Chapter 67

Directional Derivatives; Maximum and Minimum Values

DIRECTIONAL DERIVATIVES. Through $P(x, y, z)$, any point on the surface $z = f(x, y)$, pass planes parallel to the coordinate planes xOz and yOz cutting the surface in the arcs PR and PS and the plane xOy in the lines P^*M and P^*N, as shown in Fig. 67-1. The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ evaluated at $P^*(x, y)$ give, respectively the rates of change of $z = P^*P$ when y is held fixed and when x is held fixed, that is, the rates of change of z in directions parallel to the x and y axes or the slopes of the curves PR and PS at P.

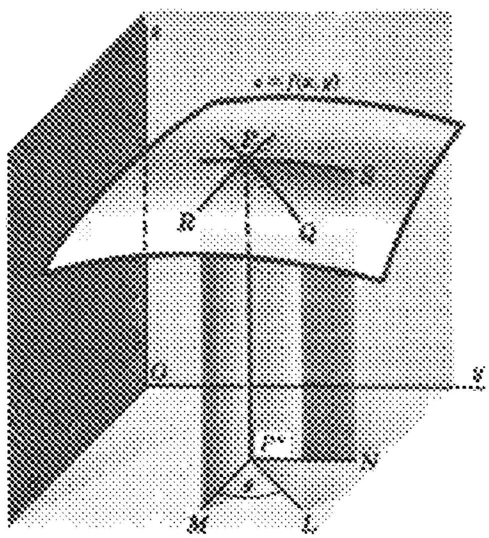

Fig. 67-1

Consider next a plane through P perpendicular to the plane xOy and making an angle θ with the x axis. Let it cut the surface in the curve PQ and the xOy plane in the line P^*L. The *directional derivative* of $f(x, y)$ at P^* in the direction θ is given by

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \tag{67.1}$$

The direction θ is the direction of the vector $(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$. The directional derivative gives the rate of change of $z = P^*P$ in the direction of P^*L or the slope of the curve PQ at P.

The directional derivative at a point P^* is a function of θ. There is a direction, determined by a vector called the *gradient* of f at P^* (Chapter 68), for which the directional derivative at P^* has a maximum value. That maximum value is the slope of the steepest tangent line that can be drawn to the surface at P. (See Problems 1 to 8.)

For a function $w = F(x, y, z)$, the directional derivative at $P(x, y, z)$ in the direction determined by the angles α, β, γ is given by

$$\frac{dF}{ds} = \frac{\partial F}{\partial x}\cos\alpha + \frac{\partial F}{\partial y}\cos\beta + \frac{\partial F}{\partial z}\cos\gamma$$

By the direction determined by α, β, and γ, we mean the direction of the vector $(\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k}$. (See Problem 9.)

RELATIVE MAXIMUM AND MINIMUM VALUES. Suppose that $z = f(x, y)$ has a relative maximum (or minimum) value at $P_0(x_0, y_0, z_0)$. Any plane through P_0 perpendicular to the plane xOy will cut the surface in a curve having a relative maximum (or minimum) point at P_0; that is, the directional derivative $\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$ of $z = f(x, y)$ must equal zero at P_0, for any value of θ. Thus, at P_0, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

The points, if any, at which $z = f(x, y)$ has a relative maximum (or minimum) value are among the points (x_0, y_0) for which $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ simultaneously. To separate the cases, we quote without proof:

Let $z = f(x, y)$ have first and second partial derivatives in a certain region including the point (x_0, y_0, z_0) at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. If $\Delta = \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) < 0$ at P_0, then $z = f(x, y)$ has

$$\text{A relative minimum at } P_0 \text{ if } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} > 0$$

or

$$\text{A relative maximum at } P_0 \text{ if } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} < 0$$

If $\Delta > 0$, P_0 yields neither a maximum nor a minimum value; if $\Delta = 0$, the nature of the critical point P_0 is undetermined. (See Problems 10 to 15.)

