Chapter 69

Double and Iterated Integrals

THE (SIMPLE) INTEGRAL $\int_a^b f(x)\,dx$ of a function $y = f(x)$ that is continuous over the finite interval $a \le x \le b$ of the x axis was defined in Chapter 38. Recall that

1. The interval $a \le x \le b$ was divided into n subintervals $h_1, h_2, \ldots, h_n$ of respective lengths $\Delta_1 x, \Delta_2 x, \ldots, \Delta_n x$ with λ_n the greatest of the $\Delta_k x$.
2. Points x_1 in h_1, x_2 in $h_2, \ldots, x_n$ in h_n were selected, and the sum $\sum_{k=1}^{n} f(x_k)\,\Delta_k x$ formed.
3. The interval was further subdivided in such a manner that $\lambda_n \to 0$ as $n \to +\infty$.
4. We defined $\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\,\Delta_k x$.

THE DOUBLE INTEGRAL. Consider a function $z = f(x, y)$ continuous over a finite region R of the xOy plane. Let this region be subdivided (see Fig. 69-1) into n subregions $R_1, R_2, \ldots, R_n$ of respective areas $\Delta_1 A, \Delta_2 A, \ldots, \Delta_n A$. In each subregion R_k, select a point $P_k(x_k, y_k)$ and form the sum

$$\sum_{k=1}^{n} f(x_k, y_k)\,\Delta_k A = f(x_1, y_1)\,\Delta_1 A + f(x_2, y_2)\,\Delta_2 A + \cdots + f(x_n, y_n)\,\Delta_n A \qquad (69.1)$$

Now, defining the diameter of a subregion to be the greatest distance between any two points within or on its boundary, and denoting by λ_n the maximum diameter of the subregions, suppose the number of subregions to be increased in such a manner that $\lambda_n \to 0$ as $n \to +\infty$. Then the *double integral* of the function $f(x, y)$ over the region R is defined as

$$\iint_R f(x, y)\,dA = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta_k A \qquad (69.2)$$

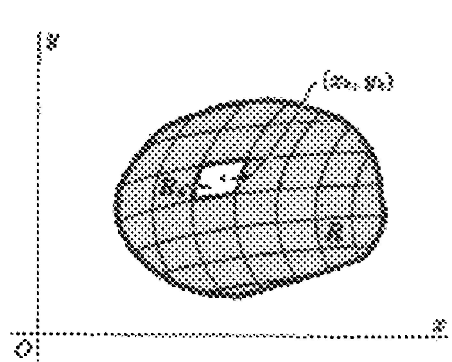

Fig. 69-1

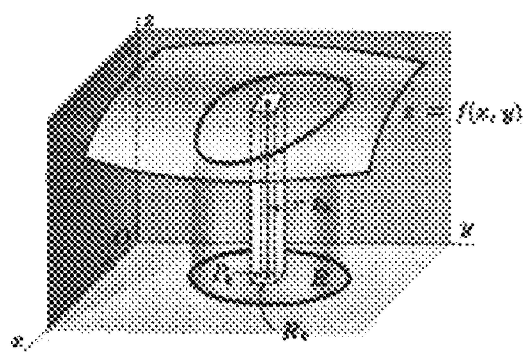

Fig. 69-2

When $z = f(x, y)$ is nonnegative over the region R, as in Fig. 69-2, the double integral (*69.2*) may be interpreted as a volume. Any term $f(x_k, y_k)\,\Delta_k A$ of (*69.1*) gives the volume of a vertical column whose parallel bases are of area $\Delta_k A$ and whose altitude is the distance z_k measured along the vertical from the selected point P_k to the surface $z = f(x, y)$. This, in turn, may be taken as an approximation of the volume of the vertical column whose lower base is the subregion R_k and whose upper base is the projection of R_k on the surface. Thus, (*69.1*) is an approximation of the volume "under the surface" (that is, the volume with lower base in the

xOy plane and upper base in the surface generated by moving a line parallel to the z axis along the boundary of R), and, intuitively, at least, (*69.2*) is the measure of this volume.

The evaluation of even the simplest double integral by direct summation is difficult and will not be attempted here.

THE ITERATED INTEGRAL. Consider a volume defined as above, and assume that the boundary of R is such that no line parallel to the x axis or to the y axis cuts it in more than two points. Draw (see Fig. 69-3) the tangents $x = a$ and $x = b$ to the boundary with points of tangency K and L, and the tangents $y = c$ and $y = d$ with points of tangency M and N. Let the equation of the plane arc LMK be $y = g_1(x)$, and that of the plane arc LNK be $y = g_2(x)$.

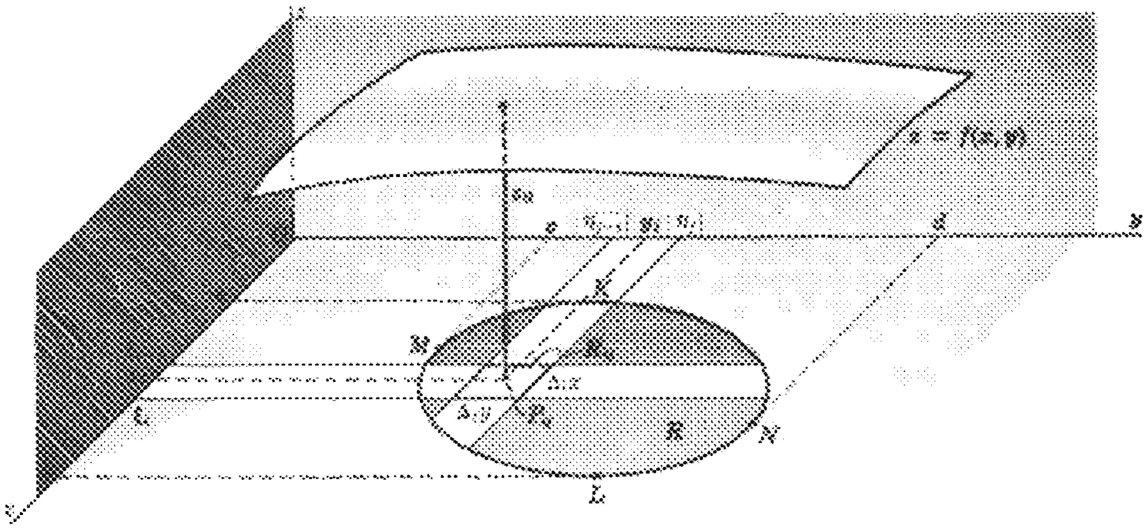

Fig. 69-3

Divide the interval $a \le x \le b$ into m subintervals $h_1, h_2, \ldots, h_m$ of respective lengths $\Delta_1 x$, $\Delta_2 x, \ldots, \Delta_m x$ by the insertion of points $x = \xi_1$, $x = \xi_2, \ldots, x = \xi_{m-1}$ (as in Chapter 38), and divide the interval $c \le y \le d$ into n subintervals $k_1, k_2, \ldots, k_n$ of respective lengths $\Delta_1 y$, $\Delta_2 y, \ldots, \Delta_n y$ by the insertion of points $y = \eta_1$, $y = \eta_2, \ldots, y = \eta_{n-1}$. Denote by λ_m the greatest $\Delta_i x$, and by μ_n the greatest $\Delta_j y$. Draw in the parallel lines $x = \xi_1, x = \xi_2, \ldots, x = \xi_{m-1}$ and the parallel lines $y = \eta_1$, $y = \eta_2, \ldots, y = \eta_{n-1}$, thus dividing the region R into a set of rectangles R_{ij} of areas $\Delta_i x\, \Delta_j y$ plus a set of nonrectangles that we shall ignore. On each subinterval h_i select a point $x = x_i$, and on each subinterval k_j select a point $y = y_j$, thereby determining in each subregion R_{ij} a point $P_{ij}(x_i, y_j)$. With each subregion R_{ij}, associate by means of the equation of the surface a number $z_{ij} = f(x_i, y_j)$, and form the sum

$$\sum_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}} f(x_i, y_j)\, \Delta_i x\, \Delta_j y \qquad (69.3)$$

Now (*69.3*) is merely a special case of (*69.1*), so if the number of rectangles is indefinitely increased in such a manner that both $\lambda_m \to 0$ and $\mu_n \to 0$, the limit of (*69.3*) should be equal to the double integral (*69.2*).

In effecting this limit, let us first choose one of the subintervals, say h_i, and form the sum

$$\left[\sum_{j=1}^{n} f(x_i, y_j)\, \Delta_j y\right] \Delta_i x \quad (i \text{ fixed})$$

of the contributions of all rectangles having h_i as one dimension, that is, the contributions of all rectangles lying in the ith column. When $n \to +\infty$, $\mu_n \to 0$ and

$$\lim_{n\to+\infty}\left[\sum_{j=1}^{n} f(x_i, y_j)\,\Delta_j y\right]\Delta_i x = \left[\int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y)\,dy\right]\Delta_i x = \phi(x_i)\,\Delta_i x$$

Now summing over the m columns and letting $m \to +\infty$, we have

$$\lim_{m\to+\infty}\sum_{i=1}^{m} \phi(x_i)\,\Delta_i x = \int_a^b \phi(x)\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\right]dx$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx \qquad (69.4)$$

Although we shall not use the brackets hereafter, it must be clearly understood that (*69.4*) calls for the evaluation of two simple definite integrals in a prescribed order: first, the integral of $f(x, y)$ with respect to y (considering x as a constant) from $y = g_1(x)$, the lower boundary of R, to $y = g_2(x)$, the upper boundary of R, and then the integral of this result with respect to x from the abscissa $x = a$ of the leftmost point of R to the abscissa $x = b$ of the rightmost point of R. The integral (*69.4*) is called an *iterated* or *repeated integral*.

It will be left as an exercise to sum first for the contributions of the rectangles lying in each row and then over all the rows to obtain the equivalent iterated integral

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy \qquad (69.5)$$

where $x = h_1(y)$ and $x = h_2(y)$ are the equations of the plane arcs MKN and MLN, respectively.

In Problem 1 it is shown by a different procedure that the iterated integral (*69.4*) measures the volume under discussion. For the evaluation of iterated integrals see Problems 2 to 6.

The principal difficulty in setting up the iterated integrals of the next several chapters will be that of inserting the limits of integration to cover the region R. The discussion here assumed the simplest of regions; more complex regions are considered in Problems 7 to 9.

Solved Problems

1. Let $z = f(x, y)$ be nonnegative and continuous over the region R of the plane xOy whose boundary consists of the arcs of two curves $y = g_1(x)$ and $y = g_2(x)$ intersecting in the points K and L, as in Fig. 69-4. Find a formula for the volume V under the surface $z = f(x, y)$.

Let the section of this volume cut by a plane $x = x_i$, where $a < x_i < b$, meet the boundary of R in the points $S(x_i, g_1(x_i))$ and $T(x_i, g_2(x_i))$, and the surface $z = f(x, y)$ in the arc UV along which $z = f(x_i, y)$. The area of this section $STUV$ is given by

$$A(x_i) = \int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y)\,dy$$

Thus, the areas of cross sections of the volume cut by planes parallel to the yOz plane are known functions $A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy$ of x, where x is the distance of the sectioning plane from the origin. By Chapter 42, the required volume is given by

$$V = \int_a^b A(x)\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\right]dx$$

This is the iterated integral of (*69.4*).

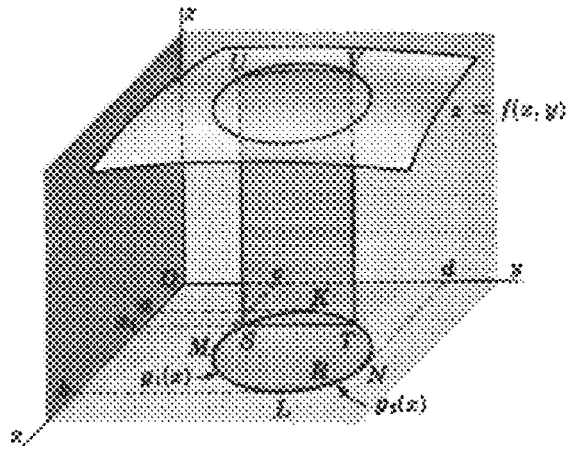

Fig. 69-4

In Problems 2 to 6, evaluate the integral at the left.

2. $$\int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 [y]_{x^2}^{x}\,dx = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

3. $$\int_1^2 \int_y^{3y} (x + y)\,dx\,dy = \int_1^2 [\tfrac{1}{2}x^2 + xy]_y^{3y}\,dy = \int_1^2 6y^2\,dy = [2y^3]_1^2 = 14$$

4. $$\int_{-1}^2 \int_{2x^2-2}^{x^2+x} x\,dy\,dx = \int_{-1}^2 [xy]_{2x^2-2}^{x^2+x}\,dx = \int_{-1}^2 (x^3 + x^2 - 2x^3 + 2x)\,dx = \tfrac{9}{4}$$

5. $$\int_0^\pi \int_0^{\cos\theta} \rho \sin\theta\,d\rho\,d\theta = \int_0^\pi [\tfrac{1}{2}\rho^2 \sin\theta]_0^{\cos\theta}\,d\theta = \tfrac{1}{2}\int_0^\pi \cos^2\theta \sin\theta\,d\theta = [-\tfrac{1}{6}\cos^3\theta]_0^\pi = \tfrac{1}{3}$$

6. $$\int_0^{\pi/2} \int_2^{4\cos\theta} \rho^3\,d\rho\,d\theta = \int_0^{\pi/2} \left[\frac{1}{4}\rho^4\right]_2^{4\cos\theta} d\theta = \int_0^{\pi/2} (64\cos^4\theta - 4)\,d\theta$$

$$= \left[64\left(\frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\right) - 4\theta\right]_0^{\pi/2} = 10\pi$$

7. Evaluate $\iint_R dA$, where R is the region in the first quadrant bounded by the semicubical parabola $y^2 = x^3$ and the line $y = x$.

The line and parabola intersect in the points $(0, 0)$ and $(1, 1)$ which establish the extreme values of x and y on the region R.

