$+$	$x^3 - 2x^2 + 4$	$\sin x$
$-$	$f'(x) = 3x^2 - 4x$	$G_1(x) = -\cos x$
$+$	$f''(x) = 6x - 4$	$G_2(x) = -\sin x$
$-$	$f'''(x) = 6$	$G_3(x) = \cos x$
$+$	$f^{(4)}(x) = 0$	$G_4(x) = \sin x$

$$\int (x^3 - 2x^2 + 4)\sin(x)dx =$$

$$(x^3 - 2x^2 + 4)(-\cos x) - (3x^2 - 4x)(-\sin x) + (6x - 4)\cos x - 6\sin x + C$$

Useful formulae

- $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C, \quad f(x) \neq 0$
- $\int f^n(x) f'(x) dx = \frac{f^{n+1}(x)}{n+1} + C, \quad n \in \mathbb{N}$
- $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C, \quad f(x) > 0$
- $\int \frac{f'(x)}{f^2(x)} dx = -\frac{1}{f(x)} + C, \quad f(x) \neq 0$

Proposition

If $\int f(x)dx = F(x) + C$, *then* $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$

Basic formulae

- $\int \tan x dx = \ln |\frac{1}{\cos x}| + C$
- $\int \cot x dx = -\ln |\frac{1}{\sin x}| + C$
- $\int \frac{1}{\sin x} dx = -\ln |\frac{1}{\sin x} + \cot x| + C$
- $\int \frac{1}{\cos x} dx = \ln |\frac{1}{\cos x} + \tan x| + C$

Examples

Example

$$\int \arctan x dx$$

Solution: $x \arctan x - \frac{1}{2} \ln(1 + x^2) + C$

Example

$$\int e^{\sqrt{x}} dx$$

Solution: $2\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}} + C$

Example

Calculate integrals:

1. $\int \frac{x^2+5x-1}{\sqrt{x}} dx$
2. $\int \frac{6x^3+x^2-2x+1}{2x-1} dx$
3. $\int \frac{1}{\sin^2(x)\cos^2(x)} dx$
4. $\int \tan^2 x dx$

Example

1. $\int (x^2 + 5)^3 \, dx$
2. $\int (22x + 11)^{2013} \, dx$
3. $\int \frac{1}{\sqrt{x+1} - \sqrt{x}} \, dx$
4. $\int \cos(\pi x + 1) \, dx$
5. $\int \cos(4x) \cos(7x) \, dx$

Example

1. $\int \sin^2 x dx$
2. $\int \sin^2(3x) dx$
3. $\int \frac{1}{x^2+4x+5} dx$
4. $\int \frac{1}{4x^2+25} dx$
5. $\int \frac{1}{x^2+x+1} dx$

Example

1. $\int \frac{1}{\sqrt{5-x^2-4x}}\,dx$
2. $\int \frac{1}{\sqrt{x^2+6x+1}}\,dx$
3. $\int \frac{1}{\sqrt{4-x^2-4x}}\,dx$
4. $\int \frac{1}{\sqrt{10x^2-7}}\,dx$

Example

1. $\int \frac{1}{x^2-6x+13}\,dx$
2. $\int \frac{x-1}{\sqrt[3]{x^2}}\,dx$
3. $\int \frac{3-2\cot^2 x}{\cos^2 x}\,dx$
4. $\int \frac{2+3x^2}{x^2(1+x^2)}\,dx$

Example

1. $\int \frac{\sqrt{1-x^2}\sqrt{1+x^2}}{\sqrt{1-x^4}}\,dx$
2. $\int \frac{\cos 2x}{\cos x - \sin x}\,dx$
3. $\int \frac{2^{x+1} - 5^{x-1}}{10^x}\,dx$
4. $\int (\sin 5x - \sin 5\alpha)\,dx$

Strategy for integrating rational functions $\frac{P(x)}{Q(x)}$

If $\deg P(x) \geq \deg Q(x)$, try long division. Then apply the method of partial fractions on the remainder.

If $Q(x)$ is the product of distinct linear factors, pick partial fractions of the form

$$\frac{A}{x-c}$$

If $Q(x)$ contains a repeated linear factor, $(x-c)^n$, pick partial fractions of the form

$$\frac{A_1}{x-c} + \frac{A_2}{(x-c)^2} + \frac{A_3}{(x-c)^3} + \ldots + \frac{A_n}{(x-c)^n}$$

If $Q(x)$ contains an unfactorable quadratic $q(x) = x^2 + cx + d$, pick a partial fraction of the form $\frac{Ax+B}{q(x)}$

If $Q(x)$ contains a repeated unfactorable quadratic $q(x)^n = (x^2 + cx + d)^n$, pick partial fractions of the form

$$\frac{A_1x + B_1}{q(x)} + \frac{A_2x + B_2}{q(x)^2} + \frac{A_3x + B_3}{q(x)^3} + \ldots + \frac{A_nx + B_n}{q(x)^n}$$

Useful identities for simple rational forms:

- $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C, \quad a \neq 0$
- $\int \frac{1}{(ax+b)^n}dx = \frac{-1}{a(n-1)(ax+b)^{n-1}} + C, \quad a \neq 0, \ n \neq 1$

Examples

Example

$$\int \frac{15x^2 - 4x - 81}{(x-3)(x+4)(x-1)} dx$$

Solution: $3 \ln|x-3| + 5 \ln|x+4| + 7 \ln|x-1| + C$

Example

$$\int \frac{x^4 - 3x^2 - 3x - 2}{x^3 - x^2 - 2x} dx$$

Solution: $\frac{x^2}{2} + x + \ln|x| - \frac{2}{3} \ln|x-2| - \frac{1}{3} \ln|x+1| + C$

Examples

Example

$$\int \frac{2x^2 - 3x + 3}{x^3 - 2x^2 + x} dx$$

Solution: $3 \ln|x| - \frac{2}{x-1} - \ln|x-1| + C$

Example

$$\int \frac{x}{x^3 + 1} dx$$

Solution: $-\frac{1}{3}\ln|x+1| + \frac{1}{6}\ln(x^2 - x + 1) + \frac{\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}} + C$

Bibliography

Examples are taken from:

I.A.Maron, *Problems In Calculus Of One Variable*, Mir Publishers, Moscow 1973