Solved Problems

1. Derive *(67.1)*.

In Fig. 67-1, let $P_1^*(x + \Delta x, y + \Delta y)$ be a second point on P^*L and denote by Δs the distance $P^*P_1^*$. Assuming that $z = f(x, y)$ possesses continuous first partial derivatives, we have, by Problem 20 of Chapter 63,

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as Δx and $\Delta y \to 0$. The average rate of change of z between the points P^* and P_1^* is

$$\begin{aligned}\frac{\Delta z}{\Delta s} &= \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta s} + \epsilon_1\frac{\Delta x}{\Delta s} + \epsilon_2\frac{\Delta y}{\Delta s} \\ &= \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta + \epsilon_1\cos\theta + \epsilon_2\sin\theta\end{aligned}$$

where θ is the angle that the line $P^*P_1^*$ makes with the x axis. Now let $P_1^* \to P^*$ along P^*L; the instantaneous rate of change of z, or the directional derivative at P^*, is

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

2. Find the directional derivative of $z = x^2 - 6y^2$ at $P^*(7, 2)$ in the direction (*a*) $\theta = 45°$, (*b*) $\theta = 135°$.

The directional derivative at any point $P^*(x, y)$ in the direction θ is

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta = 2x\cos\theta - 12y\sin\theta$$

(*a*) At $P^*(7,2)$ in the direction $\theta = 45°$, $dz/ds = 2(7)(\frac{1}{2}\sqrt{2}) - 12(2)(\frac{1}{2}\sqrt{2}) = -5\sqrt{2}$.
(*b*) At $P^*(7,2)$ in the direction $\theta = 135°$, $dz/ds = 2(7)(-\frac{1}{2}\sqrt{2}) - 12(2)(\frac{1}{2}\sqrt{2}) = -19\sqrt{2}$.

3. Find the directional derivative of $z = ye^x$ at $P^*(0,3)$ in the direction (*a*) $\theta = 30°$, (*b*) $\theta = 120°$.

Here, $dz/ds = ye^x \cos\theta + e^x \sin\theta$.
(*a*) At $(0,3)$ in the direction $\theta = 30°$, $dz/ds = 3(1)(\frac{1}{2}\sqrt{3}) + \frac{1}{2} = \frac{1}{2}(3\sqrt{3}+1)$.
(*b*) At $(0,3)$ in the direction $\theta = 120°$, $dz/ds = 3(1)(-\frac{1}{2}) + \frac{1}{2}\sqrt{3} = \frac{1}{2}(-3+\sqrt{3})$.

4. The temperature T of a heated circular plate at any of its points (x, y) is given by $T = \dfrac{64}{x^2+y^2+2}$, the origin being at the center of the plate. At the point $(1,2)$ find the rate of change of T in the direction $\theta = \pi/3$.

We have
$$\frac{dT}{ds} = -\frac{64(2x)}{(x^2+y^2+2)^2}\cos\theta - \frac{64(2y)}{(x^2+y^2+2)^2}\sin\theta$$

At $(1,2)$ in the direction $\theta = \dfrac{\pi}{3}$, $\dfrac{dT}{ds} = -\dfrac{128}{49}\dfrac{1}{2} - \dfrac{256}{49}\dfrac{\sqrt{3}}{2} = -\dfrac{64}{49}(1+2\sqrt{3})$.

5. The electrical potential V at any point (x, y) is given by $V = \ln\sqrt{x^2+y^2}$. Find the rate of change of V at the point $(3,4)$ in the direction toward the point $(2,6)$.

Here,
$$\frac{dV}{ds} = \frac{x}{x^2+y^2}\cos\theta + \frac{y}{x^2+y^2}\sin\theta$$

Since θ is a second-quadrant angle and $\tan\theta = (6-4)/(2-3) = -2$, $\cos\theta = -1/\sqrt{5}$ and $\sin\theta = 2/\sqrt{5}$. Hence, at $(3,4)$ in the indicated direction, $\dfrac{dV}{ds} = \dfrac{3}{25}\left(-\dfrac{1}{\sqrt{5}}\right) + \dfrac{4}{25}\dfrac{2}{\sqrt{5}} = \dfrac{\sqrt{5}}{25}$.