Solution 1 (Fig. 69-5): Integrating first over a horizontal strip, that is, with respect to x from $x = y$ (the line) to $x = y^{2/3}$ (the parabola), and then with respect to y from $y = 0$ to $y = 1$, we get

$$\iint_R dA = \int_0^1 \int_y^{y^{2/3}} dx\,dy = \int_0^1 (y^{2/3} - y)\,dy = [\tfrac{3}{5}y^{5/3} - \tfrac{1}{2}y^2]_0^1 = \tfrac{1}{10}$$

Solution 2 (Fig. 69-6): Integrating first over a vertical strip, that is, with respect to y from $y = x^{3/2}$ (the parabola) to $y = x$ (the line), and then with respect to x from $x = 0$ to $x = 1$, we obtain

$$\iint_R dA = \int_0^1 \int_{x^{3/2}}^{x} dy\,dx = \int_0^1 (x - x^{3/2})\,dx = [\tfrac{1}{2}x^2 - \tfrac{2}{5}x^{5/2}]_0^1 = \tfrac{1}{10}$$

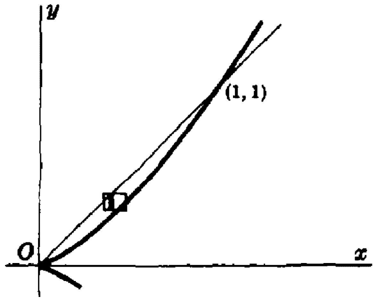

Fig. 69-5

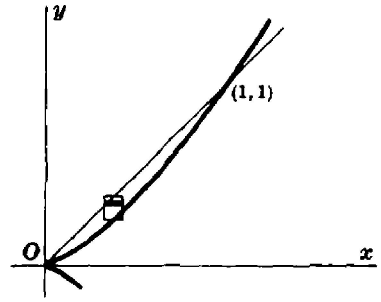

Fig. 69-6

8. Evaluate $\iint_R dA$ where R is the region between $y = 2x$ and $y = x^2$ lying to the left of $x = 1$.

Integrating first over the vertical strip (see Fig. 69-7), we have

$$\iint_R dA = \int_0^1 \int_{x^2}^{2x} dy\,dx = \int_0^1 (2x - x^2)\,dx = \tfrac{2}{3}$$

When horizontal strips are used (see Fig. 69-8), two iterated integrals are necessary. Let R_1 denote the part of R lying below AB, and R_2 the part above AB. Then

$$\iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{y/2}^{\sqrt{y}} dx\,dy + \int_1^2 \int_{y/2}^{1} dx\,dy = \tfrac{5}{12} + \tfrac{1}{4} = \tfrac{2}{3}$$

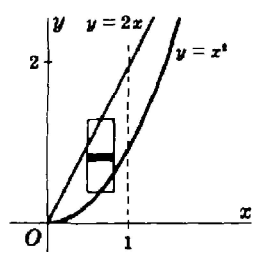

Fig. 69-7

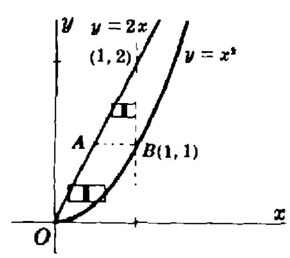

Fig. 69-8

9. Evaluate $\iint_R x^2\,dA$ where R is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$, and $x = 8$. (See Fig. 69-9.)

It is evident from Fig. 69-9 that R must be separated into two regions, and an iterated integral evaluated for each. Let R_1 denote the part of R lying above the line $y = 2$, and R_2 the part below that line. Then

$$\iint_R x^2\,dA = \iint_{R_1} x^2\,dA + \iint_{R_2} x^2\,dA = \int_2^4 \int_y^{16/y} x^2\,dx\,dy + \int_0^2 \int_y^8 x^2\,dx\,dy$$

$$= \frac{1}{3}\int_2^4 \left(\frac{16^3}{y^3} - y^3\right) dy + \frac{1}{3}\int_0^2 (8^3 - y^3)\,dy = 448$$

As an exercise, you might separate R with the line $x = 4$ and obtain

$$\iint_R x^2\,dA = \int_0^4 \int_0^x x^2\,dy\,dx + \int_4^8 \int_0^{16/x} x^2\,dy\,dx$$

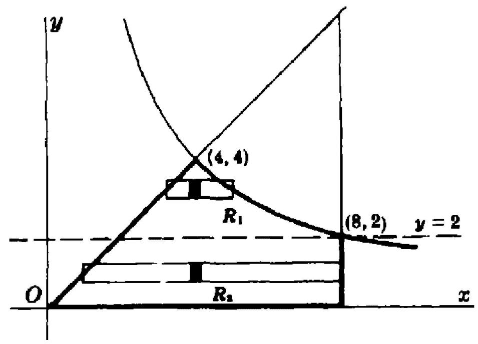

Fig. 69-9

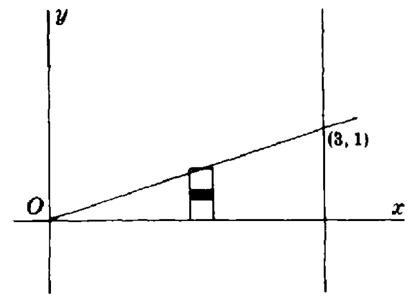

Fig. 69-10

10. Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$ by first reversing the order of integration.

The given integral cannot be evaluated directly, since $\int e^{x^2}\, dx$ is not an elementary function. The region R of integration (see Fig. 69-10) is bounded by the lines $x = 3y$, $x = 3$, and $y = 0$. To reverse the order of integration, first integrate with respect to y from $y = 0$ to $y = x/3$, and then with respect to x from $x = 0$ to $x = 3$. Thus,

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx = \int_0^3 [e^{x^2} y]_0^{x/3}\, dx$$

$$= \frac{1}{3} \int_0^3 e^{x^2} x\, dx = [\tfrac{1}{6} e^{x^2}]_0^3 = \tfrac{1}{6}(e^9 - 1)$$

Supplementary Problems

11. Evaluate the iterated integral at the left:

(a) $\int_0^1 \int_1^2 dx\, dy = 1$

(b) $\int_1^2 \int_0^3 (x + y)\, dx\, dy = 9$

(c) $\int_2^4 \int_1^2 (x^2 + y^2)\, dy\, dx = \frac{70}{3}$

(d) $\int_0^1 \int_{x^2}^x xy^2\, dy\, dx = \frac{1}{40}$

(e) $\int_1^2 \int_0^{y^{3/2}} x/y^2\, dx\, dy = \frac{3}{4}$

(f) $\int_0^1 \int_x^{\sqrt{x}} (y + y^3)\, dy\, dx = \frac{7}{60}$

(g) $\int_0^1 \int_0^{x^2} xe^y\, dy\, dx = \frac{1}{2}e - 1$

(h) $\int_2^4 \int_y^{8-y} y\, dx\, dy = \frac{32}{3}$

(i) $\int_0^{\arctan 3/2} \int_0^{2 \sec\theta} \rho\, d\rho\, d\theta = 3$

(j) $\int_0^{\pi/2} \int_0^2 \rho^2 \cos\theta\, d\rho\, d\theta = \frac{8}{3}$

(k) $\int_0^{\pi/4} \int_0^{\tan\theta \sec\theta} \rho^3 \cos^2\theta\, d\rho\, d\theta = \frac{1}{20}$

(l) $\int_0^{2\pi} \int_0^{1-\cos\theta} \rho^3 \cos^2\theta\, d\rho\, d\theta = \frac{49}{32}\pi$

12. Using an iterated integral, evaluate each of the following double integrals. When feasible, evaluate the iterated integral in both orders.

(*a*) x over the region bounded by $y = x^2$ and $y = x^3$ *Ans.* $\frac{1}{20}$

(*b*) y over the region of part (*a*) *Ans.* $\frac{1}{35}$

(*c*) x^2 over the region bounded by $y = x$, $y = 2x$, and $x = 2$ *Ans.* 4

(*d*) 1 over each first-quadrant region bounded by $2y = x^2$, $y = 3x$, and $x + y = 4$ *Ans.* $\frac{8}{3}$; $\frac{46}{3}$

(*e*) y over the region above $y = 0$ bounded by $y^2 = 4x$ and $y^2 = 5 - x$ *Ans.* 5

(*f*) $\dfrac{1}{\sqrt{2y - y^2}}$ over the region in the first quadrant bounded by $x^2 = 4 - 2y$ *Ans.* 4

13. In Problem 11(*a*) to (*h*), reverse the order of integration and evaluate the resulting iterated integral.

Chapter 70

Centroids and Moments of Inertia of Plane Areas

PLANE AREA BY DOUBLE INTEGRATION. If $f(x, y) = 1$, the double integral of Chapter 69 becomes $\iint_R dA$. In cubic units, this measures the volume of a cylinder of unit height; in square units, it measures the area of the region R. (See Problems 1 and 2.)

In polar coordinates, $A = \iint_R dA = \int_\alpha^\beta \int_{\rho_1(\theta)}^{\rho_2(\theta)} \rho\, d\rho\, d\theta$, where $\theta = \alpha$, $\theta = \beta$, $\rho_1(\theta)$, and $\rho_2(\theta)$ are chosen to cover the region R. (See Problems 3 to 5.)

CENTROIDS. The coordinates $(\bar{x}, \bar{y})$ of the centroid of a plane region R of area $A = \iint_R dA$ satisfy the relations

$$A\bar{x} = M_y \qquad \text{and} \qquad A\bar{y} = M_x$$

or

$$\bar{x}\iint_R dA = \iint_R x\, dA \qquad \text{and} \qquad \bar{y}\iint_R dA = \iint_R y\, dA$$

(See Problems 6 to 9.)

THE MOMENTS OF INERTIA of a plane region R with respect to the coordinate axes are given by

$$I_x = \iint_R y^2\, dA \qquad \text{and} \qquad I_y = \iint_R x^2\, dA$$

The polar moment of inertia (the moment of inertia with respect to a line through the origin and perpendicular to the plane of the area) of a plane region R is given by

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2)\, dA$$

(See Problems 10 to 12.)

Solved Problems

1. Find the area bounded by the parabola $y = x^2$ and the line $y = 2x + 3$.

Using vertical strips (see Fig. 70-1), we have

$$A = \int_{-1}^{3} \int_{x^2}^{2x+3} dy\, dx = \int_{-1}^{3} (2x + 3 - x^2)\, dx = 32/3 \text{ square units}$$

2. Find the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = 4 - 4x$.

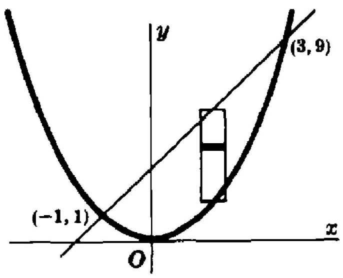

Fig. 70-1

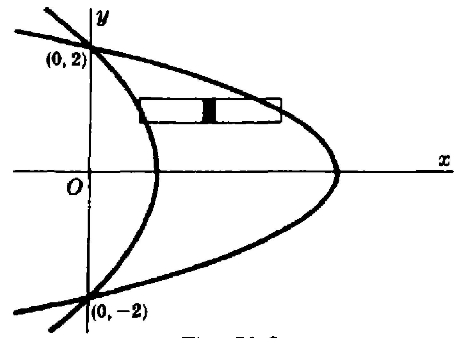

Fig. 70-2

Using horizontal strips (Fig. 70-2) and taking advantage of symmetry, we have

$$A = 2\int_0^2 \int_{1-y^2/4}^{4-y^2} dx\,dy = 2\int_0^2 [(4-y^2)-(1-\tfrac{1}{4}y^2)]\,dy$$
$$= 6\int_0^2 (1-\tfrac{1}{4}y^2)\,dy = 8 \text{ square units}$$

3. Find the area outside the circle $\rho = 2$ and inside the cardioid $\rho = 2(1+\cos\theta)$.

Owing to symmetry (see Fig. 70-3), the required area is twice that swept over as θ varies from $\theta = 0$ to $\theta = \frac{1}{2}\pi$. Thus,

$$A = 2\int_0^{\pi/2}\int_2^{2(1+\cos\theta)} \rho\,d\rho\,d\theta = 2\int_0^{\pi/2} [\tfrac{1}{2}\rho^2]_2^{2(1+\cos\theta)}\,d\theta = 4\int_0^{\pi/2} (2\cos\theta + \cos^2\theta)\,d\theta$$
$$= 4[2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta]_0^{\pi/2} = (\pi + 8) \text{ square units}$$

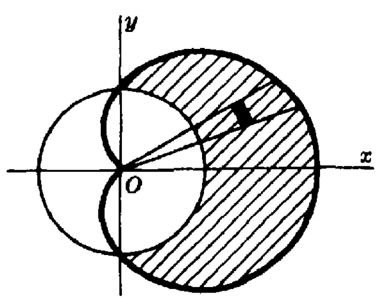

Fig. 70-3

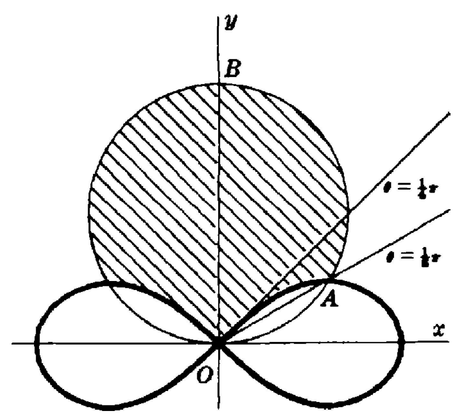

Fig. 70-4

4. Find the area inside the circle $\rho = 4\sin\theta$ and outside the lemniscate $\rho^2 = 8\cos 2\theta$.

The required area is twice that in the first quadrant bounded by the two curves and the line $\theta = \frac{1}{2}\pi$. Note in Fig. 70-4 that the arc AO of the lemniscate is described as θ varies from $\theta = \pi/6$ to $\theta = \pi/4$, while the arc AB of the circle is described as θ varies from $\theta = \pi/6$ to $\theta = \pi/2$. This area must then be considered as two regions, one below and one above the line $\theta = \pi/4$. Thus,

$$A = 2\int_{\pi/6}^{\pi/4}\int_{2\sqrt{2\cos 2\theta}}^{4\sin\theta} \rho\,d\rho\,d\theta + 2\int_{\pi/4}^{\pi/2}\int_0^{4\sin\theta} \rho\,d\rho\,d\theta$$
$$= \int_{\pi/6}^{\pi/4} (16\sin^2\theta - 8\cos 2\theta)\,d\theta + \int_{\pi/4}^{\pi/2} 16\sin^2\theta\,d\theta$$
$$= (\tfrac{8}{3}\pi + 4\sqrt{3} - 4) \text{ square units}$$

5. Evaluate $N = \int_0^{+\infty} e^{-x^2}\,dx$. (See Fig. 70-5.)

Since $\int_0^{+\infty} e^{-x^2}\,dx = \int_0^{+\infty} e^{-y^2}\,dy$, we have

$$N^2 = \int_0^{+\infty} e^{-x^2}\,dx \int_0^{+\infty} e^{-y^2}\,dy = \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}\,dx\,dy = \iint_R e^{-(x^2+y^2)}\,dA$$

Changing to polar coordinates ($x^2 + y^2 = \rho^2$, $dA = \rho\,d\rho\,d\theta$) yields

$$N^2 = \int_0^{\pi/2}\int_0^{+\infty} e^{-\rho^2}\rho\,d\rho\,d\theta = \int_0^{\pi/2} \lim_{a\to+\infty}\left[-\frac{1}{2}e^{-\rho^2}\right]_0^a d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

and $N = \sqrt{\pi}/2$.