6. Find the maximum directional derivative for the surface and point of Problem 2.

At $P^*(7,2)$ in the direction θ, $dz/ds = 14\cos\theta - 24\sin\theta$.

To find the value of θ for which $\dfrac{dz}{ds}$ is a maximum, set $\dfrac{d}{d\theta}\left(\dfrac{dz}{ds}\right) = -14\sin\theta - 24\cos\theta = 0$. Then $\tan\theta = -\frac{24}{14} = -\frac{12}{7}$ and θ is either a second- or fourth-quadrant angle. For the second-quadrant angle, $\sin\theta = 12/\sqrt{193}$ and $\cos\theta = -7/\sqrt{193}$. For the fourth-quadrant angle, $\sin\theta = -12/\sqrt{193}$ and $\cos\theta = 7/\sqrt{193}$.

Since $\dfrac{d^2}{d\theta^2}\left(\dfrac{dz}{ds}\right) = \dfrac{d}{d\theta}(-14\sin\theta - 24\cos\theta) = -14\cos\theta + 24\sin\theta$ is negative for the fourth-quadrant angle, the maximum directional derivative is $\dfrac{dz}{ds} = 14\left(\dfrac{7}{\sqrt{193}}\right) - 24\left(-\dfrac{12}{\sqrt{193}}\right) = 2\sqrt{193}$, and the direction is $\theta = 300°15'$.

7. Find the maximum directional derivative for the function and point of Problem 3.

At $P^*(0,3)$ in the direction θ, $dz/ds = 3\cos\theta + \sin\theta$.

To find the value of θ for which $\dfrac{dz}{ds}$ is a maximum, set $\dfrac{d}{d\theta}\left(\dfrac{dz}{ds}\right) = -3\sin\theta + \cos\theta = 0$. Then $\tan\theta = \frac{1}{3}$ and θ is either a first- or third-quadrant angle.

Since $\dfrac{d^2}{d\theta^2}\left(\dfrac{dz}{ds}\right) = \dfrac{d}{d\theta}(-3\sin\theta + \cos\theta) = -3\cos\theta - \sin\theta$ is negative for the first-quadrant angle, the maximum directional derivative is $\dfrac{dz}{ds} = 3\dfrac{3}{\sqrt{10}} + \dfrac{1}{\sqrt{10}} = \sqrt{10}$, and the direction is $\theta = 18°26'$.

8. In Problem 5, show that V changes most rapidly along the set of radial lines through the origin.

At any point (x_1, y_1) in the direction θ, $\frac{dV}{ds} = \frac{x_1}{x_1^2 + y_1^2}\cos\theta + \frac{y_1}{x_1^2 + y_1^2}\sin\theta$. Now V changes most rapidly when $\frac{d}{d\theta}\left(\frac{dV}{ds}\right) = -\frac{x_1}{x_1^2 + y_1^2}\sin\theta + \frac{y_1}{x_1^2 + y_1^2}\cos\theta = 0$, and then $\tan\theta = \frac{y_1/(x_1^2 + y_1^2)}{x_1/(x_1^2 + y_1^2)} = \frac{y_1}{x_1}$. Thus, θ is the angle of inclination of the line joining the origin and the point (x_1, y_1).

9. Find the directional derivative of $F(x, y, z) = xy + 2xz - y^2 + z^2$ at the point $(1, -2, 1)$ along the curve $x = t$, $y = t - 3$, $z = t^2$ in the direction of increasing z.