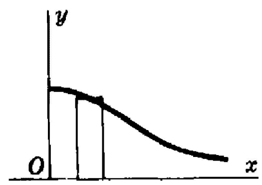

Fig. 70-5

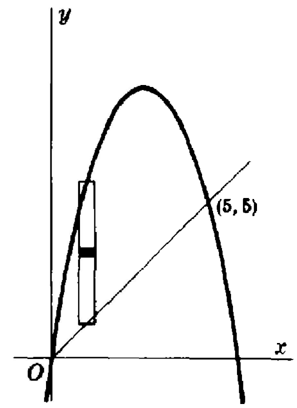

Fig. 70-6

6. Find the centroid of the plane area bounded by the parabola $y = 6x - x^2$ and the line $y = x$. (See Fig. 70-6.)

$$A = \iint_R dA = \int_0^5\int_x^{6x-x^2} dy\,dx = \int_0^5 (5x - x^2)\,dx = \tfrac{125}{6}$$

$$M_y = \iint_R x\,dA = \int_0^5\int_x^{6x-x^2} x\,dy\,dx = \int_0^5 (5x^2 - x^3)\,dx = \tfrac{625}{12}$$

$$M_x = \iint_R y\,dA = \int_0^5\int_x^{6x-x^2} y\,dy\,dx = \tfrac{1}{2}\int_0^5 [(6x - x^2)^2 - x^2]\,dx = \tfrac{625}{6}$$

Hence, $\bar{x} = M_y/A = \frac{5}{2}$, $\bar{y} = M_x/A = 5$, and the coordinates of the centroid are $(\frac{5}{2}, 5)$.

7. Find the centroid of the plane area bounded by the parabolas $y = 2x - x^2$ and $y = 3x^2 - 6x$. (See Fig. 70-7.)

$$A = \iint_R dA = \int_0^2\int_{3x^2-6x}^{2x-x^2} dy\,dx = \int_0^2 (8x - 4x^2)\,dx = \tfrac{16}{3}$$

$$M_y = \iint_R x\,dA = \int_0^2\int_{3x^2-6x}^{2x-x^2} x\,dy\,dx = \int_0^2 (8x^2 - 4x^3)\,dx = \tfrac{16}{3}$$

$$M_x = \iint_R y\,dA = \int_0^2\int_{3x^2-6x}^{2x-x^2} y\,dy\,dx = \tfrac{1}{2}\int_0^2 [(2x - x^2)^2 - (3x^2 - 6x)^2]\,dx = -\tfrac{64}{15}$$

Hence, $\bar{x} = M_y/A = 1$, $\bar{y} = M_x/A = -\frac{4}{5}$, and the centroid is $(1, -\frac{4}{5})$.

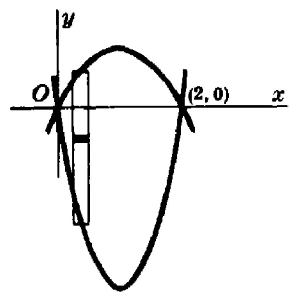

Fig. 70-7

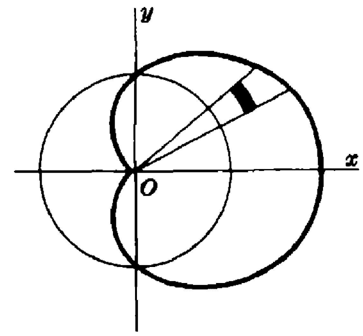

Fig. 70-8

8. Find the centroid of the plane area outside the circle $\rho = 1$ and inside the cardioid $\rho = 1 + \cos\theta$.

From Fig. 70-8 it is evident that $\bar{y} = 0$ and that $\bar{x}$ is the same whether computed for the given area or for the half lying above the polar axis. For the latter area,

$$A = \iint_R dA = \int_0^{\pi/2}\int_1^{1+\cos\theta} \rho\, d\rho\, d\theta = \frac{1}{2}\int_0^{\pi/2} [(1+\cos\theta)^2 - 1^2]\, d\theta = \frac{\pi+8}{8}$$

$$M_y = \iint_R x\, dA = \int_0^{\pi/2}\int_1^{1+\cos\theta} (\rho\cos\theta)\rho\, d\rho\, d\theta = \frac{1}{3}\int_0^{\pi/2} (3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)\, d\theta$$

$$= \frac{1}{3}\left[\frac{3}{2}\theta + \frac{3}{4}\sin 2\theta + 3\sin\theta - \sin^3\theta + \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right]_0^{\pi/2} = \frac{15\pi + 32}{48}$$

The coordinates of the centroid are $\left(\dfrac{15\pi + 32}{6(\pi + 8)}, 0\right)$.

9. Find the centroid of the area inside $\rho = \sin\theta$ and outside $\rho = 1 - \cos\theta$. (See Fig. 70-9.)

$$A = \iint_R dA = \int_0^{\pi/2}\int_{1-\cos\theta}^{\sin\theta} \rho\, d\rho\, d\theta = \frac{1}{2}\int_0^{\pi/2} (2\cos\theta - 1 - \cos 2\theta)\, d\theta = \frac{4-\pi}{4}$$

$$M_y = \iint_R x\, dA = \int_0^{\pi/2}\int_{1-\cos\theta}^{\sin\theta} (\rho\cos\theta)\rho\, d\rho\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} (\sin^3\theta - 1 + 3\cos\theta - 3\cos^2\theta + \cos^3\theta)\cos\theta\, d\theta = \frac{15\pi - 44}{48}$$

$$M_x = \iint_R y\, dA = \int_0^{\pi/2}\int_{1-\cos\theta}^{\sin\theta} (\rho\sin\theta)\rho\, d\rho\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} (\sin^3\theta - 1 + 3\cos\theta - 3\cos^2\theta + \cos^3\theta)\sin\theta\, d\theta = \frac{3\pi - 4}{48}$$

The coordinates of the centroid are $\left(\dfrac{15\pi - 44}{12(4-\pi)}, \dfrac{3\pi - 4}{12(4-\pi)}\right)$.

10. Find I_x, I_y, and I_0 for the area enclosed by the loop of $y^2 = x^2(2-x)$. (See Fig. 70-10.)

$$A = \iint_R dA = 2\int_0^2\int_0^{x\sqrt{2-x}} dy\, dx = 2\int_0^2 x\sqrt{2-x}\, dx$$

$$= -4\int_{\sqrt{2}}^0 (2z^2 - z^4)\, dz = -4\left[\frac{2}{3}z^3 - \frac{1}{5}z^5\right]_{\sqrt{2}}^0 = \frac{32\sqrt{2}}{15}$$

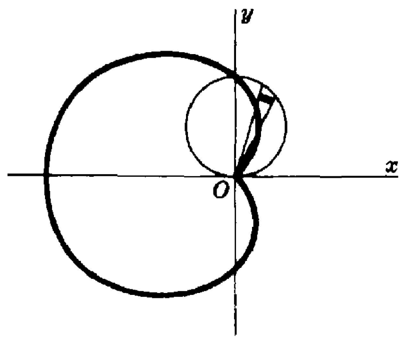

Fig. 70-9

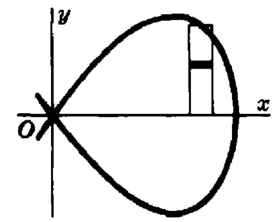

Fig. 70-10

where we have used the transformation $2 - x = z^2$. Then

$$I_x = \iint_R y^2\,dA = 2\int_0^2 \int_0^{x\sqrt{2-x}} y^2\,dy\,dx = \frac{2}{3}\int_0^2 x^3(2-x)^{3/2}\,dx$$

$$= -\frac{4}{3}\int_{\sqrt{2}}^0 (2-z^2)^3 z^4\,dz = -\frac{4}{3}\left[\frac{8}{5}z^5 - \frac{12}{7}z^7 + \frac{2}{3}z^9 - \frac{1}{11}z^{11}\right]_{\sqrt{2}}^0 = \frac{2048\sqrt{2}}{3465} = \frac{64}{231}A$$

$$I_y = \iint_R x^2\,dA = 2\int_0^2 \int_0^{x\sqrt{2-x}} x^2\,dy\,dx = 2\int_0^2 x^3\sqrt{2-x}\,dx$$

$$= -4\int_{\sqrt{2}}^0 (2-z^2)^3 z^2\,dz = -4\left[\frac{8}{3}z^3 - \frac{12}{5}z^5 + \frac{6}{7}z^7 - \frac{1}{9}z^9\right]_{\sqrt{2}}^0 = \frac{1024\sqrt{2}}{315} = \frac{32}{21}A$$

$$I_0 = I_x + I_y = \frac{13\,312\sqrt{2}}{3465} = \frac{416}{231}A$$

11. Find I_x, I_y, and I_0 for the first-quadrant area outside the circle $\rho = 2a$ and inside the circle $\rho = 4a\cos\theta$. (See Fig. 70-11.)

$$A = \iint_R dA = \int_0^{\pi/3}\int_{2a}^{4a\cos\theta} \rho\,d\rho\,d\theta = \frac{1}{2}\int_0^{\pi/3} [(4a\cos\theta)^2 - (2a)^2]\,d\theta = \frac{2\pi + 3\sqrt{3}}{3}a^2$$

$$I_x = \iint_R y^2\,dA = \int_0^{\pi/3}\int_{2a}^{4a\cos\theta} (\rho\sin\theta)^2\rho\,d\rho\,d\theta = \frac{1}{4}\int_0^{\pi/3} \{(4a\cos\theta)^4 - (2a)^4\}\sin^2\theta\,d\theta$$

$$= 4a^4\int_0^{\pi/3} (16\cos^4\theta - 1)\sin^2\theta\,d\theta = \frac{4\pi + 9\sqrt{3}}{6}a^4 = \frac{4\pi + 9\sqrt{3}}{2(2\pi + 3\sqrt{3})}a^2A$$

$$I_y = \iint_R x^2\,dA = \int_0^{\pi/3}\int_{2a}^{4a\cos\theta} (\rho\cos\theta)^2\rho\,d\rho\,d\theta = \frac{12\pi + 11\sqrt{3}}{2}a^4 = \frac{3(12\pi + 11\sqrt{3})}{2(2\pi + 3\sqrt{3})}a^2A$$

$$I_0 = I_x + I_y = \frac{20\pi + 21\sqrt{3}}{3}a^4 = \frac{20\pi + 21\sqrt{3}}{2\pi + 3\sqrt{3}}a^2A$$

12. Find I_x, I_y, and I_0 for the area of the circle $\rho = 2(\sin\theta + \cos\theta)$. (See Fig. 70-12.)

Since $x^2 + y^2 = \rho^2$,

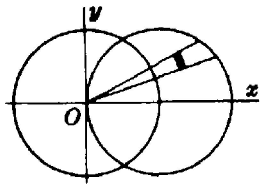

Fig. 70-11

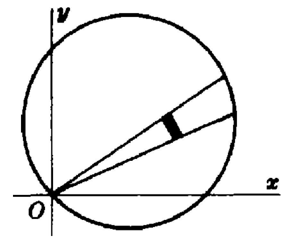

Fig. 70-12

$$I_0 = \iint_R (x^2+y^2)\,dA = \int_{-\pi/4}^{3\pi/4}\int_0^{2(\sin\theta+\cos\theta)} \rho^2\rho\,d\rho\,d\theta = 4\int_{-\pi/4}^{3\pi/4} (\sin\theta+\cos\theta)^4\,d\theta$$
$$= 4\left[\tfrac{3}{2}\theta - \cos 2\theta - \tfrac{1}{8}\sin 4\theta\right]_{-\pi/4}^{3\pi/4} = 6\pi = 3A$$

It is evident from Fig. 70-12 that $I_x = I_y$. Hence, $I_x = I_y = \frac{1}{2}I_0 = \frac{3}{2}A$.

Supplementary Problems

13. Use double integration to find the area:

(*a*) Bounded by $3x + 4y = 24$, $x = 0$, $y = 0$ *Ans.* 24 square units

(*b*) Bounded by $x + y = 2$, $2y = x + 4$, $y = 0$ *Ans.* 6 square units

(*c*) Bounded by $x^2 = 4y$, $8y = x^2 + 16$ *Ans.* $\frac{32}{3}$ square units

(*d*) Within $\rho = 2(1 - \cos\theta)$ *Ans.* 6π square units

(*e*) Bounded by $\rho = \tan\theta\sec\theta$ and $\theta = \pi/3$ *Ans.* $\frac{1}{2}\sqrt{3}$ square units

(*f*) Outside $\rho = 4$ and inside $\rho = 8\cos\theta$ *Ans.* $8(\frac{2}{3}\pi + \sqrt{3})$ square units

14. Locate the centroid of each of the following areas.

(*a*) The area of Problem 13(*a*) *Ans.* $(\frac{8}{3}, 2)$

(*b*) The first-quadrant area of Problem 13(*c*) *Ans.* $(\frac{3}{2}, \frac{8}{5})$

(*c*) The first-quadrant area bounded by $y^2 = 6x$, $y = 0$, $x = 6$ *Ans.* $(\frac{18}{5}, \frac{9}{4})$

(*d*) The area bounded by $y^2 = 4x$, $x^2 = 5 - 2y$, $x = 0$ *Ans.* $(\frac{13}{40}, \frac{26}{15})$

(*e*) The first-quadrant area bounded by $x^2 - 8y + 4 = 0$, $x^2 = 4y$, $x = 0$ *Ans.* $(\frac{3}{4}, \frac{2}{5})$

(*f*) The area of Problem 13(*e*) *Ans.* $(\frac{1}{2}\sqrt{3}, \frac{6}{5})$

(*g*) The first-quadrant area of Problem 13(*f*) *Ans.* $\left(\dfrac{16\pi + 6\sqrt{3}}{2\pi + 3\sqrt{3}}, \dfrac{22}{2\pi + 3\sqrt{3}}\right)$

15. Verify that $\frac{1}{2}\int_\alpha^\beta [g_2^2(\theta) - g_1^2(\theta)]\,d\theta = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \rho\,d\rho\,d\theta = \iint_R dA$; then infer that

$$\iint_R f(x, y)\,dA = \iint_R f(\rho\cos\theta, \rho\sin\theta)\rho\,d\rho\,d\theta$$

16. Find I_x and I_y for each of the following areas.

(*a*) The area of Problem 13(*a*) *Ans.* $I_x = 6A$; $I_y = \frac{32}{3}A$

(*b*) The area cut from $y^2 = 8x$ by its latus rectum *Ans.* $I_x = \frac{16}{5}A$; $I_y = \frac{12}{7}A$

(*c*) The area bounded by $y = x^2$ and $y = x$ *Ans.* $I_x = \frac{3}{14}A$; $I_y = \frac{3}{10}A$

(*d*) The area bounded by $y = 4x - x^2$ and $y = x$ *Ans.* $I_x = \frac{459}{70}A$; $I_y = \frac{27}{10}A$

17. Find I_x and I_y for one loop of $\rho^2 = \cos 2\theta$. *Ans.* $I_x = \left(\dfrac{\pi}{16} - \dfrac{1}{6}\right)A$; $I_y = \left(\dfrac{\pi}{16} + \dfrac{1}{6}\right)A$

18. Find I_0 for (*a*) the loop of $\rho = \sin 2\theta$ and (*b*) the area enclosed by $\rho = 1 + \cos\theta$. *Ans.* (*a*) $\frac{3}{8}A$; (*b*) $\frac{35}{24}A$

Chapter 71

Volume Under a Surface by Double Integration

THE VOLUME UNDER A SURFACE $z = f(x, y)$ or $z = f(\rho, \theta)$, that is, the volume of a vertical column whose upper base is in the surface and whose lower base is in the xOy plane, is defined by the double integral $V = \iint_R z\, dA$, the region R being the lower base of the column.