A set of direction numbers of the tangent to the curve at $(1, -2, 1)$ is $[1, 1, 2]$; the direction cosines are $[1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6}]$. The directional derivative is

$$\frac{\partial F}{\partial x}\cos\alpha + \frac{\partial F}{\partial y}\cos\beta + \frac{\partial F}{\partial z}\cos\gamma = 0\frac{1}{\sqrt{6}} + 5\frac{1}{\sqrt{6}} + 4\frac{2}{\sqrt{6}} = \frac{13\sqrt{6}}{6}$$

10. Examine $f(x, y) = x^2 + y^2 - 4x + 6y + 25$ for maximum and minimum values.

The conditions $\partial f/\partial x = 2x - 4 = 0$ and $\partial f/\partial y = 2y + 6 = 0$ are satisfied when $x = 2$, $y = -3$.

Since $f(x, y) = (x^2 - 4x + 4) + (y^2 + 6y + 9) + 25 - 4 - 9 = (x - 2)^2 + (y + 3)^2 + 12$, it is evident that $f(2, -3) = 12$ is a minimum value of the function.

Geometrically, $(2, -3, 12)$ is the minimum point of the surface $z = x^2 + y^2 - 4x + 6y + 25$.

11. Examine $f(x, y) = x^3 + y^3 + 3xy$ for maximum and minimum values.

The conditions $\partial f/\partial x = 3(x^2 + y) = 0$ and $\partial f/\partial y = 3(y^2 + x) = 0$ are satisfied when $x = 0$, $y = 0$ and when $x = -1$, $y = -1$.

At $(0, 0)$, $\frac{\partial^2 f}{\partial x^2} = 6x = 0$, $\frac{\partial^2 f}{\partial x\,\partial y} = 3$, and $\frac{\partial^2 f}{\partial y^2} = 6y = 0$. Then $\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = 9 > 0$, and $(0, 0)$ yields neither a maximum nor minimum.

At $(-1, -1)$, $\frac{\partial^2 f}{\partial x^2} = -6$, $\frac{\partial^2 f}{\partial x\,\partial y} = 3$, and $\frac{\partial^2 f}{\partial y^2} = -6$. Then $\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = -27 < 0$, and $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} < 0$. Hence, $f(-1, -1) = 1$ is the maximum value of the function.

12. Divide 120 into three parts such that the sum of their products taken two at a time is a maximum.

Let x, y, and $120 - (x + y)$ be the three parts. The function to be maximized is $S = xy + (x + y)(120 - x - y)$, and

$$\frac{\partial S}{\partial x} = y + (120 - x - y) - (x + y) = 120 - 2x - y \qquad \frac{\partial S}{\partial y} = x + (120 - x - y) - (x + y) = 120 - x - 2y$$

Setting $\frac{\partial S}{\partial x} = \frac{\partial S}{\partial y} = 0$ yields $2x + y = 120$ and $x + 2y = 120$. Simultaneous solution gives $x = 40$, $y = 40$, and $120 - (x + y) = 40$ as the three parts, and $S = 3(40^2) = 4800$. For $x = y = 1$, $S = 237$; hence, $S = 4800$ is the maximum value.

13. Find the point in the plane $2x - y + 2z = 16$ nearest the origin.

Let (x, y, z) be the required point; then the square of its distance from the origin is $D = x^2 + y^2 + z^2$. Since also $2x - y + 2z = 16$, we have $y = 2x + 2z - 16$ and $D = x^2 + (2x + 2z - 16)^2 + z^2$.

Then the conditions $\partial D/\partial x = 2x + 4(2x + 2z - 16) = 0$ and $\partial D/\partial z = 4(2x + 2z - 16) + 2z = 0$ are equivalent to $5x + 4z = 32$ and $4x + 5z = 32$, and $x = z = \frac{32}{9}$. Since it is known that a point for which D is a minimum exists, $(\frac{32}{9}, -\frac{16}{9}, \frac{32}{9})$ is that point.

14. Show that a rectangular parallelepiped of maximum volume V with constant surface area S is a cube.

Let the dimensions be x, y, and z. Then $V = xyz$ and $S = 2(xy + yz + zx)$.

The second relation may be solved for z and substituted in the first, to express V as a function of x and y. We prefer to avoid this step by simply treating z as a function of x and y. Then

$$\frac{\partial V}{\partial x} = yz + xy\frac{\partial z}{\partial x} \qquad \frac{\partial V}{\partial y} = xz + xy\frac{\partial z}{\partial y}$$

$$\frac{\partial S}{\partial x} = 0 = 2\left(y + z + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x}\right) \qquad \frac{\partial S}{\partial y} = 0 = 2\left(x + z + x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial y}\right)$$

From the latter two equations, $\frac{\partial z}{\partial x} = -\frac{y+z}{x+y}$ and $\frac{\partial z}{\partial y} = -\frac{x+z}{x+y}$. Substituting in the first two yields the conditions $\frac{\partial V}{\partial x} = yz - \frac{xy(y+z)}{x+y} = 0$ and $\frac{\partial V}{\partial y} = xz - \frac{xy(x+z)}{x+y} = 0$, which reduce to $y^2(z-x) = 0$ and $x^2(z-y) = 0$. Thus $x = y = z$, as required.

15. Find the volume V of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Let $P(x, y, z)$ be the vertex in the first octant. Then $V = 8xyz$. Consider z to be defined as a function of the independent variables x and y by the equation of the ellipsoid. The necessary conditions for a maximum are

$$\frac{\partial V}{\partial x} = 8\left(yz + xy\frac{\partial z}{\partial x}\right) = 0 \quad \text{and} \quad \frac{\partial V}{\partial y} = 8\left(xz + xy\frac{\partial z}{\partial y}\right) = 0 \tag{1}$$

From the equation of the ellipsoid, obtain $\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0$ and $\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0$. Eliminate $\partial z/\partial x$ and $\partial z/\partial y$ between these relations and (*1*) to obtain

$$\frac{\partial V}{\partial x} = 8\left(yz - \frac{c^2x^2y}{a^2z}\right) = 0 \quad \text{and} \quad \frac{\partial V}{\partial y} = 8\left(xz - \frac{c^2xy^2}{b^2z}\right) = 0$$

and, finally,

$$\frac{x^2}{a^2} = \frac{z^2}{c^2} = \frac{y^2}{b^2} \tag{2}$$

Combine (*2*) with the equation of the ellipsoid to get $x = a\sqrt{3}/3$, $y = b\sqrt{3}/3$, and $z = c\sqrt{3}/3$. Then $V = 8xyz = (8\sqrt{3}/9)abc$ cubic units.

Supplementary Problems

16. Find the directional derivative of the given function at the given point in the indicated direction:
(*a*) $z = x^2 + xy + y^2$, $(3, 1)$, $\theta = \pi/3$ (*b*) $z = x^3 + y^3 - 3xy$, $(2, 1)$, $\theta = \arctan 2/3$
(*c*) $z = y + x\cos xy$, $(0, 0)$, $\theta = \pi/3$ (*d*) $z = 2x^2 + 3xy - y^2$, $(1, -1)$, toward $(2, 1)$

Ans. (*a*) $\frac{1}{2}(7 + 5\sqrt{3})$; (*b*) $21\sqrt{13}/13$; (*c*) $\frac{1}{2}(1 + \sqrt{3})$; (*d*) $11\sqrt{5}/5$

17. Find the maximum directional derivative for each of the functions of Problem 16 at the given point.

Ans. (*a*) $\sqrt{74}$; (*b*) $3\sqrt{10}$; (*c*) $\sqrt{2}$; (*d*) $\sqrt{26}$

18. Show that the maximum directional derivative of $V = \ln\sqrt{x^2 + y^2}$ of Problem 8 is constant along any circle $x^2 + y^2 = r^2$.

19. On a hill represented by $z = 8 - 4x^2 - 2y^2$, find (*a*) the direction of the steepest grade at $(1, 1, 2)$ and (*b*) the direction of the contour line (direction for which z = constant). Note that the directions are mutually perpendicular. *Ans.* (*a*) $\arctan\frac{1}{2}$, third quadrant; (*b*) $\arctan -2$

20. Show that the sum of the squares of the directional derivatives of $z = f(x, y)$ at any of its points is constant for any two mutually perpendicular directions and is equal to the square of the maximum directional derivative.