Solved Problems

1. Find the volume in the first octant between the planes $z = 0$ and $z = x + y + 2$, and inside the cylinder $x^2 + y^2 = 16$.

From Fig. 71-1, it is evident that $z = x + y + 2$ is to be integrated over a quadrant of the circle $x^2 + y^2 = 16$ in the xOy plane. Hence,

$$V = \iint_R z\, dA = \int_0^4 \int_0^{\sqrt{16-x^2}} (x + y + 2)\, dy\, dx = \int_0^4 \left(x\sqrt{16 - x^2} + 8 - \frac{1}{2}x^2 + 2\sqrt{16 - x^2}\right) dx$$

$$= \left[-\frac{1}{3}(16 - x^2)^{3/2} + 8x - \frac{x^3}{6} + x\sqrt{16 - x^2} + 16 \arcsin \frac{1}{4}x\right]_0^4 = \left(\frac{128}{3} + 8\pi\right) \text{ cubic units}$$

2. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

From Fig. 71-2, it is evident that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the xOy plane. Hence,

$$V = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4 - y)\, dx\, dy = 2\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} (4 - y)\, dx\, dy = 16\pi \text{ cubic units}$$

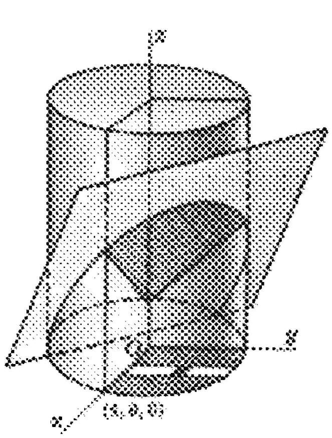

Fig. 71-1

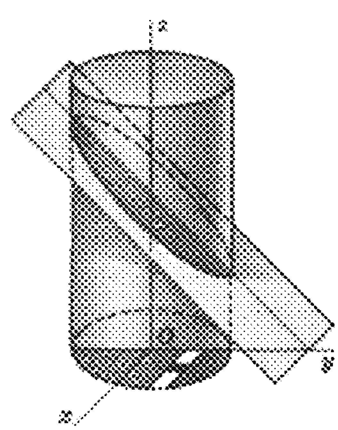

Fig. 71-2

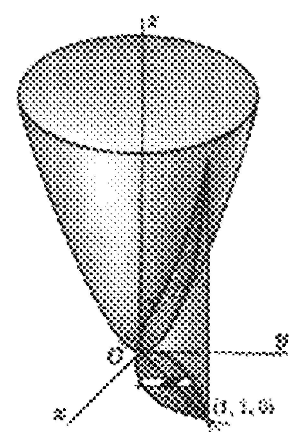

Fig. 71-3

3. Find the volume bounded above by the paraboloid $x^2 + 4y^2 = z$, below by the plane $z = 0$, and laterally by the cylinders $y^2 = x$ and $x^2 = y$. (See Fig. 71-3.)

The required volume is obtained by integrating $z = x^2 + 4y^2$ over the region R common to the parabolas $y^2 = x$ and $x^2 = y$ in the xOy plane. Hence,

$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + 4y^2)\,dy\,dx = \int_0^1 [x^2y + \tfrac{4}{3}y^3]_{x^2}^{\sqrt{x}}\,dx = \tfrac{3}{7} \text{ cubic units}$$

4. Find the volume of one of the wedges cut from the cylinder $4x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = my$. (See Fig. 71-4.)

The volume is obtained by integrating $z = my$ over half the ellipse $4x^2 + y^2 = a^2$. Hence,

$$V = 2\int_0^{a/2} \int_0^{\sqrt{a^2-4x^2}} my\,dy\,dx = m\int_0^{a/2} [y^2]_0^{\sqrt{a^2-4x^2}}\,dx = \frac{ma^3}{3} \text{ cubic units}$$

5. Find the volume bounded by the paraboloid $x^2 + y^2 = 4z$, the cylinder $x^2 + y^2 = 8y$, and the plane $z = 0$. (See Fig. 71-5.)

The required volume is obtained by integrating $z = \frac{1}{4}(x^2 + y^2)$ over the circle $x^2 + y^2 = 8y$. Using cylindrical coordinates, the volume is obtained by integrating $z = \frac{1}{4}\rho^2$ over the circle $\rho = 8\sin\theta$. Then,

$$V = \iint_R z\,dA = \int_0^{\pi}\int_0^{8\sin\theta} z\rho\,d\rho\,d\theta = \tfrac{1}{4}\int_0^{\pi}\int_0^{8\sin\theta} \rho^3\,d\rho\,d\theta$$

$$= \tfrac{1}{16}\int_0^{\pi} [\rho^4]_0^{8\sin\theta}\,d\theta = 256\int_0^{\pi} \sin^4\theta\,d\theta = 96\pi \text{ cubic units}$$

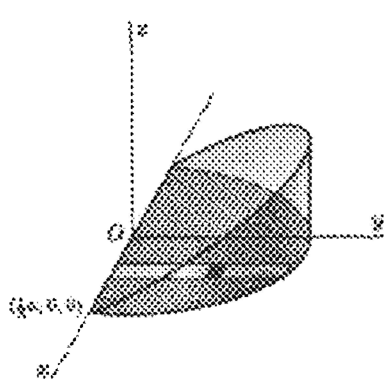

Fig. 71-4

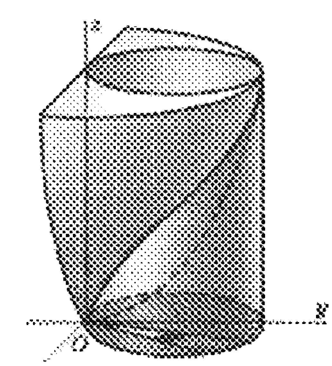

Fig. 71-5

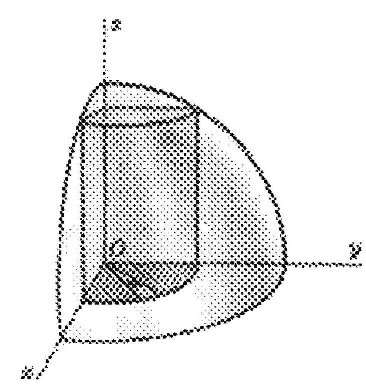

Fig. 71-6

6. Find the volume removed when a hole of radius a is bored through a sphere of radius $2a$, the axis of the hole being a diameter of the sphere. (See Fig. 71-6.)

From the figure, it is obvious that the required volume is eight times the volume in the first octant bounded by the cylinder $\rho^2 = a^2$, the sphere $\rho^2 + z^2 = 4a^2$, and the plane $z = 0$. The latter volume is obtained by integrating $z = \sqrt{4a^2 - \rho^2}$ over a quadrant of the circle $\rho = a$. Hence,

$$V = 8\int_0^{\pi/2}\int_0^{a} \sqrt{4a^2 - \rho^2}\,\rho\,d\rho\,d\theta = \tfrac{8}{3}\int_0^{\pi/2} (8a^3 - 3\sqrt{3}a^3)\,d\theta = \tfrac{4}{3}(8 - 3\sqrt{3})a^3\pi \text{ cubic units}$$

Supplementary Problems

7. Find the volume cut from $9x^2 + 4y^2 + 36z = 36$ by the plane $z = 0$. *Ans.* 3π cubic units

8. Find the volume under $z = 3x$ and above the first-quadrant area bounded by $x = 0$, $y = 0$, $x = 4$, and $x^2 + y^2 = 25$. *Ans.* 98 cubic units

9. Find the volume in the first octant bounded by $x^2 + z = 9$, $3x + 4y = 24$, $x = 0$, $y = 0$, and $z = 0$.
Ans. 1485/16 cubic units

10. Find the volume in the first octant bounded by $xy = 4z$, $y = x$, and $x = 4$. *Ans.* 8 cubic units

11. Find the volume in the first octant bounded by $x^2 + y^2 = 25$ and $z = y$. *Ans.* $\frac{125}{3}$ cubic units

12. Find the volume common to the cylinders $x^2 + y^2 = 16$ and $x^2 + z^2 = 16$. *Ans.* $\frac{1024}{3}$ cubic units

13. Find the volume in the first octant inside $y^2 + z^2 = 9$ and outside $y^2 = 3x$. *Ans.* $27\pi/16$ cubic units

14. Find the volume in the first octant bounded by $x^2 + z^2 = 16$ and $x - y = 0$. *Ans.* $\frac{64}{3}$ cubic units

15. Find the volume in front of $x = 0$ and common to $y^2 + z^2 = 4$ and $y^2 + z^2 + 2x = 16$.
Ans. 28π cubic units

16. Find the volume inside $\rho = 2$ and outside the cone $z^2 = \rho^2$. *Ans.* $32\pi/3$ cubic units

17. Find the volume inside $y^2 + z^2 = 2$ and outside $x^2 - y^2 - z^2 = 2$. *Ans.* $8\pi(4 - \sqrt{2})/3$ cubic units

18. Find the volume common to $\rho^2 + z^2 = a^2$ and $\rho = a \sin\theta$. *Ans.* $2(3\pi - 4)a^3/9$ cubic units

19. Find the volume inside $x^2 + y^2 = 9$, bounded below by $x^2 + y^2 + 4z = 16$ and above by $z = 4$.
Ans. $81\pi/8$ cubic units

20. Find the volume cut from the paraboloid $4x^2 + y^2 = 4z$ by the plane $z - y = 2$. *Ans.* 9π cubic units

21. Find the volume generated by revolving the cardioid $\rho = 2(1 - \cos\theta)$ about the polar axis.
Ans. $V = 2\pi \int\int y\rho \, d\rho \, d\theta = 64\pi/3$ cubic units

22. Find the volume generated by revolving a petal of $\rho = \sin 2\theta$ about either axis.
Ans. $32\pi/105$ cubic units

23. A square hole 2 units on a side is cut symmetrically through a sphere of radius 2 units. Show that the volume removed is $\frac{4}{3}(2\sqrt{2} + 19\pi - 54 \arctan\sqrt{2})$ cubic units.

Chapter 72

Area of a Curved Surface by Double Integration

TO COMPUTE THE LENGTH OF A(PLANAR) ARC, (1) the arc is projected on a convenient coordinate axis, thus establishing an interval on the axis, and (2) an integrand function, $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ if the projection is on the x axis or $\sqrt{1+\left(\frac{dx}{dy}\right)^2}$ if the projection is on the y axis, is integrated over the interval.

A similar procedure is used to compute the area S of a portion R^* of a surface $z = f(x, y)$: (1) R^* is projected on a convenient coordinate plane, thus establishing a region R on the plane, and (2) an integrand function is integrated over R. Then,

$$\text{If } R^* \text{ is projected on } xOy,\ S = \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

$$\text{If } R^* \text{ is projected on } yOz,\ S = \iint_R \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\, dA.$$

$$\text{If } R^* \text{ is projected on } zOx,\ S = \iint_R \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\, dA.$$

Solved Problems

1. Derive the first of the formulas for the area S of a region R^* as given above.

Consider a region R^* of area S on the surface $z = f(x, y)$. Through the boundary of R^* pass a vertical cylinder (see Fig. 72-1) cutting the xOy plane in the region R. Now divide R into n subregions

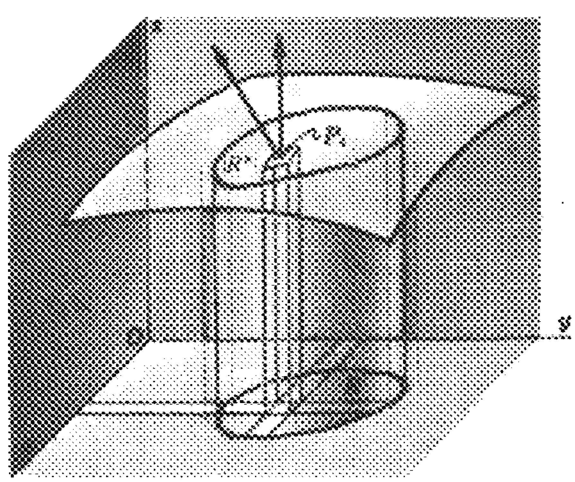

Fig. 72-1

ΔA_i (of areas ΔA_i), and denote by ΔS_i the area of the projection of ΔA_i on R^*. In each subregion ΔS_i, choose a point P_i and draw there the tangent plane to the surface. Let the area of the projection of ΔA_i on this tangent plane be denoted by ΔT_i. We shall use ΔT_i as an approximation of the corresponding surface area ΔS_i.

Now the angle between the xOy plane and the tangent plane at P_i is the angle γ_i between the z axis with direction numbers $[0, 0, 1]$, and the normal, $\left[-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right] = \left[-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right]$, to the surface at P_i; thus

$$\cos \gamma_i = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

Then (see Fig. 72-2),

$$\Delta T_i \cos \gamma_i = \Delta A_i \qquad \text{and} \qquad \Delta T_i = \sec \gamma_i \, \Delta A_i$$

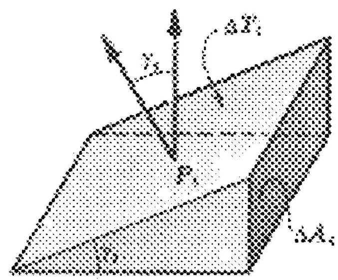

Fig. 72-2

Hence, an approximation of S is $\sum_{i=1}^{n} \Delta T_i = \sum_{i=1}^{n} \sec \gamma_i \, \Delta A_i$, and

$$S = \lim_{n \to +\infty} \sum_{i=1}^{n} \sec \gamma_i \, \Delta A_i = \iint_R \sec \gamma \, dA = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA$$

2. Find the area of the portion of the cone $x^2 + y^2 = 3z^2$ lying above the xOy plane and inside the cylinder $x^2 + y^2 = 4y$.