21. Given $z = f(x, y)$ and $w = g(x, y)$ such that $\partial z/\partial x = \partial w/\partial y$ and $\partial z/\partial y = -\partial w/\partial x$. If θ_1 and θ_2 are two mutually perpendicular directions, show that at any point $P(x, y)$, $\partial z/\partial s_1 = \partial w/\partial s_2$ and $\partial z/\partial s_2 = -\partial w/\partial s_1$.

22. Find the directional derivative of the given function at the given point in the indicated direction:
(*a*) xy^2z, $(2, 1, 3)$, $[1, -2, 2]$
(*b*) $x^2 + y^2 + z^2$, $(1, 1, 1)$, toward $(2, 3, 4)$
(*c*) $x^2 + y^2 - 2xz$, $(1, 3, 2)$, along $x^2 + y^2 - 2xz = 6$, $3x^2 - y^2 + 3z = 0$ in the direction of increasing z

Ans. (*a*) $-\frac{17}{3}$; (*b*) $6\sqrt{14}/7$; (*c*) 0

23. Examine each of the following functions for relative maximum and minimum values.

(*a*) $z = 2x + 4y - x^2 - y^2 - 3$ *Ans.* maximum $= 2$ when $x = 1$, $y = 2$
(*b*) $z = x^3 + y^3 - 3xy$ *Ans.* minimum $= -1$ when $x = 1$, $y = 1$
(*c*) $z = x^2 + 2xy + 2y^2$ *Ans.* minimum $= 0$ when $x = 0$, $y = 0$
(*d*) $z = (x - y)(1 - xy)$ *Ans.* neither maximum nor minimum
(*e*) $z = 2x^2 + y^2 + 6xy + 10x - 6y + 5$ *Ans.* neither maximum nor minimum
(*f*) $z = 3x - 3y - 2x^3 - xy^2 + 2x^2y + y^3$ *Ans.* minimum $= -\sqrt{6}$ when $x = -\sqrt{6}/6$, $y = \sqrt{6}/3$; maximum $= \sqrt{6}$ when $x = \sqrt{6}/6$, $y = -\sqrt{6}/3$
(*g*) $z = xy(2x + 4y + 1)$ *Ans.* maximum $= \frac{1}{216}$ when $x = -\frac{1}{6}$, $y = -\frac{1}{12}$

24. Find positive numbers x, y, z such that
(*a*) $x + y + z = 18$ and xyz is a maximum
(*b*) $xyz = 27$ and $x + y + z$ is a minimum
(*c*) $x + y + z = 20$ and xyz^2 is a maximum
(*d*) $x + y + z = 12$ and xy^2z^3 is a maximum

Ans. (*a*) $x = y = z = 6$; (*b*) $x = y = z = 3$; (*c*) $x = y = 5$, $z = 10$; (*d*) $x = 2$, $y = 4$, $z = 6$

25. Find the minimum value of the square of the distance from the origin to the plane $Ax + By + Cz + D = 0$. *Ans.* $D^2/(A^2 + B^2 + C^2)$

26. (*a*) The surface area of a rectangular box without a top is to be 108 ft^2. Find the greatest possible volume. (*b*) The volume of a rectangular box without a top is to be 500 ft^3. Find the minimum surface area. *Ans.* (*a*) 108 ft^3; (*b*) 300 ft^2

27. Find the point on $z = xy - 1$ nearest the origin. *Ans.* $(0, 0, -1)$

28. Find the equation of the plane through $(1, 1, 2)$ that cuts off the least volume in the first octant.

Ans. $2x + 2y + z = 6$

29. Determine the values of p and q so that the sum S of the squares of the vertical distances of the points $(0, 2)$, $(1, 3)$, and $(2, 5)$ from the line $y = px + q$ is a minimum. (*Hint*: $S = (q - 2)^2 + (p + q - 3)^2 + (2p + q - 5)^2$.) *Ans.* $p = \frac{3}{2}$; $q = \frac{11}{6}$