Solution 1: Refer to Fig. 72-3. The projection of the required area on the xOy plane is the region R enclosed by the circle $x^2 + y^2 = 4y$. For the cone,

$$\frac{\partial z}{\partial x} = \frac{1}{3}\frac{x}{z} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{1}{3}\frac{y}{z}. \qquad \text{So} \qquad 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{9z^2 + x^2 + y^2}{9z^2} = \frac{12z^2}{9z^2} = \frac{4}{3}$$

Then
$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA = \int_0^4 \int_{-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} \frac{2}{\sqrt{3}} \, dx \, dy = 2\frac{2}{\sqrt{3}} \int_0^4 \int_0^{\sqrt{4y-y^2}} dx \, dy$$

$$= \frac{4}{\sqrt{3}} \int_0^4 \sqrt{4y - y^2} \, dy = \frac{8\sqrt{3}}{3} \pi \text{ square units}$$

Solution 2: Refer to Fig. 72-4. The projection of one-half the required area on the yOz plane is the region R bounded by the line $y = \sqrt{3}z$ and the parabola $y = \frac{3}{4}z^2$, the latter obtained by eliminating x between the equations of the two surfaces. For the cone,

$$\frac{\partial x}{\partial y} = -\frac{y}{x} \qquad \text{and} \qquad \frac{\partial x}{\partial z} = \frac{3z}{x}. \qquad \text{So} \qquad 1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 = \frac{x^2 + y^2 + 9z^2}{x^2} = \frac{12z^2}{x^2} = \frac{12z^2}{3z^2 - y^2}$$

Then $S = 2\int_0^4 \int_{y/\sqrt{3}}^{2\sqrt{y}/\sqrt{3}} \frac{2\sqrt{3}z}{\sqrt{3z^2 - y^2}} \, dz \, dy = \frac{4\sqrt{3}}{3} \int_0^4 [\sqrt{3z^2 - y^2}]_{y/\sqrt{3}}^{2\sqrt{y}/\sqrt{3}} \, dy = \frac{4\sqrt{3}}{3} \int_0^4 \sqrt{4y - y^2} \, dy$

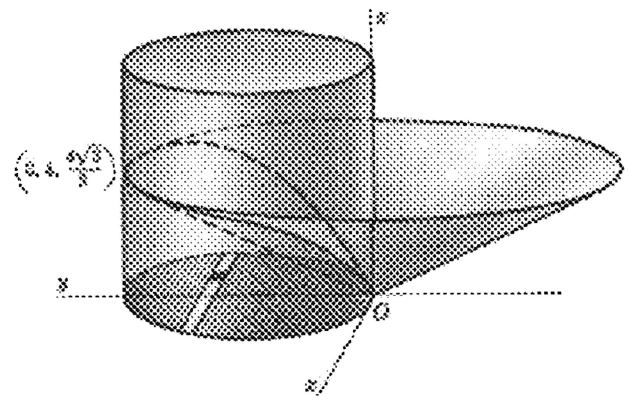

Fig. 72-3

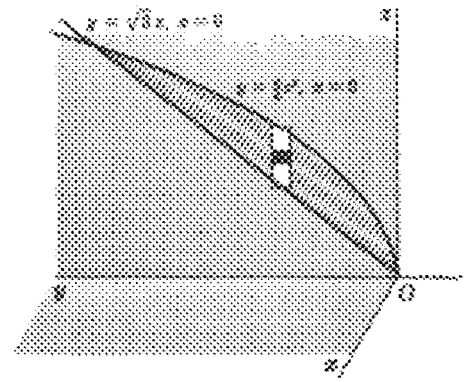

Fig. 72-4

Solution 3: Using polar coordinates in solution 1, we must integrate $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\frac{2}{\sqrt{3}}$ over the region R enclosed by the circle $\rho = 4\sin\theta$. Then,

$$S=\iint_R \frac{2}{\sqrt{3}}\,dA=\int_0^{\pi}\int_0^{4\sin\theta}\frac{2}{\sqrt{3}}\,\rho\,d\rho\,d\theta=\frac{1}{\sqrt{3}}\int_0^{\pi}[\rho^2]_0^{4\sin\theta}\,d\theta$$

$$=\frac{16}{\sqrt{3}}\int_0^{\pi}\sin^2\theta\,d\theta=\frac{8\sqrt{3}}{3}\pi \text{ square units}$$

3. Find the area of the portion of the cylinder $x^2+z^2=16$ lying inside the cylinder $x^2+y^2=16$.

Figure 72-5 shows one-eighth of the required area, its projection on the xOy plane being a quadrant of the circle $x^2+y^2=16$. For the cylinder $x^2+z^2=16$,

$$\frac{\partial z}{\partial x}=-\frac{x}{z} \quad\text{and}\quad \frac{\partial z}{\partial y}=0. \quad\text{So}\quad 1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\frac{x^2+z^2}{z^2}=\frac{16}{16-x^2}$$

Then $$S=8\int_0^4\int_0^{\sqrt{16-x^2}}\frac{4}{\sqrt{16-x^2}}\,dy\,dx=32\int_0^4 dx=128 \text{ square units}$$

4. Find the area of the portion of the sphere $x^2+y^2+z^2=16$ outside the paraboloid $x^2+y^2+z=16$.

Figure 72-6 shows one-fourth of the required area, its projection on the yOz plane being the region R bounded by the circle $y^2+z^2=16$, the y and z axes, and the line $z=1$. For the sphere,

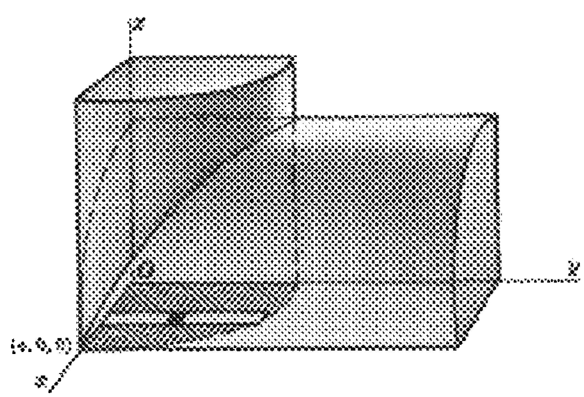

Fig. 72-5

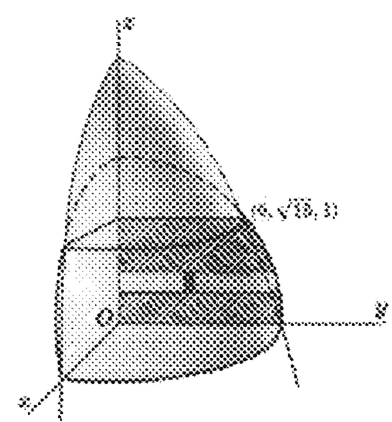

Fig. 72-6

$$\frac{\partial x}{\partial y} = -\frac{y}{x} \quad \text{and} \quad \frac{\partial x}{\partial z} = -\frac{z}{x}. \quad \text{So} \quad 1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 = \frac{x^2 + y^2 + z^2}{x^2} = \frac{16}{16 - y^2 - z^2}$$

Then $$S = 4 \iint_R \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dA = 4 \int_0^1 \int_0^{\sqrt{16 - z^2}} \frac{4}{\sqrt{16 - y^2 - z^2}}\, dy\, dz$$

$$= 16 \int_0^1 \left[\arcsin \frac{y}{\sqrt{16 - z^2}}\right]_0^{\sqrt{16 - z^2}} dz = 16 \int_0^1 \frac{1}{2}\pi\, dz = 8\pi \text{ square units}$$

5. Find the area of the portion of the cylinder $x^2 + y^2 = 6y$ lying inside the sphere $x^2 + y^2 + z^2 = 36$.

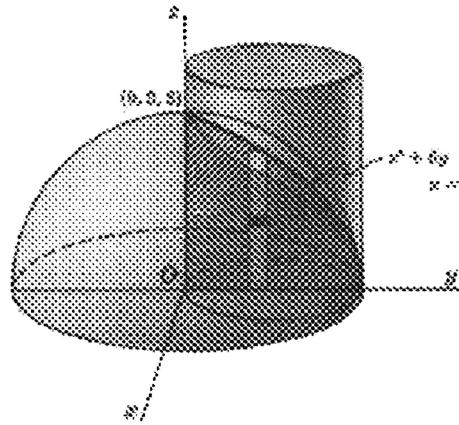

Fig. 72-7

Figure 72-7 shows one-fourth of the required area. Its projection on the yOz plane is the region R bounded by the z and y axes and the parabola $z^2 + 6y = 36$, the latter obtained by eliminating x from the equations of the two surfaces. For the cylinder,

$$\frac{\partial x}{\partial y} = \frac{3 - y}{x} \quad \text{and} \quad \frac{\partial x}{\partial z} = 0. \quad \text{So} \quad 1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 = \frac{x^2 + 9 - 6y + y^2}{x^2} = \frac{9}{6y - y^2}$$

Then $$S = 4 \int_0^6 \int_0^{\sqrt{36 - 6y}} \frac{3}{\sqrt{6y - y^2}}\, dz\, dy = 12 \int_0^6 \frac{\sqrt{6}}{\sqrt{y}}\, dy = 144 \text{ square units}$$

Supplementary Problems

6. Find the area of the portion of the cone $x^2 + y^2 = z^2$ inside the vertical prism whose base is the triangle bounded by the lines $y = x$, $x = 0$, and $y = 1$ in the xOy plane. *Ans.* $\frac{1}{2}\sqrt{2}$ square units

7. Find the area of the portion of the plane $x + y + z = 6$ inside the cylinder $x^2 + y^2 = 4$.

   Ans. $4\sqrt{3}\pi$ square units

8. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 36$ inside the cylinder $x^2 + y^2 = 6y$.

   Ans. $72(\pi - 2)$ square units

9. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 4z$ inside the paraboloid $x^2 + y^2 = z$.

Ans. 4π square units

10. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 25$ between the planes $z = 2$ and $z = 4$.

Ans. 20π square units

11. Find the area of the portion of the surface $z = xy$ inside the cylinder $x^2 + y^2 = 1$.

Ans. $2\pi(2\sqrt{2} - 1)/3$ square units

12. Find the area of the surface of the cone $x^2 + y^2 - 9z^2 = 0$ above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 6y$. *Ans.* $3\sqrt{10}\pi$ square units

13. Find the area of that part of the sphere $x^2 + y^2 + z^2 = 25$ that is within the elliptic cylinder $2x^2 + y^2 = 25$.

Ans. 50π square units

14. Find the area of the surface of $x^2 + y^2 - az = 0$ which lies directly above the lemniscate $4\rho^2 = a^2 \cos 2\theta$. *Ans.* $S = \frac{4}{a}\int\int \sqrt{4\rho^2 + a^2}\rho \, d\rho \, d\theta = \frac{a^2}{3}\left(\frac{5}{3} - \frac{\pi}{4}\right)$ square units

15. Find the area of the surface of $x^2 + y^2 + z^2 = 4$ which lies directly above the cardioid $\rho = 1 - \cos\theta$.

Ans. $8[\pi - \sqrt{2} - \ln(\sqrt{2} + 1)]$ square units

Chapter 73

Triple Integrals

CYLINDRICAL AND SPHERICAL COORDINATES. Assume that a point P has coordinates (x, y, z) in a right-handed rectangular coordinate system. The corresponding *cylindrical coordinates* of P are (r, θ, z), where (r, θ) are the polar coordinates for the point (x, y) in the xy plane. (Note the notational change here from (ρ, θ) to (r, θ) for the polar coordinates of (x, y); see Fig. 73-1.) Hence we have the relations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

In cylindrical coordinates, an equation $r = c$ represents a right circular cylinder of radius c with the z axis as its axis of symmetry. An equation $\theta = c$ represents a plane through the z axis.

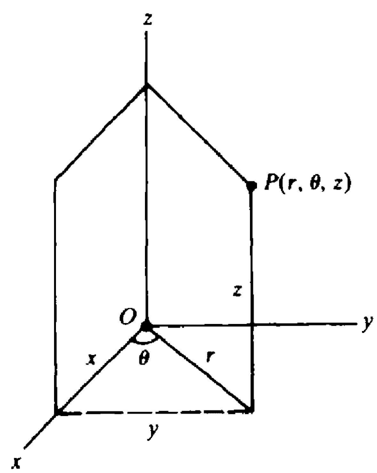

Fig. 73-1

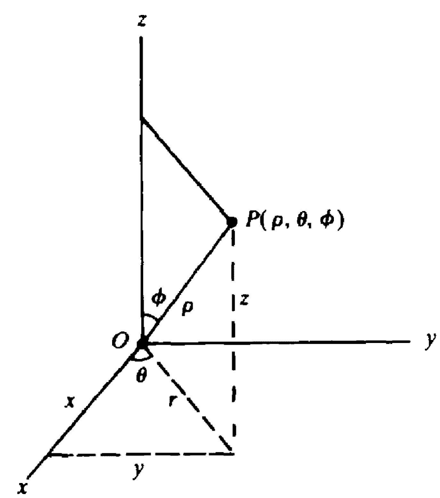

Fig. 73-2

A point P with rectangular coordinates (x, y, z) has the *spherical coordinates* (ρ, θ, ϕ), where $\rho = |OP|$, θ is the same as in cylindrical coordinates, and ϕ is the directed angle from the positive z axis to the vector **OP**. (See Fig. 73-2.) In spherical coordinates, an equation $\rho = c$ represents a sphere of radius c with center at the origin. An equation $\phi = c$ represents a cone with vertex at the origin and the z axis as its axis of symmetry.

The following additional relations hold among spherical, cylindrical, and rectangular coordinates:

$$r = \rho\sin\phi \qquad z = \rho\cos\phi \qquad \rho^2 = x^2 + y^2 + z^2$$
$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta$$

(See Problems 14 to 16.)

THE TRIPLE INTEGRAL $\iiint_R f(x, y, z)\, dV$ of a function of three independent variables over a closed region R of points (x, y, z), of volume V, on which the function is single-valued and continuous, is an extension of the notion of single and double integrals.

If $f(x, y, z) = 1$, then $\iiint_R f(x, y, z)\, dV$ may be interpreted as measuring the volume of the region R.

EVALUATION OF THE TRIPLE INTEGRAL. In rectangular coordinates,

$$\iiint_R f(x, y, z)\, dV = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

$$= \int_c^d \int_{x_1(y)}^{x_2(y)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dx\, dy, \text{ etc.}$$

where the limits of integration are chosen to cover the region R.

In cylindrical coordinates,

$$\iiint_R f(r, \theta, z)\, dV = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r, \theta, z) r\, dz\, dr\, d\theta$$

where the limits of integration are chosen to cover the region R.

In spherical coordinates,

$$\iiint_R f(\rho, \phi, \theta)\, dV = \int_\alpha^\beta \int_{\phi_1(\theta)}^{\phi_2(\theta)} \int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho, \phi, \theta)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

where the limits of integration are chosen to cover the region R.

Discussion of the definitions: Consider the function $f(x, y, z)$, continuous over a region R of ordinary space. After slicing R with planes $x = \xi_i$ and $y = \eta_j$ as in Chapter 69, let these subregions be further sliced by planes $z = \zeta_k$. The region R has now been separated into a number of rectangular parallelepipeds of volume $\Delta V_{ijk} = \Delta x_i\, \Delta y_j\, \Delta z_k$ and a number of partial parallelepipeds which we shall ignore. In each complete parallelepiped select a point $P_{ijk}(x_i, y_j, z_k)$; then compute $f(x_i, y_j, z_k)$ and form the sum

$$\sum_{\substack{i=1,\ldots,m\\ j=1,\ldots,n\\ k=1,\ldots,p}} f(x_i, y_j, z_k)\, \Delta V_{ijk} = \sum_{\substack{i=1,\ldots,m\\ j=1,\ldots,n\\ k=1,\ldots,p}} f(x_i, y_j, z_k)\, \Delta x_i\, \Delta y_j\, \Delta z_k \qquad (73.1)$$

The triple integral of $f(x, y, z)$ over the region R is defined to be the limit of (*73.1*) as the number of parallelepipeds is indefinitely increased in such a manner that all dimensions of each go to zero.

In evaluating this limit, we may sum first each set of parallelepipeds having $\Delta_i x$ and $\Delta_j y$, for fixed i and j, as two dimensions and consider the limit as each $\Delta_k z \to 0$. We have

$$\lim_{p \to +\infty} \sum_{k=1}^{p} f(x_i, y_j, z_k)\, \Delta_k z\, \Delta_i x\, \Delta_j y = \int_{z_1}^{z_2} f(x_i, y_i, z)\, dz\, \Delta_i x\, \Delta_j y$$

Now these are the columns, the basic subregions, of Chapter 69; hence,

$$\lim_{\substack{m\to+\infty\\ n\to+\infty\\ p\to+\infty}} \sum_{\substack{i=1,\ldots,m\\ j=1,\ldots,n\\ k=1,\ldots,p}} f(x_i, y_j, z_k)\, \Delta V_{ijk} = \iiint_R f(x, y, z)\, dz\, dx\, dy = \iiint_R f(x, y, z)\, dz\, dy\, dx$$

CENTROIDS AND MOMENTS OF INERTIA. The coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the *centroid of a volume* satisfy the relations

$$\bar{x}\iiint_R dV = \iiint_R x\,dV \qquad \bar{y}\iiint_R dV = \iiint_R y\,dV$$

$$\bar{z}\iiint_R dV = \iiint_R z\,dV$$

The *moments of inertia of a volume* with respect to the coordinate axes are given by

$$I_x = \iiint_R (y^2+z^2)\,dV \qquad I_y = \iiint_R (z^2+x^2)\,dV \qquad I_z = \iiint_R (x^2+y^2)\,dV$$

Solved Problems

1. Evaluate the given triple integrals:

(*a*) $\int_0^1 \int_0^{1-x} \int_0^{2-x} xyz\,dz\,dy\,dx$

$$= \int_0^1 \left[\int_0^{1-x} \left(\int_0^{2-x} xyz\,dz\right) dy\right] dx$$

$$= \int_0^1 \left[\int_0^{1-x} \left(\frac{xyz^2}{2}\bigg|_{z=0}^{z=2-x}\right) dy\right] dx = \int_0^1 \left[\int_0^{1-x} \frac{xy(2-x)^2}{2}\,dy\right] dx$$

$$= \int_0^1 \left[\frac{xy^2(2-x)^2}{4}\right]_{y=0}^{y=1-x} dx = \frac{1}{4}\int_0^1 (4x - 12x^2 + 13x^3 - 6x^4 + x^5)\,dx = \frac{13}{240}$$

(*b*) $\int_0^{\pi/2} \int_0^1 \int_0^2 zr^2 \sin\theta\,dz\,dr\,d\theta$

$$= \int_0^{\pi/2}\int_0^1 \left[\frac{z^2}{2}\right]_0^2 r^2\sin\theta\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^1 r^2\sin\theta\,dr\,d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} [r^3]_0^1 \sin\theta\,d\theta = -\frac{2}{3}[\cos\theta]_0^{\pi/2} = \frac{2}{3}$$

(*c*) $\int_0^{\pi} \int_0^{\pi/4} \int_0^{\sec\phi} \sin 2\phi\,d\rho\,d\phi\,d\theta$

$$= 2\int_0^{\pi}\int_0^{\pi/4} \sin\phi\,d\phi\,d\theta = 2\int_0^{\pi} (1 - \tfrac{1}{2}\sqrt{2})\,d\theta = (2-\sqrt{2})\pi$$

2. Compute the triple integral of $F(x, y, z) = z$ over the region R in the first octant bounded by the planes $y = 0$, $z = 0$, $x + y = 2$, $2y + x = 6$, and the cylinder $y^2 + z^2 = 4$. (See Fig. 73-3.)

Integrate first with respect to z from $z = 0$ (the xOy plane) to $z = \sqrt{4-y^2}$ (the cylinder), then with respect to x from $x = 2 - y$ to $x = 6 - 2y$, and finally with respect to y from $y = 0$ to $y = 2$. This yields

$$\iiint_R z\,dV = \int_0^2\int_{2-y}^{6-2y}\int_0^{\sqrt{4-y^2}} z\,dz\,dx\,dy = \int_0^2\int_{2-y}^{6-2y} [\tfrac{1}{2}z^2]_0^{\sqrt{4-y^2}}\,dx\,dy$$

$$= \tfrac{1}{2}\int_0^2\int_{2-y}^{6-2y} (4-y^2)\,dx\,dy = \tfrac{1}{2}\int_0^2 [(4-y^2)x]_{2-y}^{6-2y}\,dy = \tfrac{26}{3}$$

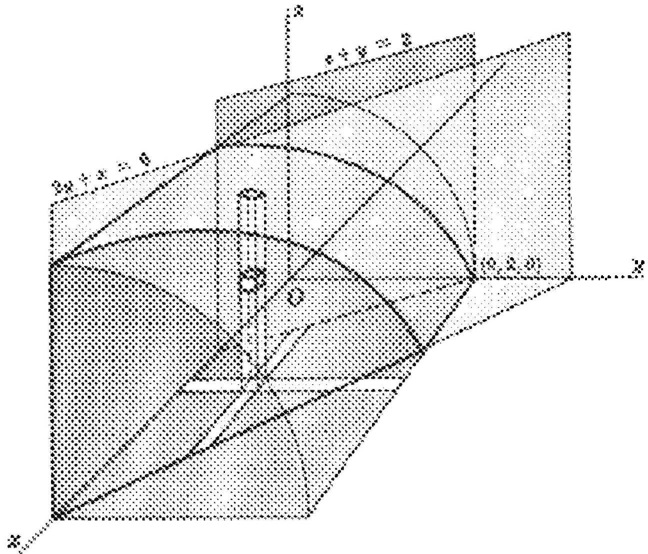

Fig. 73-3

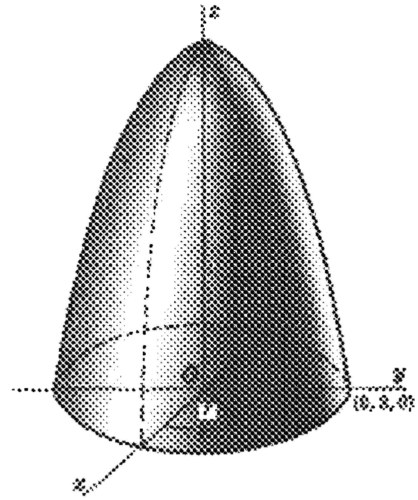

Fig. 73-4

3. Compute the triple integral of $f(r, \theta, z) = r^2$ over the region R bounded by the paraboloid $r^2 = 9 - z$ and the plane $z = 0$. (See Fig. 73-4.)

Integrate first with respect to z from $z = 0$ to $z = 9 - r^2$, then with respect to r from $r = 0$ to $r = 3$, and finally with respect to θ from $\theta = 0$ to $\theta = 2\pi$. This yields

$$\iiint_R r^2\, dV = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2(r\,dz\,dr\,d\theta) = \int_0^{2\pi}\int_0^3 r^3(9 - r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{9}{4}r^4 - \tfrac{1}{6}r^6\right]_0^3 d\theta = \int_0^{2\pi} \tfrac{243}{4}\, d\theta = \tfrac{243}{2}\pi$$

4. Show that the integrals (*a*) $4\int_0^4\int_0^{\sqrt{16-x^2}}\int_{(x^2+y^2)/4}^4 dz\,dy\,dx$, (*b*) $4\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dy\,dx\,dz$, and (*c*) $4\int_0^4\int_{y^2/4}^4\int_0^{\sqrt{4z-y^2}} dx\,dz\,dy$ give the same volume.

(*a*) Here z ranges from $z = \frac{1}{4}(x^2 + y^2)$ to $z = 4$; that is, the volume is bounded below by the paraboloid $4z = x^2 + y^2$ and above the plane $z = 4$. The ranges of y and x cover a quadrant of the circle $x^2 + y^2 = 16$, $z = 0$, the projection of the curve of intersection of the paraboloid and the plane $z = 4$ on the xOy plane. Thus, the integral gives the volume cut from the paraboloid by the plane $z = 4$.

(*b*) Here y ranges from $y = 0$ to $y = \sqrt{4z - x^2}$; that is, the volume is bounded on the left by the zOx plane and on the right by the paraboloid $y^2 = 4z - x^2$. The ranges of x and z cover one-half the area cut from the parabola $x^2 = 4z$, $y = 0$, the curve of intersection of the paraboloid and the zOx plane, by the plane $z = 4$. The region R is that of (*a*).

(*c*) Here the volume is bounded behind by the yOz plane and in front by the paraboloid $4z = x^2 + y^2$. The ranges of z and y cover one-half the area cut from the parabola $y^2 = 4z$, $x = 0$, the curve of intersection of the paraboloid and the yOz plane, by the plane $z = 4$. The region R is that of (*a*).

5. Compute the triple integral of $F(\rho, \phi, \theta) = 1/\rho$ over the region R in the first octant bounded by the cones $\phi = \frac{1}{4}\pi$ and $\phi = \arctan 2$ and the sphere $\rho = \sqrt{6}$. (See Fig. 73-5.)

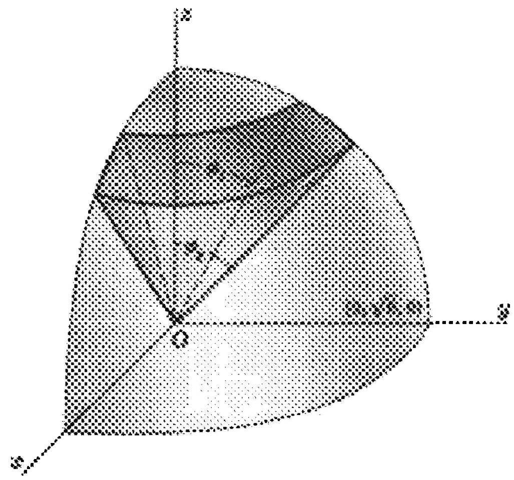

Fig. 73-5

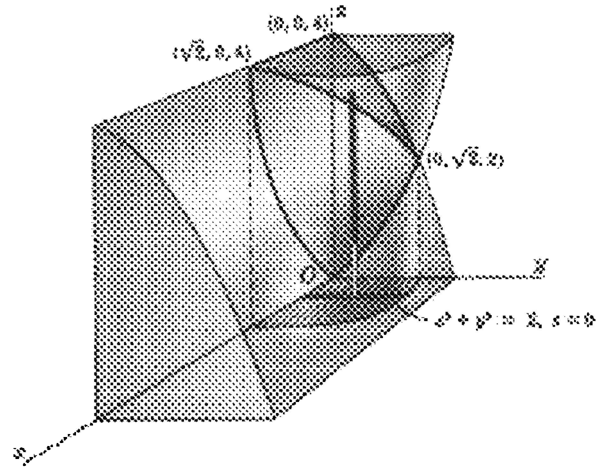

Fig. 73-6

Integrate first with respect to ρ from $\rho = 0$ to $\rho = \sqrt{6}$, then with respect to ϕ from $\phi = \frac{1}{4}\pi$ to $\phi = \arctan 2$, and finally with respect to θ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$. This yields

$$\iiint_R \frac{1}{\rho}\, dV = \int_0^{\pi/2} \int_{\pi/4}^{\arctan 2} \int_0^{\sqrt{6}} \frac{1}{\rho}\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 3 \int_0^{\pi/2} \int_{\pi/4}^{\arctan 2} \sin\phi \, d\phi \, d\theta$$

$$= -3 \int_0^{\pi/2} \left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{2}} \right) d\theta = \frac{3\pi}{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}} \right)$$

6. Find the volume bounded by the paraboloid $z = 2x^2 + y^2$ and the cylinder $z = 4 - y^2$. (See Fig. 73-6.)

Integrate first with respect to z from $z = 2x^2 + y^2$ to $z = 4 - y^2$, then with respect to y from $y = 0$ to $y = \sqrt{2 - x^2}$ (obtain $x^2 + y^2 = 2$ by eliminating x between the equations of the two surfaces), and finally with respect to x from $x = 0$ to $x = \sqrt{2}$ (obtained by setting $y = 0$ in $x^2 + y^2 = 2$) to obtain one-fourth of the required volume. Thus,

$$V = 4 \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{2x^2+y^2}^{4-y^2} dz \, dy \, dx = 4 \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} [(4 - y^2) + (2x^2 + y^2)] \, dy \, dx$$

$$= 4 \int_0^{\sqrt{2}} \left[4y - 2x^2 y - \frac{2y^3}{3} \right]_0^{\sqrt{2-x^2}} dx = \frac{16}{3} \int_0^{\sqrt{2}} (2 - x^2)^{3/2} \, dx = 4\pi \text{ cubic units}$$

7. Find the volume within the cylinder $r = 4\cos\theta$ bounded above by the sphere $r^2 + z^2 = 16$ and below by the plane $z = 0$. (See Fig. 73-7.)

Integrate first with respect to z from $z = 0$ to $z = \sqrt{16 - r^2}$, then with respect to r from $r = 0$ to $r = 4\cos\theta$, and finally with respect to θ from $\theta = 0$ to $\theta = \pi$ to obtain the required volume. Thus,

$$V = \int_0^{\pi} \int_0^{4\cos\theta} \int_0^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta = \int_0^{\pi} \int_0^{4\cos\theta} r\sqrt{16 - r^2} \, dr \, d\theta$$

$$= -\frac{64}{3} \int_0^{\pi} (\sin^3\theta - 1) \, d\theta = \frac{64}{9}(3\pi - 4) \text{ cubic units}$$

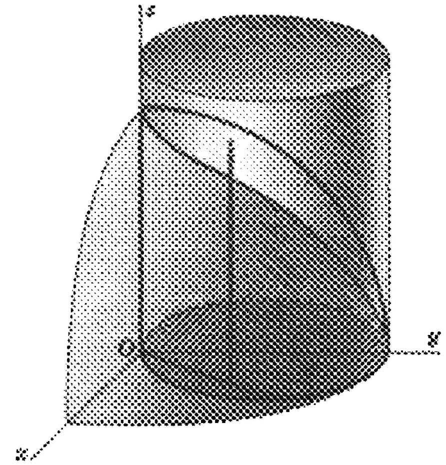

Fig. 73-7

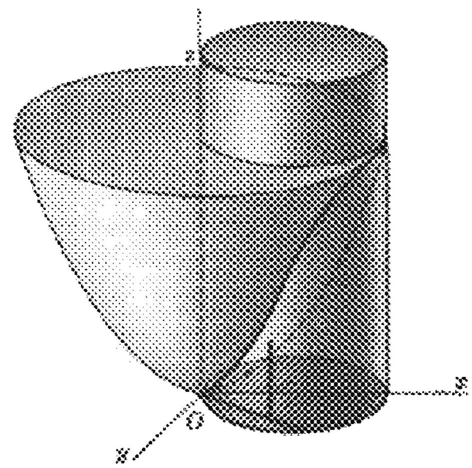

Fig. 73-8

8. Find the coordinates of the centroid of the volume within the cylinder $r = 2\cos\theta$, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$. (See Fig. 73-8.)

$$V = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta$$

$$= \tfrac{1}{2}\int_0^{\pi/2}[r^4]_0^{2\cos\theta}\,d\theta = 8\int_0^{\pi/2}\cos^4\theta\,d\theta = \tfrac{3}{2}\pi$$

$$M_{yz} = \iiint_R x\,dV = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2}(r\cos\theta)r\,dz\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^4\cos\theta\,dr\,d\theta = \tfrac{64}{5}\int_0^{\pi/2}\cos^6\theta\,d\theta = 2\pi$$

Then $\bar{x} = M_{yz}/V = \frac{4}{3}$. By symmetry, $\bar{y} = 0$. Also,

$$M_{xy} = \iiint_R z\,dV = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} zr\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2\cos\theta} r^5\,dr\,d\theta$$

$$= \tfrac{32}{3}\int_0^{\pi/2}\cos^6\theta\,d\theta = \tfrac{5}{3}\pi$$

and $\bar{z} = M_{xy}/V = \frac{10}{9}$. Thus, the centroid has coordinates $(\frac{4}{3}, 0, \frac{10}{9})$.

9. For the right circular cone of radius a and height h, find (a) the centroid, (b) the moment of inertia with respect to its axis (c), the moment of inertia with respect to any line through its vertex and perpendicular to its axis, (d) the moment of inertia with respect to any line through its centroid and perpendicular to its axis, an (e) the moment of inertia with respect to any diameter of its base.

Take the cone as in Fig. 73-9, so that its equation is $r = \dfrac{a}{h}z$. Then

$$V = 4\int_0^{\pi/2}\int_0^a\int_{hr/a}^h r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^a\left(hr - \frac{h}{a}r^2\right)dr\,d\theta$$

$$= \frac{2}{3}ha^2\int_0^{\pi/2}d\theta = \frac{1}{3}\pi ha^2$$

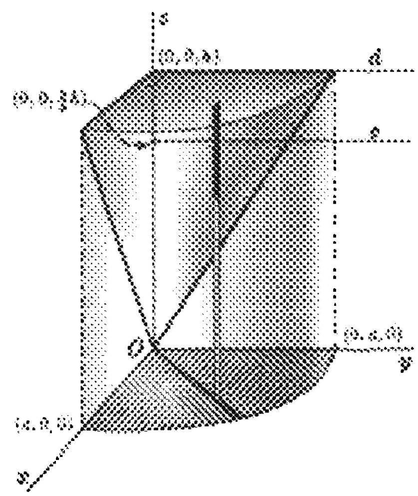

Fig. 73-9

(*a*) The centroid lies on the z axis, and we have

$$M_{xy} = \iiint_R z\,dV = 4\int_0^{\pi/2}\int_0^a\int_{hr/a}^h zr\,dz\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^a \left(h^2r - \frac{h^2}{a^2}r^3\right) dr\,d\theta = \frac{1}{2}h^2a^2\int_0^{\pi/2} d\theta = \frac{1}{4}\pi h^2a^2$$

Then $\bar{z} = M_{xy}/V = \frac{3}{4}h$, and the centroid has coordinates $(0, 0, \frac{3}{4}h)$.

(*b*) $$I_z = \iiint_R (x^2 + y^2)\,dV = 4\int_0^{\pi/2}\int_0^a\int_{hr/a}^h (r^2)r\,dz\,dr\,d\theta = \tfrac{1}{10}\pi ha^4 = \tfrac{3}{10}a^2V$$

(*c*) Take the line as the y axis. Then

$$I_y = \iiint_R (x^2 + z^2)\,dV = 4\int_0^{\pi/2}\int_0^a\int_{hr/a}^h (r^2\cos^2\theta + z^2)r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^a \left[\left(hr^3 - \frac{h}{a}r^4\right)\cos^2\theta + \frac{1}{3}\left(h^3r - \frac{h^3}{a^3}r^4\right)\right] dr\,d\theta$$

$$= \frac{1}{5}\pi ha^2\left(h^2 + \frac{1}{4}a^2\right) = \frac{3}{5}\left(h^2 + \frac{1}{4}a^2\right)V$$

(*d*) Let the line c through the centroid be parallel to the y axis. By the parallel-axis theorem,

$$I_y = I_c + V(\tfrac{3}{4}h)^2 \qquad \text{and} \qquad I_c = \tfrac{3}{5}(h^2 + \tfrac{1}{4}a^2)V - \tfrac{9}{16}h^2V = \tfrac{3}{80}(h^2 + 4a^2)V$$

(*e*) Let d denote the diameter of the base of the cone parallel to the y axis. Then

$$I_d = I_c + V(\tfrac{1}{4}h)^2 = \tfrac{3}{80}(h^2 + 4a^2)V + \tfrac{1}{16}h^2V = \tfrac{1}{20}(2h^2 + 3a^2)V$$

10. Find the volume cut from the cone $\phi = \frac{1}{4}\pi$ by the sphere $\rho = 2a\cos\phi$. (See Fig. 73-10.)

$$V = 4\iiint_R dV = 4\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{2a\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{32a^3}{3}\int_0^{\pi/2}\int_0^{\pi/4} \cos^3\phi\sin\phi\,d\phi\,d\theta = 2a^3\int_0^{\pi/2} d\theta = \pi a^3 \text{ cubic units}$$

11. Locate the centroid of the volume cut from one nappe of a cone of vertex angle 60° by a sphere of radius 2 whose center is at the vertex of the cone.

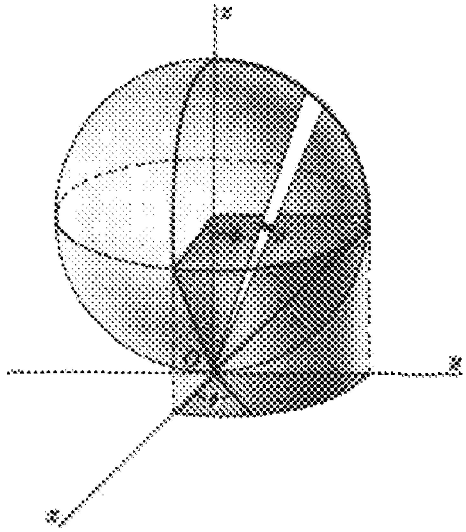

Fig. 73-10

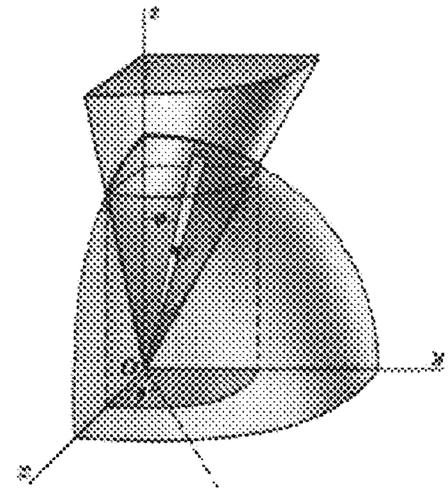

Fig. 73-11

Take the surfaces as in Fig. 73-11, so that $\bar{x} = \bar{y} = 0$. In spherical coordinates, the equation of the cone is $\phi = \pi/6$, and the equation of the sphere is $\rho = 2$. Then

$$V = \iiint_R dV = 4\int_0^{\pi/2}\int_0^{\pi/6}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{32}{3}\int_0^{\pi/2}\int_0^{\pi/6} \sin\phi \, d\phi \, d\theta$$

$$= -\frac{32}{3}\left(\frac{\sqrt{3}}{2} - 1\right)\int_0^{\pi/2} d\theta = \frac{8\pi}{3}(2 - \sqrt{3})$$

$$M_{xy} = \iiint_R z \, dV = 4\int_0^{\pi/2}\int_0^{\pi/6}\int_0^2 (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/6} \sin 2\phi \, d\phi \, d\theta = \pi$$

and $\bar{z} = M_{xy}/V = \frac{3}{8}(2 + \sqrt{3})$.

12. Find the moment of inertia with respect to the z axis of the volume of Problem 11.

$$I_z = \iiint_R (x^2 + y^2)\, dV = 4\int_0^{\pi/2}\int_0^{\pi/6}\int_0^2 (\rho^2 \sin^2\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{128}{5}\int_0^{\pi/2}\int_0^{\pi/6} \sin^3\phi \, d\phi \, d\theta = \frac{128}{5}\left(\frac{2}{3} - \frac{3}{8}\sqrt{3}\right)\int_0^{\pi/2} d\theta = \frac{8\pi}{15}(16 - 9\sqrt{3}) = \frac{5 - 2\sqrt{3}}{5} V$$

Supplementary Problems

13. Describe the curve determined by each of the following pairs of equations in cylindrical coordinates.
(*a*) $r = 1, z = 2$ (*b*) $r = 2, z = \theta$ (*c*) $\theta = \pi/4, r = \sqrt{2}$ (*d*) $\theta = \pi/4, z = r$

Ans. (*a*) circle of radius 1 in plane $z = 2$ with center having rectangular coordinates $(0, 0, 2)$; (*b*) helix on right circular cylinder $r = 2$; (*c*) vertical line through point having rectangular coordinates $(1, 1, 0)$; (*d*) line through origin in plane $\theta = \pi/4$, making an angle of 45° with xy plane

14. Describe the curve determined by each of the following pairs of equations in spherical coordinates.

(a) $\rho = 1,\ \theta = \pi$ (b) $\theta = \dfrac{\pi}{4},\ \phi = \dfrac{\pi}{6}$ (c) $\rho = 2,\ \phi = \dfrac{\pi}{4}$

Ans. (a) circle of radius 1 in xz plane with center at origin; (b) halfline of intersection of plane $\theta = \pi/4$ and cone $\phi = \pi/6$; (c) circle of radius $\sqrt{2}$ in plane $z = \sqrt{2}$ with center on z axis

15. Transform each of the following equations in either rectangular, cylindrical, or spherical coordinates into equivalent equations in the two other coordinate systems.
(a) $\rho = 5$ (b) $z^2 = r^2$ (c) $x^2 + y^2 + (z-1)^2 = 1$

Ans. (a) $x^2 + y^2 + z^2 = 25$, $r^2 + z^2 = 25$; (b) $z^2 = x^2 + y^2$, $\cos^2\phi = \frac{1}{2}$ (that is, $\phi = \pi/4$ or $\phi = 3\pi/4$); (c) $r^2 + z^2 = 2z$, $\rho = 2\cos\phi$

16. Evaluate the triple integral on the left in each of the following:

(a) $\displaystyle\int_0^1 \int_1^2 \int_2^3 dz\,dx\,dy = 1$

(b) $\displaystyle\int_0^1 \int_{x^2}^{x} \int_0^{xy} dz\,dy\,dx = \frac{1}{24}$

(c) $\displaystyle\int_0^6 \int_0^{12-2y} \int_0^{4-2y/3-x/3} x\,dz\,dx\,dy = 144 \left[= \int_0^{12} \int_0^{6-x/2} \int_0^{4-2y/3-x/3} x\,dz\,dy\,dx\right]$

(d) $\displaystyle\int_0^{\pi/2} \int_0^4 \int_0^{\sqrt{16-z^2}} (16 - r^2)^{1/2} rz\,dr\,dz\,d\theta = \frac{256}{5}\pi$

(e) $\displaystyle\int_0^{2\pi} \int_0^{\pi} \int_0^5 \rho^4 \sin\phi\,d\rho\,d\phi\,d\theta = 2500\pi$

17. Evaluate the integral of Problem 16(b) after changing the order to $dz\,dx\,dy$.

18. Evaluate the integral of Problem 16(c), changing the order to $dx\,dy\,dz$ and to $dy\,dz\,dx$.

19. Find the following volumes, using triple integrals in rectangular coordinates:
(a) Inside $x^2 + y^2 = 9$, above $z = 0$, and below $x + z = 4$ *Ans.* 36π cubic units
(b) Bounded by the coordinate planes and $6x + 4y + 3z = 12$ *Ans.* 4 cubic units
(c) Inside $x^2 + y^2 = 4x$, above $z = 0$, and below $x^2 + y^2 = 4z$ *Ans.* 6π cubic units

20. Find the following volumes, using triple integrals in cylindrical coordinates:
(a) The volume of Problem 4
(b) The volume of Problem 19(c)
(c) That inside $r^2 = 16$, above $z = 0$, and below $2z = y$ *Ans.* $64/3$ cubic units

21. Find the centroid of each of the following volumes:
(a) Under $z^2 = xy$ and above the triangle $y = x$, $y = 0$, $x = 4$ in the plane $z = 0$ *Ans.* $(3, \frac{9}{5}, \frac{9}{8})$
(b) That of Problem 19(b) *Ans.* $(\frac{1}{2}, \frac{3}{4}, 1)$
(c) The first-octant volume of Problem 19(a) *Ans.* $\left(\dfrac{64 - 9\pi}{16(\pi - 1)}, \dfrac{23}{8(\pi - 1)}, \dfrac{73\pi - 128}{32(\pi - 1)}\right)$
(d) That of Problem 19(c) *Ans.* $(\frac{8}{3}, 0, \frac{10}{9})$
(e) That of Problem 20(c) *Ans.* $(0, 3\pi/4, 3\pi/16)$

22. Find the moments of inertia I_x, I_y, I_z of the following volumes:
(a) That of Problem 4 *Ans.* $I_x = I_y = \frac{32}{3}V$; $I_z = \frac{16}{3}V$
(b) That of Problem 19(b) *Ans.* $I_x = \frac{5}{2}V$; $I_y = 2V$; $I_z = \frac{13}{10}V$
(c) That of Problem 19(c) *Ans.* $I_x = \frac{55}{18}V$; $I_y = \frac{175}{18}V$; $I_z = \frac{80}{9}V$
(d) That cut from $z = r^2$ by the plane $z = 2$ *Ans.* $I_x = I_y = \frac{7}{3}V$; $I_z = \frac{2}{3}V$

23. Show that, in cylindrical coordinates, the triple integral of a function $f(r, \theta, z)$ over a region R may be represented by

$$\int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r, \theta, z) r \, dz \, dr \, d\theta$$

(*Hint*: Consider, in Fig. 73-12, a representative subregion of R bounded by two cylinders having Oz as axis and of radii r and $r + \Delta r$, respectively, cut by two horizontal planes through $(0, 0, z)$ and $(0, 0, z + \Delta z)$, respectively, and by two vertical planes through Oz making angles θ and $\theta + \Delta\theta$, respectively, with the xOz plane. Take $\Delta V = (r\,\Delta\theta)\,\Delta r\,\Delta z$ as an approximation of its volume.)

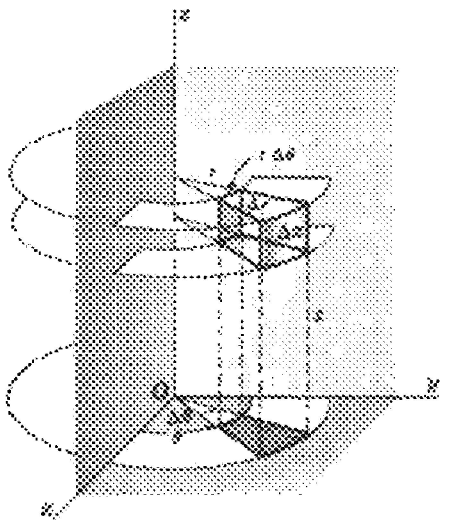

Fig. 73-12

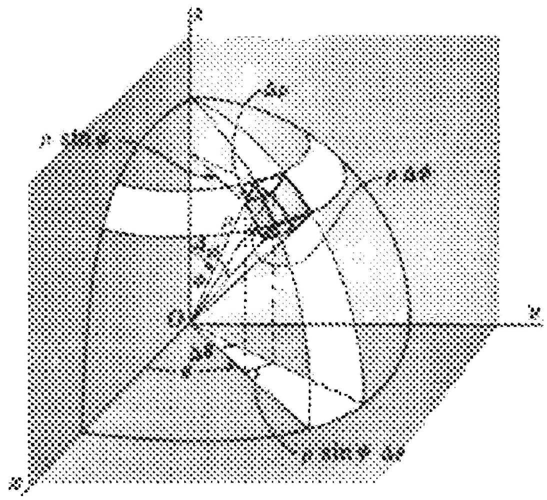

Fig. 73-13

24. Show that, in spherical coordinates, the triple integral of a function $f(\rho, \phi, \theta)$ over a region R may be represented by

$$\int_{\alpha}^{\beta} \int_{\phi_1(\theta)}^{\phi_2(\theta)} \int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho, \phi, \theta) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(*Hint*: Consider, in Fig. 73-13, a representative subregion of R bounded by two spheres centered at O, of radii ρ and $\rho + \Delta\rho$, respectively, by two cones having O as vertex, Oz as axis, and semivertical angles ϕ and $\phi + \Delta\phi$, respectively, and by two vertical planes through Oz making angles θ and $\theta + \Delta\theta$, respectively, with the zOy plane. Take $\Delta V = (\rho\,\Delta\phi)(\rho \sin\phi\,\Delta\theta)(\Delta\rho) = \rho^2 \sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$ as an approximation of its volume.)

Chapter 74

Masses of Variable Density

HOMOGENEOUS MASSES have been treated in previous chapters as geometric figures by taking the density $\delta = 1$. The mass of a homogeneous body of volume V and density δ is $m = \delta V$.

For a nonhomogeneous mass whose density δ varies continuously from point to point, an element of mass dm is given by:

$\delta(x, y)\, ds$ for a material curve (e.g., a piece of fine wire)

$\delta(x, y)\, dA$ for a material two-dimensional plate (e.g., a thin sheet of metal)

$\delta(x, y, z)\, dV$ for a material body

Solved Problems

1. Find the mass of a semicircular wire whose density varies as the distance from the diameter joining the ends.

Take the wire as in Fig. 74-1, so that $\delta(x, y) = ky$. Then, from $x^2 + y^2 = r^2$,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \frac{r}{y}\, dx$$

and

$$m = \int \delta(x, y)\, ds = \int_{-r}^{r} ky \frac{r}{y}\, dx = kr \int_{-r}^{r} dx = 2kr^2 \text{ units}$$

2. Find the mass of a square plate of side a if the density varies as the square of the distance from a vertex.

Take the square as in Fig. 74-2, and let the vertex from which distances are measured be at the origin. Then $\delta(x, y) = k(x^2 + y^2)$ and

$$m = \iint_R \delta(x, y)\, dA = \int_0^a \int_0^a k(x^2 + y^2)\, dx\, dy = k \int_0^a \left(\tfrac{1}{3}a^3 + ay^2\right) dy = \tfrac{2}{3}ka^4 \text{ units}$$

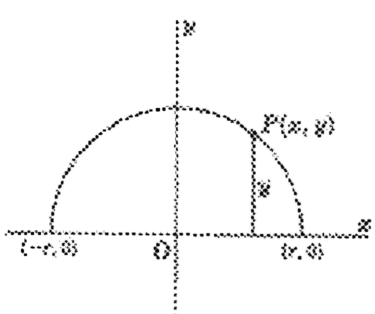

Fig. 74-1

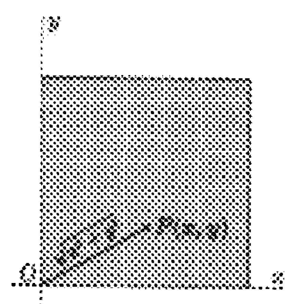

Fig. 74-2

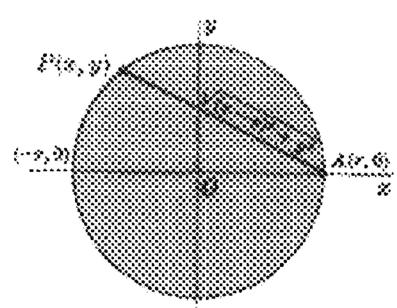

Fig. 74-3

3. Find the mass of a circular plate of radius r if the density varies as the square of the distance from a point on the circumference.

Take the circle as in Fig. 74-3, and let $A(r, 0)$ be the fixed point on the circumference. Then $\delta(x, y) = k[(x-r)^2 + y^2]$ and

$$m = \iint_R \delta(x, y)\, dA = 2\int_{-r}^{r}\int_0^{\sqrt{r^2-x^2}} k[(x-r)^2 + y^2]\, dy\, dx = \tfrac{3}{2}k\pi r^4 \text{ units}$$

4. Find the center of mass of a plate in the form of the segment cut from the parabola $y^2 = 8x$ by its latus rectum $x = 2$ if the density varies as the distance from the latus rectum. (See Fig. 74-4.)

Here, $\delta(x, y) = 2 - x$ and, by symmetry, $\bar{y} = 0$. For the upper half of the plate,

$$m = \iint_R \delta(x, y)\, dA = \int_0^4 \int_{y^2/8}^{2} k(2-x)\, dx\, dy = k\int_0^4 \left(2 - \frac{y^2}{4} + \frac{y^4}{128}\right) dy = \frac{64}{15}k$$

$$M_y = \iint_R \delta(x, y)x\, dA = \int_0^4 \int_{y^2/8}^{2} k(2-x)x\, dx\, dy = k\int_0^4 \left[\frac{4}{3} - \frac{y^4}{64} + \frac{y^6}{(24)(64)}\right] dy = \frac{128}{35}k$$

and $\bar{x} = M_y/m = \frac{6}{7}$. The center of mass has coordinates $(\frac{6}{7}, 0)$.

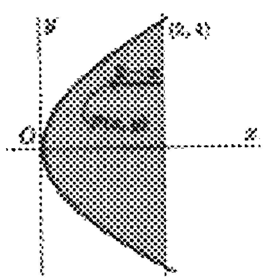

Fig. 74-4

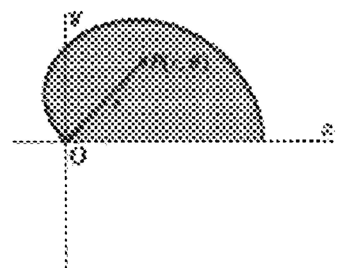

Fig. 74-5

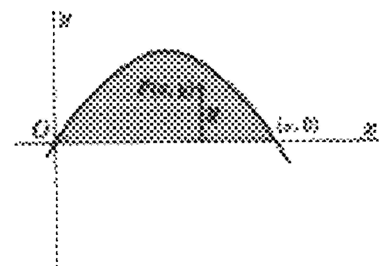

Fig. 74-6

5. Find the center of mass of a plate in the form of the upper half of the cardioid $r = 2(1 + \cos\theta)$ if the density varies as the distance from the pole. (See Fig. 74-5.)

$$m = \iint_R \delta(r, \theta)\, dA = \int_0^{\pi}\int_0^{2(1+\cos\theta)} (kr)r\, dr\, d\theta = \tfrac{8}{3}k\int_0^{\pi} (1+\cos\theta)^3\, d\theta = \tfrac{20}{3}k\pi$$

$$M_x = \iint_R \delta(r, \theta)y\, dA = \int_0^{\pi}\int_0^{2(1+\cos\theta)} (kr)(r\sin\theta)r\, dr\, d\theta$$

$$= 4k\int_0^{\pi} (1+\cos\theta)^4 \sin\theta\, d\theta = \tfrac{128}{5}k$$

$$M_y = \iint_R \delta(r, \theta)x\, dA = \int_0^{\pi}\int_0^{2(1+\cos\theta)} (kr)(r\cos\theta)r\, dr\, d\theta = 14k\pi$$

Then $\bar{x} = \dfrac{M_y}{m} = \dfrac{21}{10}$, $\bar{y} = \dfrac{M_x}{m} = \dfrac{96}{25\pi}$, and the center of mass has coordinates $\left(\dfrac{21}{10}, \dfrac{96}{25\pi}\right)$.

6. Find the moment of inertia with respect to the x axis of a plate having for edges one arch of the curve $y = \sin x$ and the x axis if its density varies as the distance from the x axis. (See Fig. 74-6.)

$$m = \iint_R \delta(x, y)\, dA = \int_0^{\pi} \int_0^{\sin x} ky\, dy\, dx = \tfrac{1}{2}k \int_0^{\pi} \sin^2 x\, dx = \tfrac{1}{4}k\pi$$

$$I_x = \iint_R \delta(x, y) y^2\, dA = \int_0^{\pi} \int_0^{\sin x} (ky)(y^2)\, dy\, dx = \tfrac{1}{4}k \int_0^{\pi} \sin^4 x\, dx = \tfrac{3}{32}k\pi = \tfrac{3}{8}m$$

7. Find the mass of a sphere of radius a if the density varies inversely as the square of the distance from the center.

Take the sphere as in Fig. 74-7. Then $\delta(x, y, z) = \dfrac{k}{x^2 + y^2 + z^2} = \dfrac{k}{\rho^2}$ and

$$m = \iiint_R \delta(x, y, z)\, dV = 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{a} \frac{k}{\rho^2} \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta$$

$$= 8ka \int_0^{\pi/2} \int_0^{\pi/2} \sin \phi\, d\phi\, d\theta = 8ka \int_0^{\pi/2} d\theta = 4k\pi a \text{ units}$$

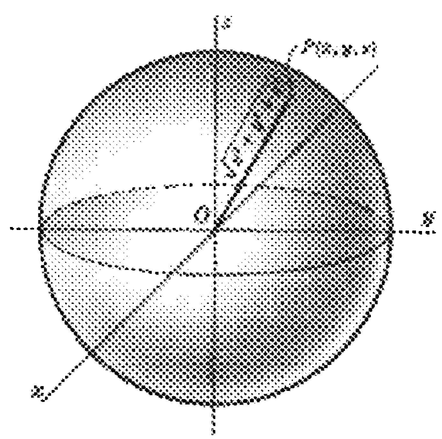

Fig. 74-7

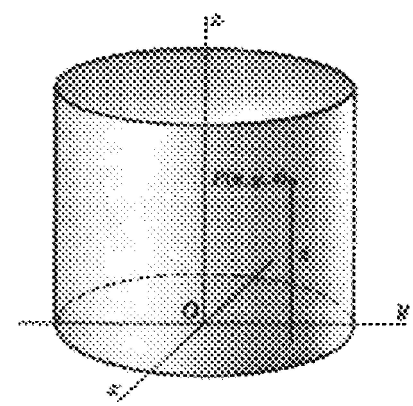

Fig. 74-8

8. Find the center of mass of a right circular cylinder of radius a and height h if the density varies as the distance from the base.

Take the cylinder as in Fig. 74-8, so that its equation is $r = a$ and the volume in question is that part of the cylinder between the planes $z = 0$ and $z = h$. Clearly, the center of mass lies on the z axis. Then

$$m = \iiint_R \delta(z, r, \theta)\, dV = 4 \int_0^{\pi/2} \int_0^{a} \int_0^{h} (kz) r\, dz\, dr\, d\theta = 2kh^2 \int_0^{\pi/2} \int_0^{a} r\, dr\, d\theta$$

$$= kh^2 a^2 \int_0^{\pi/2} d\theta = \tfrac{1}{2}k\pi h^2 a^2$$

$$M_{xy} = \iiint_R \delta(z, r, \theta) z\, dV = 4 \int_0^{\pi/2} \int_0^{a} \int_0^{h} (kz^2) r\, dz\, dr\, d\theta = \tfrac{4}{3}kh^3 \int_0^{\pi/2} \int_0^{a} r\, dr\, d\theta$$

$$= \tfrac{2}{3}kh^3 a^2 \int_0^{\pi/2} d\theta = \tfrac{1}{3}k\pi h^3 a^2$$

and $\bar{z} = M_{xy}/m = \tfrac{2}{3}h$. Thus the center of mass has coordinates $(0, 0, \tfrac{2}{3}h)$.